TESTS FOR PRIMALITY BASED ON SYLVESTER’S CYCLOTOMIC NUMBERS

MORGAN WARD
TESTS FOR PRIMALITY BASED
ON SYLVESTER'S CYCLOMATIC NUMBERS

MORGAN WARD

Introduction. Lucas, Carmichael [1] and others have given tests for primality of the Fermat and Mersenne numbers which utilize divisibility properties of the Lucas sequences \((U)\) and \((V)\); in this paper we are concerned only with the first sequence;

\[
(U): U_0, U_1, U_2, \ldots, U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \ldots.
\]

Here \(\alpha\) and \(\beta\) are the roots of a suitably chosen quadratic polynomial \(x^2 - Px + Q\), with \(P\) and \(Q\) coprime integers. (For an account of these tests, generalizations and references to the early literature, see Lehmer's Thesis [2]).

I develop here a test for primality of a less restrictive nature which utilizes a divisibility property of the Sylvester cyclotomic sequence [3]:

\[
(Q): Q_0 = 0, Q_1 = 1, Q_2, \ldots, Q_n = \prod_{1 \leq r < n \atop (r, n) = 1} (\alpha - e^{2\pi ir/n} \beta), \ldots
\]

Here \(\alpha\) and \(\beta\) have the same meaning as before. \((U)\) and \((Q)\) are closely connected [4]; in fact

\[
(1.1) \quad U_n = \prod_{d|n} Q_d.
\]

The divisibility property is expressed by the following theorem proved in § 3 of this paper.

Theorem. If \(m\) is an odd number dividing some cyclotomic number \(Q_n\) whose index \(n\) is prime to \(m\), then every divisor of \(m\) greater than one has the same rank of apparition \(n\) in the Lucas sequence \((U)\) connected with \((Q)\).

Here the rank of apparition or rank, of any number \(d\) in \((U)\) means as usual the least positive index \(x\) such that \(U_x \equiv 0 \pmod{d}\).

The following primality test is an immediate corollary.

Primality test. If \(m\) is odd, greater than two, and divides some cyclotomic number \(Q_n\) whose index \(n\) is both prime to \(m\) and greater than the square root of \(m\), then \(m\) is a prime number except in two trivial cases: \(m = (n-1)^2\), \(n-1\) a prime greater than 3, or \(m = n^2 - 1\) with \(n - 1\) and \(n + 1\) both primes.

Received January 14, 1959.
The primality tests of Lucas and Carmichael are the special case when \( n = m \pm 1 \) is a power of two (which allows \( Q_n \) to be expressed in terms of \( V_n \)) with \( X^2 - Px + Q \) suitably specialized.

2. Notations. We denote the rational field by \( R \), and the ring of rational integers by \( I \). The polynomial

\[
f(x) = x^2 - Px + Q,
\]

is assumed to have distinct roots \( \alpha \) and \( \beta \).

We denote the root field of \( f(x) \) by \( \mathcal{K} \) and the ring of its integers by \( \mathcal{I} \). Thus \( \mathcal{K} \) is either \( R \) itself, or a simple quadratic extension of \( R \).

Let \( p \) be an odd prime of \( I \), and \( \wp \) a prime ideal factor of \( p \) in \( \mathcal{I} \). Every element \( \rho \) of \( \mathcal{K} \) may be put in the form \( \rho = \alpha/a \) with \( \alpha \) in \( \mathcal{K} \) and \( a \) in \( I \). The totality of such \( \rho \) with \( (a, p) = 1 \) forms a subring \( \mathcal{K}_p \) of \( \mathcal{K} \). Evidently \( \mathcal{K} \supset \mathcal{K}_p \supset \mathcal{I} \supset I \). If we extend \( \wp \) into \( \mathcal{K}_p \) in the obvious way, we obtain a prime ideal \( \wp \). The homomorphic image of \( \mathcal{K}_p \) modulo \( \wp \) is a field, \( \mathcal{K}_p / \wp \). We denote the mapping of \( \mathcal{K}_p \) onto \( \mathcal{K} \) by \( \mathcal{I} \).

Let \( F_n(z) \) denote the cyclotomic polynomial of degree \( \phi(n) \). \( F_n(z) \) has coefficients in \( I \), and if \( n \) is greater than one, then (Lehmer [2], Carmichael [1])

\[
Q_n = \beta^{\phi(n)} F_n\left( \frac{\alpha}{\beta} \right),
\]

Furthermore

\[
z^n - 1 = \prod_{d|n} F_n(z).
\]

3. Proof of theorem. Let \( m \) be an odd number greater than one which divides some term of \( (Q) \) whose index \( n \) is prime to \( m \), so that

\[
Q_n \equiv 0 \pmod{m},
\]

(3.1) 

Throughout the next three lemmas, \( p \) stands for a fixed prime factor of \( m \).

**Lemma 1.** If \( \wp \) is any ideal factor of \( p \) in \( \mathcal{I} \), then

\[
(Q, p) = (\alpha, \wp) = (\beta, \wp) = (1).
\]

**Proof.** It suffices to prove that \( (Q, p) = (1) \). Assume the contrary. Then \( (p, P) = 1 \). Since \( U_1 = 1 \) and \( U_{x+2} = P U_{x+1} - Q U_x \equiv P U_{x+1} \pmod{p} \), it follows by induction that \( U_n \not\equiv 0 \pmod{p} \). Then by (1.1), \( Q_n \not\equiv 0 \).
(mod $p$). But $p$ divides $m$ so that by (3.1) $Q_n \equiv 0$ (mod $p$) a contradiction.

**Lemma 2.** The rank of apparition of $p$ in $(U)$ is $n$.

*Proof.* Since $U_n \equiv 0$ (mod $p$), $p$ has a positive rank of apparition in $(U)$, $r$ say. Then $r$ divides $n$. But by (1.1), $U_r = \prod_{d \mid n} Q_d$. Hence $Q_d \equiv 0$ (mod $p$) for some $d$ dividing both $r$ and $n$. Clearly, if $d = n$, then $r = n$ and we are finished. Assume that $d$ is less than $n$.

The number $\alpha / \beta = \alpha / Q$ is in $\mathcal{F}$ by Lemma 1. Let $\tau$ be its image in $\mathcal{F}$ under the mapping $\psi$. Then by (2.2) and Lemma 1 $F_n(\tau) = F_d(\tau) = 0$ in $\mathcal{F}$. Consequently the resultant of the polynomials $F_n(z)$ and $F_d(z)$ is zero in $\mathcal{F}$. Therefore its inverse image under the mapping is in $\mathcal{I}$. But this resultant is evidently in $I$. Therefore it must be divisible by $p$. But by formula (2.3), since $d < n$ the resultant of $F_n(z)$ and $F_d(z)$ must divide the discriminant $\pm n^{n-1}$ of $z^n - 1$. Thus $n \equiv 0$ (mod $p$) so that $(n, m) = 0 \pmod{p}$ which contradicts (3.1) and completes the proof.

**Lemma 3.** The rank of apparition in $(U)$ of any positive power of $p$ which divides $m$ is $n$.

*Proof.* Let $p^k$ divide $m$, $k \geq 1$ and let the rank of $p^k$ in $(U)$ be $r$. Now $U_n = \prod_{d \mid n} Q_d \equiv 0$ (mod $p^k$). But by Lemma 2, each $Q_d$ with $d < n$ is prime to $p$. Hence $r$ must equal $n$.

The theorem proper now follows easily. For let $m'$ be any divisor of $m$ other than one. By Lemma 3, every prime power dividing $m'$ has rank of apparition $n$ in $(U)$. But the rank of apparition of $m'$ in $(U)$ is the least common multiple of the ranks of the prime powers of maximal order dividing $m'$. (Carmichael [1]). Hence $m'$ also has rank of apparition $n$ in $(U)$.

4. **Proof of primality test.** Assume that (3.1) holds for some $n$ greater than $\sqrt{m}$. If $m$ is not a prime, it has a prime factor $\leq \sqrt{m}$. Let $p$ be the smallest such factor, and let

\begin{equation}
(4.1) \quad m = pq, \quad q \geq 3.
\end{equation}

Then $p$ has rank $n$ in $(U)$ by Lemma 3. But by a classical result of Lucas, $U_{p+1} \equiv 0$ (mod $p$). Hence $n$ divides $p + 1$. If $n$ is less than $p + 1$, $\sqrt{m} < p \leq \sqrt{m}$, a contradiction. Hence $n = p + 1$. If $p = \sqrt{m}$, then $m = (n - 1)^2$ and $n - 1$ is a prime. Since $m$ is odd, $n \geq 4$. This is the first trivial case.

If $p < \sqrt{m}$, then $q \geq p + 2$ and $m \geq p(p + 2)$. But if $m > p(p + 2)$,
then \( n^2 > m \geq (p + 1)^2 = n^2 \), a contradiction. Hence \( m = p(p + 2) \) where \( p + 2 \) has no prime factor smaller than \( p \). Hence \( p + 2 \) is a prime and \( m = n^2 - 1 \) with both \( n - 1 \) and \( n + 1 \) primes. This is the second trivial case. In every other case then, \( m \) must be a prime.

5. Conclusion. The two trivial cases can actually occur. For if \( P = 22 \) and \( Q = 3 \), then \( Q_6 = \alpha^2 - \alpha\beta + \beta^2 = P^2 - 3Q = 475 \). Hence \( Q_6 \equiv 0 \pmod{25} \) and \( 25 = (6 - 1)^2 \). Again, if \( P = 17 \) and \( Q = 3 \), then \( Q_6 = 280 \). Hence \( Q_6 \equiv 0 \pmod{35} \) and \( 35 = 6^2 - 1 = 5 \times 7 \). It is worth noting that these trivial cases cannot occur if \( \alpha \) and \( \beta \) are rational integers. (See [1], Theorem XII and remark.)

To illustrate the theorem, note that if \( P = 2 \) and \( Q = 1 \), \( Q_6 = 73 \). Since \( \sqrt{73} < 9 \) and \( (9, 73) = 1 \), \( Q_6 = 73 \) is a prime. But for \( P = 3 \) and \( Q = 1 \), \( Q_6 = 91 \). But \( 9 < \sqrt{91} \) so the test is inapplicable. As a matter of fact, \( 91 \) is the product of two primes. Evidently the test may be extended to cover such a case. That is, if \( Q_n \equiv 0 \pmod{m} \), \( (n, m) = 1 \) and \( n > \sqrt{m} \), \( m \) will usually be either a prime, or the product of two primes.

References

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.
Edmond Darrell Cashwell and C. J. Everett, *The ring of number-theoretic functions* ......................................................... 975
Heinz Otto Cordes, *On continuation of boundary values for partial differential operators* .................................................. 987
Philip C. Curtis, Jr., *n-parameter families and best approximation* ........ 1013
Uri Fixman, *Problems in spectral operators* ........................................ 1029
I. S. Gál, *Uniformizable spaces with a unique structure* .................... 1053
John Mitchell Gary, *Higher dimensional cyclic elements* ................. 1061
Richard P. Gosselin, *On Diophantine approximation and trigonometric polynomials* ......................................................... 1071
Gilbert Helmberg, *Generating sets of elements in compact groups* ....... 1083
Daniel R. Hughes and John Griggs Thompson, *The H-problem and the structure of H-groups* ........................................ 1097
James Patrick Jans, *Projective injective modules* ........................... 1103
Samuel Karlin and James L. McGregor, *Coincidence properties of birth and death processes* .................................................. 1109
Samuel Karlin and James L. McGregor, *Coincidence probabilities* ......... 1141
J. L. Kelley, *Measures on Boolean algebras* .................................... 1165
John G. Kemeny, *Generalized random variables* ................................ 1179
Donald G. Malm, *Concerning the cohomology ring of a sphere bundle* .... 1191
Marvin David Marcus and Benjamin Nelson Moyls, *Transformations on tensor product spaces* .............................................. 1215
Charles Alan McCarthy, *The nilpotent part of a spectral operator* ........ 1223
Kotaro Oikawa, *On a criterion for the weakness of an ideal boundary component* ............................................................... 1233
Barrett O'Neill, *An algebraic criterion for immersion* ......................... 1239
Murray Harold Protter, *Vibration of a nonhomogeneous membrane* ....... 1249
Victor Lenard Shapiro, *Intrinsic operators in three-space* .................. 1257
Morgan Ward, *Tests for primality based on Sylvester's cyclotomic numbers* ................................................................. 1269
L. E. Ward, *A fixed point theorem for chained spaces* ....................... 1273
Alfred B. Willcox, Šilov type C algebras over a connected locally compact abelian group ....................................................... 1279
Jacob Feldman, *Correction to “Equivalence and perpendicularly of Gaussian processes”* .................................................... 1295