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VIBRATION OF A NONHOMOGENEOUS MEMBRANE

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1. **Introduction.** We consider a simply connected two dimensional domain D with a nonhomogeneous membrane M stretched across D and fixed at the boundary Γ . Let $p(x, y) \geq 0$ be the density function of the membrane. We shall be concerned with the first eigenvalue λ_0 of the equation

$$(1) \quad u_{xx} + u_{yy} + \lambda p(x, y)u = 0$$

subject to the condition $u = 0$ on Γ . Let K be the circle with boundary C on which a homogeneous membrane M_1 of the same mass as M is stretched. Let λ_1 be the first eigenvalue of

$$(2) \quad v_{xx} + v_{yy} + \lambda v = 0$$

with $v = 0$ on C . In a recent paper Nehari [1] established the following interesting result.

THEOREM. (Nehari) *If $\log p(x, y)$ is subharmonic then*

$$(3) \quad \lambda_0 \geq \lambda_1 .$$

Nehari further showed that relaxation to the condition that $p(x, y)$ be subharmonic is not possible. In fact for the case that D is a circle and $p(x, y)$ is superharmonic the inequality in (3) is shown to be reversed.

It is the purpose of this paper to establish comparison theorems for the first eigenvalue of homogeneous and nonhomogeneous membranes of the same shape. That is, we shall consider the first eigenvalue of equations (1) and (2) in the same domain D subject to the boundary condition $u = 0$ and $v = 0$ on Γ respectively. We denote the first eigenvalue of the latter problem by μ and consider comparisons between λ_0 and μ . We of course have the completely trivial comparison

$$\lambda_0 \geq \mu$$

if $0 \leq p(x, y) \leq 1$ throughout D . Nehari's result pertained to the case where $p(x, y)$ had average value 1 and thus we wish to obtain relations between λ_0 and μ for density functions which may become large.

A general technique for obtaining lower bounds for the first eigenvalue for a homogeneous membrane in a domain D follows from the

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inclusion principle. If D is contained in D_0 then the first eigenvalue for D is larger than that for D_0 . If D is bounded then we can enclose D in a rectangle or circle for which the first eigenvalue is known. This technique is also possible for nonhomogeneous membranes as will be readily seen from the basic inequalities established in § 2. In § 3 comparison theorems are established when the density function is assumed to satisfy various conditions involving the behavior of the second derivative of $p(x, y)$. Section 4 discusses comparison theorems between two nonhomogeneous membranes.

2. Basic inequalities. Let u be any function which vanishes on Γ , and let $a(x, y)$ be an arbitrary C^2 function in D . We apply Green's theorem to the expression

$$\iint_D au(u_{xx} + u_{yy})dxdy$$

and obtain

$$(4) \iint_D au(u_{xx} + u_{yy})dxdy = - \iint_D a(u_x^2 + u_y^2)dxdy + \frac{1}{2} \iint_D u^2(a_{xx} + a_{yy})dxdy$$

The boundary integrals vanishing in virtue of $u = 0$ on Γ . Further we let $P(x, y), Q(x, y)$ be arbitrary C' functions in D and note that

$$(5) \iint_D [Pu^2]_x + [Qu^2]_y dxdy = 0 .$$

Performing the differentiations in (5) and adding the result to (4) we get

$$\begin{aligned} & - \iint_D au(u_{xx} + u_{yy})dxdy \\ & = + \iint_D \left\{ a(u_x^2 + u_y^2) + 2Pu u_x + 2Qu u_y \right. \\ & \quad \left. + \left[P_x + Q_y - \frac{1}{2}(a_{xx} + a_{yy}) \right] u^2 \right\} dxdy . \end{aligned}$$

If u were the first eigenfunction and λ the first eigenvalue of the nonhomogeneous membrane, then (1) would hold and the above expression would be

$$(6) \iint_D \left\{ a(u_x^2 + u_y^2) + 2Pu u_x + 2Qu u_y \right. \\ \left. + \left[P_x + Q_y - \frac{1}{2}(a_{xx} + a_{yy}) - a\lambda p \right] u^2 \right\} dxdy = 0$$

On the other hand this integrand is a quadratic form in u_x, u_y, u . It will be a positive definite form if $a > 0$ and

$$(7) \quad P_x + Q_y \geq \frac{1}{a}(P^2 + Q^2) + \frac{1}{2}(a_{xx} + a_{yy}) + ap\lambda .$$

If a, P, Q, λ happen to satisfy (7) then clearly it is impossible that (6) holds. However if (7) holds for any value $\bar{\lambda}$, it obviously holds for $0 \leq \lambda \leq \bar{\lambda}$ and thus (6) cannot hold for any function $u(x, y)$ with $0 \leq \lambda \leq \bar{\lambda}$. This implies that $\bar{\lambda}$ is a lower bound for the first eigenvalue of (1).

We shall therefore be concerned with the possibility of selection of functions P, Q, a such that inequality (7) holds for some value $\bar{\lambda}$. For convenience we assume the bounded domain D is in the first quadrant. We select the function $a(x, y)$ to be

$$a(x, y) = \sin \alpha x \sin \beta y$$

where α and β are constants selected so that $a(x, y)$ is positive throughout \bar{D} . We define the quantities

$$m_0 = \min_{\bar{D}} a$$

and $M_0 = m_0^{-1}$. Inequality (7) is implied by the inequality

$$(8) \quad P_x + Q_y \geq M_0(P^2 + Q^2) + \frac{1}{2}(a_{xx} + a_{yy}) + ap\lambda$$

and if we define

$$P_1 = M_0P, Q_1 = M_0Q$$

(8) is equivalent to

$$(9) \quad P_{1x} + Q_{1y} \geq P_1^2 + Q_1^2 + \frac{1}{2}M_0(a_{xx} + a_{yy}) + M_0ap\lambda .$$

Let $\phi(x, y)$ be the first eigenfunction for equation (2) in the domain D subject to the condition $v = 0$ on Γ . That is,

$$\phi_{xx} + \phi_{yy} + \mu\phi = 0 .$$

We make the following selection:

$$P_1 = -\frac{\phi_x}{\phi}, \quad Q_1 = -\frac{\phi_y}{\phi}$$

and obtain from (9)

$$(10) \quad \mu \geq M_0 \sin \alpha x \sin \beta y \left[-\frac{1}{2}(\alpha^2 + \beta^2) + \lambda p(x, y) \right]$$

Define the quantity

$$N_0 = \max_{\bar{D}} p(x, y) \sin \alpha x \sin \beta y$$

and we obtain the following result.

THEOREM 1. *Let λ_0 be the first eigenvalue for the nonhomogeneous membrane with density function $p(x, y)$ spanning a domain D and μ the first eigenvalue for the homogeneous membrane spanning the same domain. Then*

$$(11) \quad \lambda_0 \geq \frac{\mu + \frac{1}{2}(\alpha^2 + \beta^2)}{M_0 N_0} .$$

The theorem is an immediate consequence of inequality (10) which exhibits the positive definiteness of the integrand (6). Inequality (11) is a statement that (10) must be violated.

We note that (11) is a useful relation if N_0 is particularly small; hence this states that $p(x, y)$ should be small near the center of the membrane, but may be large near the outer edge and still (11) will be a significant lower bound for λ_0 . The basic distinction between (11) and other results lies in the fact that $p(x, y)$ has no restriction except positivity.

A word should be said about the selection of the function $a(x, y)$. We chose for this function the first eigenfunction for the equation (2) applied to a rectangle which contains D in its interior. We could have selected for $a(x, y)$ the first eigenfunction for any including domain, e.g., a circle, equilateral triangle, etc. with a resulting inequality similar to (11). Finally the selection $a \equiv 1$ yields the standard result

$$\lambda_0 \geq \frac{\mu}{\max_{\bar{D}} p(x, y)} .$$

3. Bounds with condition on the density function. We return to inequality (7) and the selection of a , P , and Q . We recall that these functions may be arbitrary except that $a(x, y)$ must be positive. We make the choice

$$(12) \quad a(x, y) = \frac{1}{p(x, y)}$$

Then (7) becomes

$$P_x + Q_y \geq p(x, y)(P^2 + Q^2) + \frac{1}{2}\Delta\left(\frac{1}{p}\right) + \lambda .$$

We define

$$(13) \quad p_0 = \max_{\bar{D}} p(x, y)$$

and select

$$P = -\frac{\phi_x}{p_0\phi}, \quad Q = -\frac{\phi_y}{p_0\phi}$$

where, as before, ϕ is the first eigenfunction of (2) for the domain D . We obtain

$$\frac{\mu}{p_0} \geq \frac{1}{2}\Delta\left(\frac{1}{p}\right) + \lambda .$$

If we assume the function $1/p$ is superharmonic and set

$$(14) \quad N_1 = -\max_{\bar{D}} \frac{1}{2}\Delta\left(\frac{1}{p}\right)$$

we obtain the following result.

THEOREM 2. *Let λ_0 be the first eigenvalue for the nonhomogeneous membrane with density function $p(x, y)$ and μ the corresponding first eigenvalue for the homogeneous membrane spanning the same domain D . If $1/p$ is superharmonic in D we have the inequality*

$$(15) \quad \lambda_0 \geq \frac{\mu}{p_0} + N_1$$

where p_0 and N_1 are given by (13) and (14) respectively.

It is possible to obtain a comparison theorem for the case where $\log p$ is subharmonic. To see this we make the choice

$$a(x, y) = \log \frac{1}{p}$$

and we assume $0 < p(x, y) < 1$ in \bar{D} . With this selection we take

$$P = -\frac{\phi_x}{p_0\phi}, \quad Q = -\frac{\phi_y}{p_0\phi}$$

as before and obtain

$$\frac{\mu}{p_0} \geq \frac{1}{2} \Delta \left(\log \frac{1}{p} \right) + \lambda p \log \frac{1}{p}.$$

We assume $\log p$ is subharmonic and define

$$(16) \quad N_2 = \frac{1}{2} \min_{\bar{D}} \Delta(\log p)$$

$$(17) \quad N_3 = \max_{\bar{D}} p \log \frac{1}{p}.$$

THEOREM 3. *Let λ_0 and μ be as in Theorem 2. If $\log p$ is subharmonic in D then*

$$\lambda_0 \geq \frac{\mu}{p_0 N_3} + \frac{N_2}{N_3}$$

where N_2 and N_3 are given by (16) and (17).

A final application of this type which we exhibit results from the selection

$$a = e^{\alpha p(x,y)}$$

where α is a constant which remains to be chosen. If we suppose that p is strictly superharmonic and select α so that

$$\frac{1}{2} \Delta p + \alpha(p_x^2 + p_y^2) \leq 0$$

we obtain the relation

$$\lambda_0 \geq \mu \max_{\bar{D}} \left(\frac{e^{-\alpha p}}{p} \right).$$

4. Comparison of two nonhomogeneous membranes. Let $q(x, y)$ be a second density function corresponding to a membrane spanning D and let ν be the first eigenvalue for

$$(18) \quad w_{xx} + w_{yy} + \nu q(x, y)w = 0$$

with boundary condition $w = 0$ on Γ . We denote the corresponding first eigenfunction by $\psi(x, y)$. It is possible to compare λ_0 and ν when the functions p and q satisfy various relations. Let

$$(19) \quad q_0 = \max_{\bar{D}} q(x, y)$$

$$(20) \quad r_0 = \max_{\bar{D}} \frac{p(x, y)}{q(x, y)}$$

and

$$(21) \quad N_4 = - \max_{\bar{D}} \Delta \left(\frac{q}{p} \right).$$

We make the selections

$$a = \frac{q}{p}, \quad P = - \frac{\sqrt{r_x}}{r_0 \sqrt{r}}, \quad Q = - \frac{\sqrt{r_y}}{r_0 \sqrt{r}}$$

and find

$$\frac{\nu q}{r_0} \geq \frac{1}{2} \Delta \left(\frac{q}{p} \right) + q\lambda.$$

THEOREM 4. *Let λ_0 and ν be the first eigenvalue corresponding to density functions p and q respectively. If q/p is superharmonic then we have the inequality*

$$\lambda_0 \geq \frac{\nu}{r_0} + \frac{1}{2} \frac{N_4}{q_0}$$

where q_0 , r_0 and N_4 are given by (19), (20) and (21).

Additional inequalities, analogous to those obtained in §§ 2 and 3 may be obtained by other selections for a , P and Q .

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