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**CHARACTERISTIC SUBGROUPS OF MONOMIAL GROUPS**

RALPH BOYETT CROUCH

# CHARACTERISTIC SUBGROUPS OF MONOMIAL GROUPS

R. B. CROUCH

**1. Introduction.** Let  $U$  be a set,  $o(U) = B = \lambda'_u$ ,  $u \geq 0$ , where  $o(U)$  means the number of elements of  $U$ . Let  $H$  be a fixed group. A monomial substitution  $y$  is a transformation that maps every  $x$  of  $U$  in a one-to-one fashion into an  $x$  of  $U$  multiplied on the left by an element  $h_x$  of  $H$ . Multiplication of substitutions means successive applications. The set of all monomial substitutions forms the monomial group  $\Sigma$ . Ore [5] has studied this group for finite  $U$ , and some of his results have been generalized to general  $U$  in [2], [3], and [4].

This paper determines the structure of the characteristic subgroups for the case when  $U$  is infinite in the cases where normal subgroups and automorphisms are known. The method used makes clear how corresponding theorems for the case where  $U$  is finite might be proved but does not list these results.

**2. Definitions and preliminaries.** Let  $d$  be the cardinal of the integers. Let  $B$  be an infinite cardinal;  $B^+$ , the successor of  $B$ ;  $U$ , a set such that  $o(U) = B$ ; and  $C$  such that  $d \leq C \leq B^+$ . Let  $H$  be a fixed group and  $e$  the identity of  $H$ . Denote by  $\Sigma = \Sigma(H; B, d, C)$  the monomial group of  $U$  over  $H$  whose elements are of the form

$$(1) \quad y = \begin{pmatrix} \cdots & x_\varepsilon & \cdots \\ \cdots & h_\varepsilon x_{i_\varepsilon} & \cdots \end{pmatrix}$$

where only a finite number of the  $h_\varepsilon$  are not  $e$  and the number of  $x$  not mapped into themselves is less than  $C$ . Any element of  $\Sigma$  may be written in the form

$$y = \begin{pmatrix} \cdots & x_\varepsilon & \cdots \\ \cdots & h_\varepsilon x_\varepsilon & \cdots \end{pmatrix} \begin{pmatrix} \cdots & x_\varepsilon & \cdots \\ \cdots & e x_{i_\varepsilon} & \cdots \end{pmatrix}$$

or  $y = vs$  where  $v$  sends every  $x$  into itself and every  $h$  of  $s$  is  $e$ . Elements of the form of

$$v = \begin{pmatrix} \cdots & x_\varepsilon & \cdots \\ \cdots & h_\varepsilon x_\varepsilon & \cdots \end{pmatrix} = [\cdots, h_\varepsilon, \cdots]$$

are *multiplications* and all such elements form a normal subgroup, the *basis groups*  $V(B, d) = V$  of  $\Sigma$ . The  $h_\varepsilon$  of  $y$  are called the *factors* of  $y$ . Elements of the form of  $s$  are *permutations* and all such elements form a subgroup, the *permutation group*,  $S(B, C) = S$  of  $\Sigma(H; B, d, C)$ . Cycles

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of  $s$  will also be written as  $(x_1, \dots, x_n)$  and  $(\dots, x_{-1}, x_0, x_1, \dots)$ . Baer [1] has shown that the normal subgroups of  $S(B, C)$  are the alternating group,  $A=A(B, d)$ , and  $S(B, D)$  where  $d \leq D \leq C$ . Let  $E$  be the identity of  $\Sigma$ ,  $I$  the identity of  $S$ .

**3. Characteristic subgroups of  $\Sigma(H; B, d, C)$ ,  $d \leq C < B^+$ .** The normal subgroups of  $\Sigma(H; B, d, C)$  are known [2], [3]. They are classified first as to whether or not they are contained in the basis group  $V$ .

If  $N$  is normal in  $\Sigma$  and  $N \subset V$  its elements are multiplications with only a finite number of non-identity factors which are contained in a normal subgroup  $G$  of  $H$ . The set of all possible products of factors of all elements of  $N$  form a normal subgroup  $G_1$  of  $H$ . The group  $G/G_1$  is Abelian and  $G/G_1$  is in the center of  $H/G_1$ .

If  $M$  is normal in  $\Sigma$  and  $M \not\subset V$  then  $M \cap S = P \neq E$  is a normal subgroup of  $S$ . The group  $N = M \cap V$  is as above except that  $G = H$ . It becomes necessary to consider the cases where  $P = S(B, D)$  with  $d \leq D \leq C$  and  $P = A(B, d)$ . When  $P = S(B, D)$  then  $M = N \cup P$ .

If  $M$  is normal in  $\Sigma$ ,  $M \not\subset V$ ,  $P = A(B, d)$ ,  $M \cap V = N$ ,  $M/N \cong A(B, d)$  then  $M = N \cup A(B, d)$ .

If  $M$  is normal in  $\Sigma$ ,  $M \not\subset V$ ,  $P = A(B, d)$ ,  $M \cap V = N$ ,  $M/N \not\cong A(B, d)$  then  $M = N \cup A(B, d) \cup L$  where  $L$  is the cyclic group generated by  $[e, a](1, 2)$  with  $a^2 \in G_1$ ,  $a \notin G_1$ .

The converses of these theorems are true. That is, if one starts with the proper subgroups of  $H$  and constructs  $N$  or  $M$  as above the resulting group is normal in  $\Sigma$ .

The automorphisms of  $\Sigma(H; B, d, C)$  are known [4]. A mapping  $\theta$  is an automorphism of  $\Sigma(H; B, d, C)$  if and only if  $\theta = T^+ I_{(s^+)} I_{(v^+)}$  where  $T^+$ ,  $I_{(s^+)}$ ,  $I_{(v^+)}$  are automorphisms of  $\Sigma$  defined as follows. Let  $T$  be any automorphism of  $H$ . Then

$$yT^+ = vst^+ = [h_1, \dots, h_s, \dots]sT^+ = [h_1^T, \dots, h_s^T, \dots]s.$$

Let  $s^+ \in S(B, B^+)$ . Then  $I_{(s^+)}$  is defined by  $yI_{(s^+)} = s^+y(s^+)^{-1}$ . Let  $v^+ \in V(B, B^+)$  if  $C = d$ ,  $v^+ \in V(B, d)$  if  $d < C$  then  $I_{(v^+)}$  is defined by  $yI_{(v^+)} = v^+y(v^+)^{-1}$ .

**THEOREM 1.** *If  $N$  is a subgroup of  $\Sigma(H; B, d, C)$  contained in the basis group then  $N$  is characteristic in  $\Sigma$  if and only if  $N$  is normal in  $\Sigma$ , (hence is as described above) and  $G, G_1$  are characteristic in  $H$ .*

*Proof.* Assume  $N$  is characteristic in  $\Sigma$ . Then  $N$  is normal in  $\Sigma$  and its structure is known. Choose  $\theta = T^+$  with  $T$  arbitrary in the automorphism group of  $H$  and  $v$  arbitrary in  $N$ . Then

$$\begin{aligned} v\theta &= [e, \dots, e, e, g_{i_1}, e, \dots, e, g_{i_n}, e, \dots]T^+ \\ &= [e, \dots, g_{i_1}^T, e, \dots, e, g_{i_n}^T, e, \dots] . \end{aligned}$$

The elements  $g_{i_1}^T$  must be in  $G$ . This shows  $G$  is characteristic in  $H$ . Furthermore  $g_{i_1}^T g_{i_2}^T \dots g_{i_n}^T = (g_{i_1} \dots g_{i_n})^T$  must be in  $G_1$  and since  $g_{i_1} \dots g_{i_n}$  is arbitrary in  $G_1$ ,  $G_1$  is characteristic in  $H$ .

Conversely, if  $N \subset V(B, d)$ ,  $N$  is normal in  $\Sigma$ ,  $G, G_1$  are characteristic in  $H$  then  $N$  is characteristic in  $\Sigma$ . To see this let  $v_1$  be arbitrary in  $N$ . Then  $v_1\theta = v_1 T I_{(s^+)I_{(v^+)}} = v_2 I_{(s^+)I_{(v^+)}}$ . The non-identity factors of  $v_2$  are in  $G$  and their product in  $G_1$  by  $G, G_1$  characteristic in  $H$ . Now  $v_2 I_{(s^+)I_{(v^+)}} = (v^+)(s^+)v_2(s^+)^{-1}(v^+)^{-1}$ . The effect of  $I_{(s^+)}$  on  $v_2$  is to permute the non-identity factors so  $(v^+)(v_3)(v^+)^{-1}$  is now to be considered with  $v_3$  in  $N$ . Since  $G$  is normal in  $H$  in  $G/G_1$  is in the center of  $H/G_1$ ,  $(v^+)v_3(v^+)^{-1}$  will be in  $N$ .

**THEOREM 2.** *Let  $M = N \cup P$  be a normal subgroup of  $\Sigma(H; B, d, C)$ ,  $d \leq C < B^+$ , where  $N$  is as described above,  $P = S(B, D)$ . Then  $M$  is characteristic in  $\Sigma$  if and only if  $G_1$  is characteristic in  $H$ .*

*Proof.* By an argument similar to that used in Theorem 1,  $G_1$  is characteristic in  $H$ .

Conversely, if  $y = v_1 s_1$  is arbitrary in  $M$  then

$$v_1 s_1 \theta = v_1 s_1 T^+ I_{(s^+)I_{(v^+)}} = v_2 s_1 I_{(s^+)I_{(v^+)}} .$$

Since  $G_1$  is characteristic in  $H$ ,  $v_2$  belongs to  $N$ . Now consider

$$(v^+)(s^+)v_2 s_1 (s^+)^{-1}(v^+)^{-1} = (v^+)v_3 (s^+)s_1 (s^+)^{-1}(v^+)^{-1} = (v^+)v_3 s_2 (v^+)^{-1} .$$

The multiplication  $v_3$  is in  $N$  since the factors are still in  $H$ , and the product of the factors is still in  $G_1$  since  $H/G_1$  is Abelian. The permutation  $s_2$  is in  $P$  since  $P$  is normal in  $S(B, B^+)$ . It is now convenient to consider two cases. If  $C=d$  the permutation  $s_2$  is finite and  $(v^+)v_3 s_2 (v^+)^{-1} = (v^+)v_3 v_4 s_2$  where the factors of  $v_4$  differ from the inverse of those in  $(v^+)$  in only a finite number of places. Therefore  $(v^+)v_3 v_4$  will have a finite number of factors of the form  $k_{i_\varepsilon} h_\varepsilon k_{i_\varepsilon}^{-1}$ . If  $k_{i_\varepsilon} \neq k_{i_\varepsilon}$  then  $k_{i_\varepsilon} h_\varepsilon k_{i_\varepsilon}, k_{i_\varepsilon} \neq k_{i_\varepsilon}$ , will be a factor of  $(v^+)v_3 v_4$ . Since  $H/G_1$  is Abelian the product of the factors is in  $G_1$ . Therefore,  $(v^+)v_3 v_4 s_2 = v_5 s_2$  belongs to  $M$ . If  $C > d$  then  $(v^+), v_4$  have only a finite number of non-identity factors and the same argument holds. Therefore  $(v^+)v_3 v_4 s_2$  belongs to  $M$ .

**THEOREM 3.** *Let  $M = N \cup A(B, d)$  be a normal subgroup of  $\Sigma(H; B, d, C)$ ,  $d \leq C < B^+$ . Then  $M$  is characteristic in  $\Sigma$  if and only if  $G_1$  is characteristic in  $H$ .*

*Proof.* The argument used in the proof of Theorem 1 may be used to show that  $G_1$  is characteristic in  $H$  if  $M$  is characteristic in  $\Sigma$ .

Conversely, if  $y = v_1s_1$  is arbitrary in  $M$  then

$$\begin{aligned} y\theta &= v_1s_1\theta = v_1s_1T^+I_{(s^+)}I_{(v^+)} = v_2s_1I_{(s^+)}I_{(v^+)} = (v^+)(s^+)v_2s_1(s^+)^{-1}(v^+)^{-1} \\ &= (v^+)v_3(s^+)s_1(s^+)^{-1}(v^+)^{-1} = (v^+)v_3s_2(v^+)^{-1} = (v^+)v_3v_4s_2. \end{aligned}$$

Now  $v_2 \in N$  by  $G_1$  characteristic in  $H$  and  $v_3$  will be in  $N$  by  $H/G_1$  Abelian. Since  $A(B, d)$  is normal in  $S(B, B^+)$ ,  $s_2$  belongs to  $A(B, d)$ . The factors of  $v_4$  differ from the inverse of those in  $v$  in only a finite number of places since  $s_2$  moves only a finite number of  $x$ . Therefore,  $(v^+)v_3v_4 \in N$ ,  $s_2 \in A(B, d)$  and  $M$  is characteristic in  $\Sigma$ .

**THEOREM 4.** *Let  $M_1 = N \cup A \cup L$  be a normal subgroup of  $\Sigma(H; B, d, C)$ ,  $d \leq C < B^+$ . Let  $L$  be generated by  $y = [e, a](1, 2)$  with  $a^2 \in G_1$ ,  $a \notin G_1$ . Then  $M_1$  is characteristic in  $\Sigma$  if and only if  $G_1$  is characteristic in  $H$ , and  $a^x$  belongs to the coset  $aG_1$  for all automorphisms  $T$  of  $H$ .*

*Proof.* By considering  $v \in N$  and  $\theta = T^+$  we see that  $G_1$  is characteristic in  $H$ . By considering  $y = [e, a](1, 2)$  of  $M_1$  and  $\theta = T^+$  we see that  $[e, a^T](1, 2)$  must belong to  $M_1$ . This means  $a^T$  belongs to  $aG_1$ .

Conversely, if  $v_1s_1 \in M_1$  then

$$\begin{aligned} v_1s_1\theta &= v_1s_1T^+I_{(s^+)}I_{(v^+)} = v_2s_1I_{(s^+)}I_{(v^+)} = (v^+)(s^+)v_2s_1(s^+)^{-1}(v^+)^{-1} \\ &= (v^+)v_3(s^+)s_1(s^+)^{-1}(v^+)^{-1} = (v^+)v_3s_2(v^+)^{-1} = (v^+)v_3v_4s_2. \end{aligned}$$

Now  $v_2s_1$  is in  $M_1$  by  $G_1$  characteristic if the product of the factors of  $v_1$  is in  $G_1$  and by  $a^x$  in  $aG_1$  if the product of the factors is in  $aG_1$ . The multiplication  $v_3$  has only a finite number of non-identity factors because  $v_2$  has only a finite number of non-identity factors. Since  $s_1$  is finite,  $s_2$  is a finite permutation and is even or odd as  $s_1$  is even or odd. Therefore,  $v_4$  has only a finite number of factors different from the inverse of the factors of  $(v^+)$ . The factors of  $(v^+)v_3v_4$  have their product in  $G_1$  or  $aG_1$  according as  $v_3$  has its product in  $G_1$  or  $aG_1$ . Therefore, if  $s_1$  was even  $s_2$  is even,  $v_1$  had the product of its factors in  $G_1$  and so does  $(v^+)v_3v_4$ . If  $s_1$  was odd so is  $s_2$  and  $v_1$  had the product of its factors in  $aG_1$  and so does  $(v^+)v_3v_4$ . That is,  $M_1$  is characteristic.

**4. Characteristic subgroups of  $\Sigma_A(H; B, d, d)$ .** The normal subgroups of  $\Sigma_A(H; B, d, d)$  are precisely those of  $\Sigma(H; B, d, d)$  that are contained in  $\Sigma_A(H; B, d, d)$  [2, p. 210]. The automorphism of  $\Sigma_A(H; B, d, d)$  are those of  $\Sigma(H; B, d, d)$  restricted to  $\Sigma(H; B, d, d)$  [4, p. 84].

**THEOREM 5.** *Let  $N$  be a subgroup of  $\Sigma_A(H; B, d, d)$  contained in the basis group. Then  $N$  is characteristic in  $\Sigma_A$  if and only if  $N$  is normal in  $\Sigma_A$  and  $G, G_1$  are characteristic in  $H$ .*

**THEOREM 6.** *Let  $M$  be a subgroup of  $\Sigma_A(H; B, d, d)$ ,  $M \not\subseteq V(B, d)$ . Then  $M$  is characteristic in  $\Sigma_A$  if and only if  $M$  is normal, i.e.  $M = N \cup A$ , and  $G_1$  is characteristic in  $H$ .*

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