A CLASS OF HYPER-FC-GROUPS

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1. Introduction. An element $g$ of an arbitrary group $G$ is called an FC element if it has a finite number of conjugates in $G$. The set of all FC elements of $G$ forms a characteristic subgroup $H$ of $G$ (see Baer [1]). The upper FC-series of $G$, introduced by Haimo [4] as the FC-chain, may be defined by

\[ H_0 = \{1\}, \]
\[ H_{i+1}/H_i = H(G/H_i), \]

the subgroup of all FC elements of $G/H_i$. The upper FC-series is continued transfinitely in the usual way, by defining

\[ H_\alpha = \bigcup_{\beta < \alpha} H_\beta, \]

when $\alpha$ is a limit ordinal. If $H_\gamma = G$, but $H_\delta \neq G$, for all $\delta < \gamma$, we say that the group $G$ is hyper-FC of FC-class $\gamma$, following McLain [7].

A group $G$ in which the transfinite upper central series

\[ \{1\} = Z_0 \leq Z_1 \leq \cdots \leq Z_\alpha \leq \cdots \]

reaches the whole group is called a ZA-group (Kurosh [6]), and we may say that $G$ has class $\alpha$ if $Z_\alpha = G$, but $Z_\beta \neq G$, for all $\beta < \alpha$. Glushkov [3] and McLain [7] have given constructions for a ZA-group of any given class. The main object of this note is to construct groups of given FC-class.

2. Constructions and proofs.

Definition. We say that a group $G$ is of type $Q_\alpha$ if

1. $G$ has FC-class $\alpha$, with upper FC-series
   \[ \{1\} = H_0 \leq H_1 \leq \cdots \leq H_\alpha = G, \]

2. $H_{\gamma+1}/H_\gamma$ is infinite, for all $\gamma < \alpha$, and

3. $H_{\gamma+1}/H_\gamma$ has the unit subgroup for its centre, for all $\gamma < \alpha$.

Thus the group with only one element is of type $Q_0$, and, in constructing a group $G$ of type $Q_\alpha$, we may assume the existence of a group $G_\beta$ of type $Q_\beta$, for each $\beta < \alpha$. If $\alpha$ is a limit ordinal, we define $G$ to be the ordinary (restricted) direct product of the groups $G_\beta$, for all $\beta < \alpha$. Then $G$ has the properties 1 — 3, and thus has type $Q_\alpha$. For the case $\alpha = \beta + 1$ we shall construct $G$ by ‘wreathing’ the regular
representation of $G_\beta$ with a certain kind of infinite centreless $FC$-group of permutations of the positive integers. (For convenience, we say that a group is centreless if its centre consists of the unit element alone.)

**Definition.** A faithful representation of a group $G$ by permutations of the positive integers will be called a *special* representation of $G$ if

(i) the stabiliser of each integer has finite index in $G$ and  
(ii) the intersection of the stabilisers of the elements of any set of all but a finite number of these integers is the unit subgroup.

**Definition.** An infinite centreless $FC$-group possessing a special representation will be called a group of type $F$.

To construct an example of a group of type $F$, let $D = B_1 \times B_2 \times \cdots$ be the ordinary direct product of an infinite sequence of finite centreless groups $B_i$, $i = 1, 2, \cdots$. Let $D_n = B_{n+1} \times B_{n+2} \times \cdots$, let $k_n$ be the order of $D/D_n$ and let the elements of $D/D_n$, in an arbitrary order, be $X_1^n, X_2^n, \cdots, X_{k_n}^n$.

For each element $g \in D$ and each $n = 1, 2, \cdots$, define the permutation $\pi_{gn}$ of the integers $1, 2, \cdots, k_n$ by the rule

(1) \[ \pi_{gn}(i) = j \text{ when } gX_i^n = X_j^n. \]

Now, for each $g \in G$, define the permutation $\pi_g$ of the positive integers by the rule

(2) \[ \pi_g(i + \sum_{j=1}^{n-1} k_j) = \pi_{gn}(i) + \sum_{j=1}^{n-1} k_j, \]

for all $i = 1, 2, \cdots, k_n$, and $n = 1, 2, \cdots$. The systems of transitivity in this permutation representation of $D$ are the sets $T_n$ of integers $m$ such that $\sum_{i=1}^{n-1} k_i < m \leq \sum_{i=1}^n k_i$, for $n = 1, 2, \cdots$. If $m \in T_n$, then the subgroup $D_n$ of $D$ is contained in the stabiliser of $m$. Hence the stabiliser in $D$ of each positive integer has finite index in $D$. On the other hand, suppose $g$ is in the stabiliser in $D$ of all but a finite number of the positive integers. Then there is a number $n_0$ such that $g$ is in the stabiliser of each integer of each system $T_n$ with $n \geq n_0$. So if $i$ is any integer in the range $1 \leq i \leq k_n$, $n \geq n_0$, we know that $g$ is in the stabiliser of $i + \sum_{j=1}^{n-1} k_j$, and this means that $gX_i^n = X_j^n$. Thus $g \in D_n$. But the subgroups $D_n$, with $n \geq n_0$, intersect in the unit subgroup of $D$. So $g = 1$. We observe also that the permutation representation of $D$ defined by (1) and (2) is faithful. Thus we have a special representation of the infinite centreless $FC$-group $D$, which is therefore a group of type $F$.

**Lemma.** If $G_\beta$ is a group of type $Q_\beta$ and $J$ is a group of type $F$,
A CLASS OF HYPER-FC-GROUPS

then a group $G$ formed by wreathing the regular representation of $G_\beta$ with a special representation $R$ of $J$ is a group of type $Q_{\beta+1}$.

Proof. The wreath group $G$ may be regarded as a semi-direct product

$$G = KE, \ K \cap E = 1,$$

where $K = \prod_{i=1}^{\infty} A_i$ is the direct product of a sequence of groups, each isomorphic to $G_\beta$, and $E$ is isomorphic to $J$. The automorphisms of $K$ induced by elements of $E$ permute the subgroups $A_i$, $i = 1, 2, \ldots$, realizing the special representation $R$ of $J \simeq E$. Associated with $G$ is a set of isomorphisms $\theta_{ij}$, $i, j = 1, 2, \ldots$ such that $\theta_{ij}(A_i) = A_j$, and if $a \in A_i$, $g \in E$ and $g^{-1}A_ig = A_j$, then $g^{-1}ag = \theta_{ij}(a)$. $\theta_{ii}$ is the identity automorphism, for all $i$. (A brief general description of wreath groups, and further references, may be found in Hall [5].)

Let $C_i$ be the set of all elements $g$ in $E$ such that $g^{-1}A_ig = A_i$. Then $C_i$ is the centraliser in $E$ of each element of $A_i$. Since the representation $R$ is special, the subgroup $C_i$ of $E$ has finite index in $E$, for each $i$, and the unit element is the only element of $E$ common to all the subgroups of any set of all but a finite number of the $C$'s.

For all $\gamma \leq \beta$, put $H_\gamma = H_\gamma(K)$, the $\gamma$th term of the upper $FC$-series of $K$. If possible, let $\tau + 1$ be the least such ordinal for which $H_{\tau+1}(G) \neq H_{\tau+1}$. Now any element $k$ of $K$ can be written as the product of a finite number of elements $a_{i\nu} \in A_{i\nu}$, $\nu = 1, 2, \ldots, n$, and the subgroup $C(k) = \bigcap_{\nu=1}^{n} C_{i\nu}$ has finite index in $E$. But $C(k)$ is contained in the centraliser of $k$ in $E$, so $g^{-1}kg$, with $g \in E$, is finite valued. Hence

$$H_{\tau+1}(G) \cap K = H_{\tau+1}.$$

Suppose $kg \in H_{\tau+1}(G)$, where $k \in K$ and $g \in E$, $g \neq 1$. Let $\sigma + 1$ be the least ordinal in the range $\tau + 1 \leq \sigma + 1 \leq \beta$ such that $k \in H_{\sigma+1}$. Now $H_\sigma$ is a characteristic subgroup of $K$, and hence is normal in $G$, and both $kH_\sigma$ and $kgH_\sigma$ are $FC$ elements of $G/H_\sigma$. Hence $gH_\sigma$ is $FC$ in $G/H_\sigma$.

We can choose an infinite sequence of distinct positive integers, $\mu_1, \mu_2, \ldots$, such that $g^{-1}A_{\mu_i}g \neq A_{\mu_i}$, for all $i = 1, 2, \ldots$, for otherwise $g$ would belong to all but a finite number of the $C$'s. Moreover, since $C_i$ has finite index in $E$, for each $i$, we can choose the sequence $\mu_1, \mu_2, \ldots$ so that distinct terms belong to distinct systems of transitivity in the representation $R$ of $E$. By relabelling the subgroups $A_i$, $i = 1, 2, \ldots$, we may arrange that the sequence $\mu_1, \mu_2, \ldots$ is just the sequence of odd positive integers. So if $n$ is any odd positive integer, and $g^{-1}A_ng = A_n$, then $n$ is even. Since $\sigma < \beta$, we can choose
\( a_n \in A_n - H_\sigma(A_n), \) for \( n = 1, 3, \cdots \). Let \( a_n^{-1} = g^{-1} a_n g \), and define
\[
c_n = g^{-1} g^a_n = a_n^{-1} a_n, \quad n = 1, 3, \cdots.
\]
Then
\[
c_n^{-1} c_m = (g^a_n)^{-1} g^a_m = a_n^{-1} a_n a_m^{-1} a_m.
\]
If \( n \neq m \), the four integers \( n, \hat{n}, m \) and \( \hat{m} \) are all distinct and thus \( (g^a_n)^{-1} g^a_m \notin H_\sigma \). Thus \( gH_\sigma \) is not FC in \( G/H_\sigma \), contrary to what we have already proved.

It follows that the upper FC-series of \( G \) is
\[
\{1\} = H_0 \leq H_1 \leq \cdots \leq H_\beta = K < G,
\]
for \( G/K \cong E \cong J \), and \( J \) is an FC-group. Moreover \( J \) is infinite and centreless, and the factors \( H_{\gamma+1}/H_\gamma \) are infinite and centreless, for all \( \gamma < \beta \), since \( G_\beta \) is a group of type \( Q_\beta \), and \( K \) is a direct product of groups isomorphic with \( G_\beta \). Thus \( G \) is a group of type \( Q_{\beta+1} \), as required.

We have now shown how to construct a group of type \( Q_\alpha \), given groups of type \( Q_\beta \) for all \( \beta < \alpha \), whether \( \alpha \) is a limit ordinal or not. So, by transfinite induction, we have:

THEOREM. There exist groups of type \( Q_\alpha \), for any ordinal \( \alpha \).

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REFERENCES


BROWN UNIVERSITY
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Richard Arens, *Extensions of Banach algebras* ........................................... 1
Fred Guenther Brauer, *Spectral theory for linear systems of differential equations* .......................................................... 17
Herbert Busemann and Ernst Gabor Straus, *Area and normality* ................. 35
Ralph Boyett Crouch, *Characteristic subgroups of monomial groups* .......... 85
Richard J. Driscoll, *Existence theorems for certain classes of two-point boundary problems by variational methods* .................................................. 91
A. M. Duguid, *A class of hyper-FC-groups* .................................................. 117
Adriano Mario Garsia, *The calculation of conformal parameters for some imbedded Riemann surfaces* .......................................................... 121
Irving Leonard Glicksberg, *Homomorphisms of certain algebras of measures* .......................................................... 167
Branko Grünbaum, *Some applications of expansion constants* ................... 193
John Hilzman, *Error bounds for an approximate solution to the Volterra integral equation* .......................................................... 203
Charles Ray Hobby, *The Frattini subgroup of a p-group* ............................ 209
Milton Lees, *von Newmann difference approximation to hyperbolic equations* .......................................................... 213
Azriel Lévy, *Axiom schemata of strong infinity in axiomatic set theory* ........ 223
Benjamin Muckenhoupt, *On certain singular integrals* .................................. 239
Kotaro Oikawa, *On the stability of boundary components* .......................... 263
J. Marshall Osborn, *Loops with the weak inverse property* ......................... 295
Paulo Ribenboim, *Un théorème de réalisation de groupes réticulés* .......... 305
Daniel Saltz, *An inversion theorem for Laplace-Stieltjes transforms* ........ 309
Berthold Schweizer and Abe Sklar, *Statistical metric spaces* ..................... 313
Morris Weisfeld, *On derivations in division rings* ..................................... 335
Bertram Yood, *Faithful *-representations of normed algebras* .................... 345