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THE FRATTINI SUBGROUP OF A p -GROUP

CHARLES RAY HOBBY

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The Frattini subgroup $\Phi(G)$ of a group G is defined as the intersection of all maximal subgroups of G . It is well known that some groups cannot be the Frattini subgroup of any group. Gaschütz [3, Satz 11] has given a necessary condition for a group H to be the Frattini subgroup of a group G in terms of the automorphism group of H . We shall show that two theorems of Burnside [2] limiting the groups which can be the derived group of a p -group have analogues that limit the groups which can be Frattini subgroups of p -groups.

We first state the two theorems of Burnside.

THEOREM A. *A non-abelian group whose center is cyclic cannot be the derived group of a p -group.*

THEOREM B. *A non-abelian group, the index of whose derived group is p^2 , cannot be the derived group of a p -group.*

We shall prove the following analogues of the theorems of Burnside.

THEOREM 1. *If H is a non-abelian group whose center is cyclic, then H cannot be the Frattini subgroup $\Phi(G)$ of any p -group G .*

THEOREM 2. *A non-abelian group H , the index of whose derived group is p^2 , cannot be the Frattini subgroup $\Phi(G)$ of any p -group G .*

We shall require four lemmas, the first two of which are due to Blackburn and Gaschütz, respectively.

LEMMA 1. [1, Lemma 1] *If N is a normal subgroup of the p -group G such that the order of N is p^2 , then the centralizer of N in G has index at most p in G .*

LEMMA 2. [3, Satz 2] *If $H = \Phi(G)$ for a p -group G and N is a subgroup of H that is normal in G , then $\Phi(G/N) = \Phi(G)/N$.*

LEMMA 3. *If $N = \{a\} \times \{b\}$ is a subgroup of order p^3 normal in the p -group G such that N is contained in $\Phi(G)$, and if $\{a\}$ is a group of order p^2 in the center of $\Phi(G)$, then N is in the center of $\Phi(G)$.*

Proof. N normal in G implies that N contains a group C of order p which is in the center of G . If C is not contained in $\{a\}$ the proof

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is trivial, hence we may assume $C = \{a^p\}$. Since an element of order p in a p -group cannot be conjugate to a power of itself the possible conjugates of b under G are

$$b, ba^p, \dots, ba^{(p-1)p}.$$

The index of the centralizer of b in G is equal to the number of conjugates of b under G , hence is at most p . Thus b is in the center of $\Phi(G)$, and the lemma follows.

LEMMA 4. *If H is a non-abelian group of order p^3 then there is no p -group G such that $\Phi(G) = H$.*

Proof. If $H = \Phi(G)$ for a p -group G , then H is normal in G and must contain a group N of index p which is also normal in G . Then N is a group of order p^2 , hence (Lemma 1) the centralizer C of N in G has index at most p in G . Therefore C contains H , and N is in the center of H . Since the center of H has order p this is a contradiction, and the lemma follows.

We can now prove Theorems 1 and 2.

Proof of Theorem 1. We proceed by induction on the order of H . The theorem is true if H has order p^3 (Lemma 4). Suppose H is group of minimal order for which the theorem is false, and let C of a subgroup of H of order p which is contained in the center of G . Then (Lemma 2)

$$\Phi(G/C) = \Phi(G)/C = H/C.$$

Thus the induction hypothesis implies that H/C cannot be a non-abelian group with cyclic center. We consider two cases: H/C is abelian; or, the center of H/C is non-cyclic.

Case 1. Suppose H/C is abelian. Since H is not abelian, and C has order p , we conclude that C is the derived group of H . Thus H/C , which coincides with its center, is not cyclic, and we are in Case 2.

Case 2. Suppose that the center Z of H/C is non-cyclic. The elements of order p in Z form a characteristic subgroup P of Z . Since Z is not cyclic, P is also not cyclic and hence has order at least p^2 . Thus we can find subgroups \bar{M} and \bar{N} of P which are normal in G/C and have orders p and p^2 , respectively, where \bar{M} is contained in \bar{N} . Let M and N be the subgroups of G which map on \bar{M} and \bar{N} . Then M and N are subgroups of H which contain C and are normal in G ; M and N have orders p^2 and p^3 , respectively, and M is contained in N .

We see from Lemma 1 that the centralizer of M in G has index at

most p in G , hence M is in the center of H , which is cyclic. Also, N is abelian since N is contained in H and the index of M in N is p . Now \bar{N} is contained in P , hence is not cyclic. Therefore N is a non-cyclic group which (Lemma 3) is in the center of H . Since the center of H is cyclic this is a contradiction, and the proof is complete.

Proof of Theorem 2. We denote the derived group of a group K by K' . Suppose G is a p -group such that $\Phi(G) = H$ where $H' \neq \{1\}$ and $(H:H') = p^3$. Let N be a normal subgroup of G which has index p in H' . Then G/N is a p -group such that (Lemma 2)

$$\Phi(G/N) = \Phi(G)/N = H/N.$$

But $(H/N)' = H'/N \neq \{1\}$, and the order of H/N is

$$(H:N) = (H:H')(H':N) = p^3.$$

Thus H/N is a non-abelian group of order p^3 which is the Frattini subgroup of the p -group G/N . This is impossible (Lemma 4) and the theorem follows.

REMARK 1. The only properties of the Frattini subgroup used in the proof of Theorems 1 and 2 are the following: $\Phi(G)$ is a characteristic subgroup of G which is contained in every subgroup of index p in G ; and, $\Phi(G/N) = \Phi(G)/N$ whenever N is normal in G and contained in $\Phi(G)$. Thus if we have a rule ψ which assigns a unique subgroup $\psi(G)$ to every p -group G , then Theorems 1 and 2 will hold after replacing "the Frattini subgroup $\Phi(G)$ " by "the subgroup $\psi(G)$ " if $\psi(G)$ satisfies the following conditions.

- (1) $\psi(G)$ is a characteristic subgroup of G .
- (2) $\psi(G)$ is contained in $\Phi(G)$.
- (3) $\psi(G/N) = \psi(G)/N$ if N is normal in G and N is contained in $\psi(G)$.

In particular, if $\psi(G) = G'$, the derived group of G , we have the theorems of Burnside. The proofs are unchanged.

REMARK 2. Blackburn [1] has used Theorem A to characterize the groups having two generators which are the derived group of a p -group. Using Theorem 1 it is easy to see that Blackburn's proof establishes the following

THEOREM 3. *If $H = \Phi(G)$ for a p -group G and if H has at most two generators, then H contains a cyclic normal subgroup N such that H/N is cyclic.*

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