FAITHFUL *-REPRESENTATIONS OF NORMED ALGEBRAS

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1. Introduction. Let $B$ be a complex Banach algebra with an involution $x \mapsto x^*$ in which, for some $k > 0$, $||xx^*|| \geq k \ ||x|| \ ||x^*||$ for all $x$ in $B$. Kaplansky [8, p. 403] explicitly made note of the conjecture that all such $B$ are symmetric. An equivalent formulation is the conjecture that all such $B$ are $B^*$-algebras in an equivalent norm. In 1947 an affirmative answer had already been provided by Arens [1] for the commutative case. We consider in § 2 the general (non-commutative) case. It is shown that the answer is affirmative if $k$ exceeds the sole real root of the equation $4t^3 - 2t^2 + t - 1 = 0$. This root lies between .676 and .677. In any case these algebras are characterized spectrally as those Banach algebras with involution for which self-adjoint elements have real spectrum and there exists $c > 0$ such that $\rho(h) \geq c \ ||h||$, $h$ self-adjoint (where $\rho(h)$ is the spectral radius of $h$).

A basic question concerning a given complex Banach algebra $B$ with an involution is whether or not it has a faithful*-representation as operators on a Hilbert space. In § 3 we give a necessary and sufficient condition entirely in terms of algebraic and linear space notions in $B$. This is that $\rho(h) = 0$ implies $h = 0$ for $h$ self-adjoint and that $R \cap (-R) = (0)$. Here $R$ is the set of all self-adjoint elements linearly accessible [11, p. 448] from the set of all finite sums of elements of the form $x^*x$. This is related to a previous criterion of Kelley and Vaught [10] which however involves topological notions (in particular, the assumption that the involution is continuous).

If $B$ is semi-simple with minimal one-sided ideals a simpler discussion of *-representations (§ 5) is possible even if $B$ is incomplete. For example if $B$ is primitive then $B$ has a faithful*-representation if and only if $xx^* = 0$ implies $x^*x = 0$. The incomplete case has features not present in the Banach algebra case. In the former case, unlike the latter, a*-representation may be discontinuous. A class of examples is provided in § 5.

2. Arens*-algebras. Let $B$ be a complex normed algebra with an involution $x \mapsto x^*$. An involution is a conjugate linear anti-automorphism of period two. Elements for which $x = x^*$ are called self-adjoint (s. a.) and the set of s. a. elements is denoted by $H$. Let $\mathcal{H}$ be a Hilbert space and...
be the algebra of all bounded linear operators on £. By a*-representation of $B$ we mean a homomorphism $x \rightarrow T_x$ of $B$ into some $\mathcal{B}(\hat{\mathcal{H}})$ where $T_x^*$ is the adjoint of $T_x$. A*-representation which is one-to-one is called faithful.

We shall be mainly, but not exclusively, interested in the case where $B$ is complete (a Banach algebra). In § 2 we shall assume throughout that $B$ is a Banach algebra with an involution $x \rightarrow x^*$. As in [5, p. 8] we set $\chi_{oy} = x + y - xy$ and say that $x$ is quasi-regular with quasi-inverse $y$ if $\chi_{oy} = yx = 0$. The quasi-inverse of $x$ is unique, if it exists, and is denoted by $x'$. As, for example, in [16, p. 617] we define the spectrum of $x$, $sp(x)$, to be the set consisting of all complex numbers $\lambda \neq 0$ such that $\lambda^{-1}x$ is not quasi-regular, plus $\lambda = 0$ provided there does not exist a subalgebra of $B$ with an identity element and containing $x$ as an invertible element. (The treatment of zero as a spectral value plays no role below.) The spectral radius $\rho(x)$ if $x$ is defined to be $\sup |\lambda|$ for $\lambda \in sp(x)$.

We say that $B$ is an Arens*-algebra [1] if there exists $k > 0$ such that $||x^*x|| \geq k ||x|| ||x^*||$, $x \in B$. As usual, we say that $B$ is a $B^*$-algebra if $||x^*x|| = ||x||^2$, $x \in B$.

2.1. LEMMA. Let $B$ an Arens*-algebra with $||xx^*|| \geq k ||x|| ||x^*||$, $x \in B$. Then for each s.a. element $h$, $\rho(h) \geq k ||h||$ and $sp(h)$ is real.

That the spectrum of a s.a. element $h$ is real is shown in [1, p. 273]. By use of the inequality $||h^na|| \geq k ||h^na-1||^2$ as in [16, p. 626] it follows that $\rho(h) \geq k ||h||$. We shall show (Theorem 2.4) that the spectral conditions of Lemma 2.1 imply that $B$ is an Arens*-algebra.

2.2. LEMMA. Suppose that for each s.a. element $h$, $\rho(h) \geq c ||h||$ and $sp(h)$ is real, where $c > 0$. Let $h$ be s.a., $sp(h) \subset [-a, b]$ where $a \geq 0$, $b \geq 0$ and let $r > 0$. Then

1) $||(-t^{-1}h)|| < r$ if $t > (1 - cr)b/cr$ and $t > (1 + cr)a/cr$,

2) $||(t^{-1}h)|| < r$ if $t > (1 - cr)a/cr$ and $t > (1 + cr)b/cr$.

Note that (2) follows from (1) as applied to the element $h$. By [18, Theorem 3.4] the involution is continuous on $B$. Therefore $h$ generates a closed*-subalgebra $B_0$. Let $\mathcal{M}$ be the space of regular maximal ideals of $B_0$. For $t > a$ set $u = (-t^{-1}h)$. By [8, Theorem 4.2], $u \in B_0$. It is readily seen that $u$ is s.a. Since $-t^{-1}h + u + t^{-1}hu = 0$ we have, for each $M \in \mathcal{M}$, $u(M) = h(M)/(t + h(M))$. By, [8, p. 402] the spectrum of $h$ is the same whether computed in $B$ or in $B_0$ so that $-a \leq h(M) \leq b$. Since $\lambda/(t + \lambda)$ is an increasing function of $\lambda$ we see that $-a/(t - a) \leq u(M) \leq b/(t + b)$. Now $\rho(u) = \sup |u(M)|$, $M \in \mathcal{M}$. Therefore, since $u$ is s.a.,

$$(2.1) \quad c ||u|| \leq \rho(u) \leq \max [a/(t - a), b/(t + b)].$$
From formula (2.1), \(\|u\| < r\) if \(a/(t - a) < cr\) and \(b/(t + b) < cr\). This yields (1).

Note that, under the given hypotheses, \(c \leq 1\).

2.3. **Lemma.** Let \(x\) and \(y\) be quasi-regular. Then \(x + y\) is quasi-regular if and only if \(x'y'\) is quasi-regular.

The formulas \(x'\circ(x + y)\circ y' = x'y'\) and \(x + y = x\circ(x'y')\circ y\) yield the desired result. Let \(r > 0\). If \(\|x'\| < r\) and \(\|y'\| < r^{-1}\) it follows from Lemma 2.3 and [12, p. 66] that \((x + y)'\) exists.

Consider the situation of Lemma 2.2 and let \(h_k\) be s. a., \(k = 1, 2\) where \(N = \max(\rho(h_1), \rho(h_2))\). By Lemma 2.2, \(\|t^{-1}h_k\|' < 1\) and \(\|(-t^{-1}h_k)'\| < 1\) if \(t > (1 + c)N/c\). Then, by Lemma 2.3,

\[
sp(h_1 + h_2) \subset \left[-(1 + c)N/c, (1 + c)N/c\right].
\]

Suppose next that \(sp(h_k) \subset [0, \infty), k = 1, 2\). Then \(\|t^{-1}h_k''\| < 1\) if \(t > (1 + c)N/c\) and \(\|(-t^{-1}h_k)'\| < 1\) if \(t > (1 - c)N/c\). Then by Lemma 2.3,

\[
sp(h_1 + h_2) \subset \left[-(1 - c)N/c, (1 - c)N/c\right].
\]

2.4. **Theorem.** Suppose that for each s. a. element \(h\), \(\rho(h) \geq c\|h\|\) and \(sp(h)\) is real, where \(c > 0\). Then \(B\) is an Arens*-algebra with \(\|xx^*\| \geq k\|x\|\|x^*\|, x \in B\), where \(k\) can be chosen to be \(c^3/(1 + c)(1 + 2c^2)\).

Let \(x = u + iv\) where \(u\) and \(v\) are s. a. Then \(xx = u^2 + v^2 + i(2uv - vu), xx^* = u^2 + v^2 + i(vu - uv)\) and \(xx^* + x^*x = 2u^2 + 2v^2\). We next compare \(\rho(u') = [\rho(u)]'\) and \(\rho(v')\) with \(\rho(xx^*)\). For this purpose we may suppose that \(\rho(u) \geq \rho(v)\) for otherwise we can replace \(x\) by \(ix = -v + iu\). If \(\lambda \neq 0\) then \(\lambda \in sp(xx^*)\) if and only if \(\lambda \in sp(x^*x)\). Thus \(\rho(xx^*) = \rho(x^*x)\).

By (2.2), \(sp(xx^* + x^*x) \subset \left[-(1 + c)\rho(xx^*)/c, (1 + c)\rho(xx^*)/c\right].\) Now \(2u^2 = xx^* + x^*x - 2v^2\). Let \(r > 0, t > 0\). By Lemma 2.2,

\[
\|t^{-1}(xx^* + x^*x)'\| < r, t > (1 + cr)(1 + c)\rho(xx^*)/c^2r.
\]

Since \(sp(-2v^2) \subset (-\infty, 0]\) and \(\rho(2v^2) \leq \rho(2u^2)\), by Lemma 2.2 we have, for \(t > 0\),

\[
\|t^{-1}(-2v^2)'\| < r^{-1}, t > (r - c)\rho(2u^2)/c.
\]

we select \(c < r < 2c\). For such \(r\), Lemma 2.3 and formulas (2.4) and (2.5) show that \(t^{-1}(2u^2)'\) exists if \(t > \max\{(1 + cr)(1 + c)\rho(xx^*)/c^2r, (r - c)\rho(2u^2)/c\}\). Now \((r - c)/c < 1\) and \(sp(2u^2) \subset [0, \infty)\). Therefore, letting \(r \rightarrow 2c\), we have

\[
\rho(2u^2) \leq (1 + 2c^2)(1 + c)\rho(xx^*)/(2c^3).
\]

On the other hand \(\|x\| \leq \|u\| + \|v\| \leq [\rho(u) + \rho(v)])/c \leq 2\rho(u)/c \) and \(\|x^*\| \leq 2\rho(u)/c\). Therefore, by (2.6),
(2.7) \[ ||x|| ||x^*|| \leq 4\rho(u^2)/c^2 \leq (1 + 2c)(1 + c)\rho(xx^*)/c^2. \]

But \( \rho(xx^*) \leq ||xx^*|| \). This together with (2.7) completes the proof.

2.5. **Corollary.** Under the hypotheses of Theorem 2.4, the norm of the involution as an operator on \( B \) does not exceed \((1 + c)(1 + 2c^2)/c^5\).

In (2.7) we may replace \( ||x|| ||x^*|| \) by \( ||x^*||^2 \) and \( \rho(xx^*) \) by \( ||x|| ||x^*|| \).

This gives \( ||x^*|| \leq (1 + c)(1 + 2c^2)||x||/c^5 \).

We denote by \( P(N) \) the set of \( x \in B \) such that \( sp(x*x) \subseteq [0, \infty) \).  

2.6. **Lemma.** For an Arens*-algebra \( B \) the following are equivalent.

(a) \( B \) is a \( B^* \)-algebra in an equivalent norm.

(b) \( N = (0) \).

(c) \( P = B \).

Suppose that \( N = (0) \). Let \( y \in B \). Since the involution on \( B \) is continuous, the element \( y^*y \) generates a closed*-subalgebra \( B_y \). Let \( \mathfrak{M} \) be the space of regular maximal ideals of \( B_y \). By [1, p. 279] the commutative algebra \( B_y \) is *-isomorphic to \( C(\mathfrak{M}) \). Also \( sp(y^*y) \) is real. Hence there exist \( u, v \in B_y \) such that \( u(M) = \sup (y^*y(M), 0) \) and \( v(M) = -\inf (y^*y(M), 0) \), \( M \in \mathfrak{M} \). Then \( u \) and \( v \) are s. a., \( y^*y = u - v \) and \( uv = 0 \). As in [14, p. 281], \( (yv)^*(yv) = -v^2 \) so that \( yv = 0 \) by hypothesis. Then \( v = 0 \) and \( sp(y^*y) \subseteq [0, \infty) \).

A theorem of Gelfand and Neumark [13] asserts that if \( B \) is semi-simple, has a continuous involution, is symmetric \((B = P)\) and has an identity then there exists a faithful*-representation \( x \rightarrow T_x \) of \( B \). This theorem is also valid when \( B \) has no identity [4, Theorem 2.16]. In our situation, \( B \) is semi-simple [18, Lemma 3.5] and the involution is continuous. Thus a faithful*-representation exists. This representation is bi-continuous by [18, Corollary 4.4].

That (a) implies (b) follows from the well-known fact that any \( B^* \)-algebra is symmetric [14, p. 207 and p. 281].

The equation \( 4t^3 - 2t^2 + t - 1 = 0 \) has exactly one real root \( a \). This root \( a \) lies between .676 and .677.

2.7. **Theorem.** Suppose that for each s. a. element \( h \), \( \rho(h) \geq c||h|| \) and \( sp(h) \) is real, where \( c > 0 \). Then there is an equivalent norm for \( B \) in which \( B \) is a \( B^* \)-algebra if \( c > a \).

Suppose that \( sp(x*x) \subseteq (-\infty, 0] \). By Lemma 2.6 it is sufficient to show that \( x = 0 \). Suppose that \( x \neq 0 \). By Theorem 2.4 it is clear that \( xx^* \neq 0 \) and \( \rho(xx^*) \neq 0 \). Set \( x = u + iv \) where \( u \) and \( v \) are s. a. As in the proof of Theorem 2.4, \( xx^* + xx^* = 2u^2 + 2v^2 \) and we may assume that \( \rho(u) \geq \rho(v) \). Since \( sp(u^2) \subseteq [0, \infty), sp(v^2) \subseteq [0, \infty) \) formula 2.3 shows that \( sp(2u^2 + 2v^2) \subseteq \left[-(1 - c)\rho(2u^2)/c, (1 + c)\rho(2u^2)/c \right] \). Let \( r > 0, t > 0 \). From Lemma 2.2,
\[ || -t^{-1}(2u^2 + 2v^2) || < r \text{ if } t > (1 - cr)(1 + c)r(2u^2)/(c^2r) \text{ and } t > (1 + cr)(1 - c)r(2u^2)/(c^2r). \]

We write \( x^*x = 2u^2 + 2v^2 + (-xx^*) \). By Lemma 2.2, \( || -t^{-1}(-xx^*) || < r^{-1} \) if \( t > 0 \) and \( t > (r - cr)(1 + c)r(2u^2)/(c^2r) \). By Lemma 2.3, \( (-t^{-1}x^*)' \) exists if \( t > \max \{ (1 + cr)(1 - c)r(2u^2)/c^2r, (1 + cr)(1 - c)r(2u^2)/c^2r, (r - c)r(2u^2)/c^2r \} \). Since \( sp(-xx^*) \subset (-\infty, 0] \), \( \rho(x^*x) \) cannot exceed this maximum. Now select \( r, 1 \leq r < 2c \) which is possible since \( c > a \). Then \( (r - c)/c = 1 \) and \( (1 + cr)(1 - c) \geq (1 - cr)(1 + c) \). Therefore \( \rho(x^*x) \leq (1 + cr)(1 - c)r(2u^2)/c^2r. \)

Letting \( r \to 2c \) we obtain

\[ \rho(x^*x) \leq (1 + 2c^3)(1 - c)r(2u^2)/2c^3. \tag{2.8} \]

Next we express \(-2u^2 = 2v^2 + (-xx^* - xx^*). \) By formula (2.3), \( sp(-xx^* - xx^*) \subset [-1, (1 - c)r(2u^2)/c^2r]. \) Recall that \( \rho(2v^2) \leq \rho(2u^2) \). Repeating the above reasoning we see that for \( r > 0, t > 0, (-t^{-1}(-2u^2)) \) exists for \( t > \max \{ (1 + cr)(1 + c)r(2u^2)/(c^2r), (1 + cr)(1 - c)r(2u^2)/c^2r, (r - c)r(2u^2)/c^2r \} \). But \( sp(-2u^2) \subset (-\infty, 0] \). Then by the argument above we obtain

\[ \rho(2v^2) \leq (1 + 2c^3)(1 - c)r(x^*x)/2c^3. \tag{2.9} \]

From formulas (2.8) and (2.9) we see that \( (1 + 2c^3)(1 - c) \geq 2c^3 \) or \( 4c^3 - 2c^3 + c - 1 \leq 0 \). This gives \( c \leq a \) which is impossible by hypothesis.

Thus if \( c > a \) we have \( N = (0) \). We subsequently show (Corollary 2.11) that, in any case, \( N \) and \( P \) are closed in an Arens*-algebra \( B \).

Following Rickart [16, p. 625] we say that \( B \) is an \( A^* \)-algebra if there exists in \( B \) an auxiliary normed-algebra norm \( || x || \) (\( B \) need not be complete in this norm) such that, for some \( c > 0, || x^*x || \geq c || x ||^2 \). He raises the question of whether every \( A^* \)-algebra has a faithful*-representation.

2.8. COROLLARY. An \( A^* \)-algebra \( B \) where \( || x^*x || \geq c \geq c || x ||^2, \ x \in B, \) in the auxiliary norm has a faithful*-representation if \( c > a \).

Observe that \( || x^* || \geq c || x ||^2 \) so that \( || x^* || \leq c^{-1} || x ||, \ x \in B. \) Thus the involution on \( B \) is continuous in the topology provided by the norm \( || x || \). Let \( B_o \) be the completion of \( B \) in the norm \( || x || \). We extend the function \( || x || \) from \( B \) to \( B_o \) by continuity. Likewise the involution \( x \to x^* \) can be extended by continuity to provide a continuous involution \( y \to y^* \) on \( B_o \). We then have \( || y^*y || \geq c || y ||^2, \ y \in B_o. \) As in [16, p. 626] we obtain \( \rho(h) \geq c || h || \) for \( h \) s. a. in \( B_o \) where \( \rho(h) \) is the spectral radius computed for \( h \) as an element of the Banach algebra \( B_o, \rho(h) = \lim || h^n ||^{1/n}. \) Also \( || y^*y || \geq c \geq || y^* || || y ||, \ y \in B_o, \) so that \( B_o \) is an Arens*-algebra. Hence, by Lemma 2.1, the spectrum of each s. a. element of \( B_o \) is real. By Theorem 2.7, \( B_o \) is a \( B^* \)-algebra in an equivalent norm. Therefore \( B \) has the desired faithful*-representation.

We have no information on the truth or falsity of Theorem 2.7 for \( c \leq a. \)
To prove Theorem 2.7 without restriction on the size of $c$ one can assume
without loss of generality that $B$ has an identity. For suppose that $B$
has no identity, $\| x^* x \| \geq k \| x^* \| \| x \|$, $x \in B$. Adjoin an identity $e$
to $B$ to form the algebra $B_1$ with the norm defined in $B_1$ by the rule
\[ \| \lambda e + x \| = \sup_{\| y \| = 1} (\| y \| + xy) . \]
Then $B_1$ is a Banach algebra with the involution $(\lambda e + x)^* = \lambda e + x^*[1, p.
275]$. By changing in minor ways arguments in [14, p. 207] we see that $B_1$
is an Arens*-algebra. There is a constant $K$ such that $\| x^* \| \leq K \| x \|$, $x \in B$. Choose $0 < r < 1$. Given $\lambda e + x \in B_1$ there exists $y \in B$, $\| y \| = 1$, such that
\[ r^2 \| \lambda e + x \|^2 < \| \lambda y + xy \|^2 \leq K \| (\lambda y + xy)^* \| \| \lambda y + xy \| \]
\[ \leq Kk^{-1} \| y^*(\lambda e + x)^*(\lambda e + x)y \| \]
\[ \leq K^2k^{-1} \| (\lambda e + x)^*(\lambda e + x) \| . \]

Then
\[ \| (\lambda e + x)^*(\lambda e + x) \| \geq k K^{-2} \| \lambda e + x \|^2 \geq (kK^{-2}) \| \lambda e + x \| \| (\lambda e + x)^* \|. \]

We use this fact later.

Some results on spectral theory in Arens*-algebras were obtained by
Newburgh [15]. In a $B^*$-algebra $\rho(x)$ is a continuous function on the set $H$
of s.a. elements since $\rho(h) = \| h \|$, $h \in H$. This property holds for all
Arens*-algebras.

2.9. Theorem. In any Arens*-algebra, $\rho(x)$ is a continuous function
on $H$.

We assume that $\rho(h) \geq c \| h \|$ and $sp(h)$ is real, $h \in H$. We shall use
the following principle [12, p. 67]. If $y'$ exists and $\| z \| < (1 + \| y' \|)^{-1}$
then $(y + z)'$ exists.

Let $h \in H$, $h \neq 0$. Select $t > \rho(h)$ and set $u = (t^{-1}h)'$. We proceed as
in the proof of Lemma 2.2. Let $B_0$ be the closed*-subalgebra generated by $h$
and let $\mathfrak{M}$ be its space of regular maximal ideals. Then $u \in B_0$. Since
\[ t^{-1}h \circ u = 0 \] we obtain, for each $M \in \mathfrak{M}$, $u(M) = h(M) / (h(M) - t)$. Since
$\lambda / (\lambda - t)$ is a decreasing function of $\lambda$, $\sup | u(M) |$ can be majorized by
\[ \rho(h) / (t - \rho(h)) . \] Then $1 + \| u \|^{-1} \geq (1 + c^{-1}\rho(u))^{-1} \geq c(t - \rho(h))(ct + (1 - c)\rho(h)) = a(t)$, say.

Therefore $t^{-1}h + t^{-1}h_1$ is quasi-regular if $\| t^{-1}h \| < a(t)$ or if
\[ ct^2 - c[\rho(h) + \| h_1 \|]t - (1 - c)\rho(h) \| h_1 \| > 0 . \]

We apply this to $h_1 \in H$, $\| h_1 \| < \rho(h)$. The larger zero $d$ of the left hand
side of (2.10) is given by
\[ (2.11) \quad 2d = \rho(h) + \| h_1 \| + [(\rho(h) - \| h_1 \|)^2 + 4c^{-1}\rho(h) \| h_1 \|^{1/2}] \cdot \]

The radical term of (2.11) is majorized by \( \rho(h) - \| h_1 \| + 2(c^{-1}\rho(h) \| h_1 \|)^{1/2} \). Hence \( \hat{d} \leq \rho(h) + (c^{-1}\rho(h) \| h_1 \|)^{1/2} \). Thus \( t \notin sp(h + h_1) \) if \( t > \rho(h) + (c^{-1}\rho(h) \| h_1 \|)^{1/2} \).

Likewise \( t \notin sp(-h - h_1) \) under the same condition. This shows that

\[ (2.12) \quad \rho(h + h_1) \leq \rho(h) + (c^{-1}\rho(h) \| h_1 \|)^{1/2} \cdot \]

provided \( h_i \in H \) and \( \| h_i \| < \rho(h) \).

Note that \( \rho(h + h_1) \geq c \| h + h_1 \| \geq c (\| h \| - \| h_1 \|) \geq c(\rho(h) - \| h_1 \|). \) Therefore if \( \| h_i \| < c(\rho(h) - \| h_1 \|) \) or equivalently if \( \| h_i \| < c\rho(h)(1 + c) \) we have \( \| h_i \| < \rho(h + h_1) \). We may then apply the above analysis to the pair of s.a. elements \((h + h_1), -h_2, \) to obtain (if \( \| h_i \| < c\rho(h)(1 + c) \))

\[ (2.13) \quad \rho(h) \leq \rho(h_1 + h_2) + (c^{-1}\rho(h + h_1) \| h_1 \|)^{1/2} \cdot \]

From (2.12), \( \rho(h + h_1) \leq [c^{-1/2} + 1] \rho(h) \). Inserting this estimate in the radical term of (2.13) we obtain

\[ (2.14) \quad \rho(h) \leq \rho(h + h_1) + (c^{-1} + c^{-3/2})(\rho(h) \| h_1 \|)^{1/2} \cdot \]

Combining (2.12) and (2.14) we obtain

\[ | \rho(h + h_1) - \rho(h) | \leq [(c^{-1} + c^{-3/2})\rho(h) \| h_1 \|]^{1/2} \cdot \]

provided \( \| h_i \| < c\rho(h)(1 + c) \).

This show that \( \rho(x) \) is continuous on \( H \) at \( x = h \). Clearly we have continuity on \( H \) at \( x = 0 \).

For \( x \) s.a. in an Arens*-algebra let \([a(x), b(x)]\) be the smallest closed interval containing \( sp(x) \).

2.10. COROLLARY. For an Arens*-algebra \( B \), \( a(x) \) and \( b(x) \) are continuous functions of \( x \) on \( H \).

As remarks above indicate, there is no loss of generality in supposing that \( B \) has an identity \( e \). Let \( h \) be s.a. Choose \( \lambda > 0 \) such that \( sp(\lambda e + h) \subset [1, \infty) \). Let \( h_n \to h \), where each \( h_n \) is s.a., and choose \( 0 < \varepsilon < 1 \). We have \( \rho(\lambda e + h) = b(\lambda e + h) = \lambda + b(h) \). By the “spectral continuity theorem” (see e.g. [15, Theorem 1]) for all \( n \) sufficiently large \( sp(\lambda e + h_n) \subset (1 - \varepsilon, b(\lambda e + h) + \varepsilon) \). Also for all \( n \) sufficiently large \( \rho(\lambda e + h_n) - \rho(\lambda e + h) < \varepsilon \) by Theorem 2.9. Since, for such \( n \), \( sp(\lambda e + h_n) \subset (0, \infty) \), then \( \lambda + b(h_n) = \rho(\lambda e + h_n) \to \lambda + b(h) \). Therefore \( b(h_n) \to b(h) \). A similar argument shows that \( a(h_n) \to a(h) \).

2.11. COROLLARY. For an Arens*-algebra \( B \), \( N \) and \( P \) are closed sets.

This follows directly from the continuity of the involution on \( B \) and Corollary 2.10. Likewise the set \( H^+ \) of all s.a. elements whose spectrum is non-negative is closed.
3. Faithful*-representations. Let $B$ be a Banach algebra with an involution $x \to x^*$. Our aim here is to give necessary and sufficient conditions for $B$ to possess a faithful*-representation. Our criterion (Theorem 3.4) is in terms of algebraic and linear space properties of $B$. A criterion of Kelley and Vaught [10] is largely topological in nature. To discuss this we first prove a simple lemma. We adopt the following notation. Let $R_0$ be the collection of all finite sums of elements of $B$ of the form $x^*x$. Let $R = \{ x \in H \mid \text{there exists } y \in R_0 \text{ such that } ty + (1 - t)x \in R_0, 0 < t \leq 1 \}$. In the notation of Klee [11, p. 448], $R = \text{lin } R_0$ (computed in the real linear space $H$, the union in $H$ of $R_0$ and the points of $H$ linearly accessible from $R_0$). Let $P$ be the closure in $B$ of $R_0$. If $B$ has an identity $e$ and the involution is continuous then $H$ is closed, $e$ is an interior point of $R_0$ [10] and $R = P$ [11, p. 448]. If $B$ has no identity or if the involution is not assumed continuous we see no relation, in general, between $R$ and $P$ other than $R \subseteq P$.

3.1. Lemma. Suppose that $B$ has a continuous involution $x \to x^*$ and an identity $e$. Then there is an equivalent Banach algebra norm $|| x ||$, where $|| x^* || = || x ||$, $x \in B$, and $|| e || = 1$.

We first introduce an equivalent norm $|| x ||_0$ in which $|| x^* ||_0 = || x ||_0$, $x \in B$, by setting $|| x ||_0 = \max (|| x ||, || x^* ||)$. Let $L_x(R_0)$ be the operator on $B$ defined by left (right) multiplication by $x$: $L_x(y) = xy$ and $R_x(y) = yx$. Let $|| L_x ||$ be the norm of $L_x$ as an operator on $B$ where the norm $|| y ||_0$ is used for $B$. $|| R_x ||$ is defined in the same way. We set $|| x ||_1 = \max (|| L_x ||, || R_x ||)$. Then $|| x + y ||_1 \leq || x ||_1 + || y ||_1$ and $|| xy ||_1 \leq || x ||_1 || y ||_1$. Clearly $|| x ||_1 \leq || x ||_0$. Moreover $|| L_x || \geq || x ||_0 || e ||_0$ and the norms $|| x ||_0$ and $|| x ||_1$ are equivalent. Trivially $|| e ||_1 = 1$. Also $|| L_x || = \sup_{|| y ||_0 = 1} || x^* y ||_0 = \sup_{|| y ||_0 = 1} || y^* x ||_0 = || R_x ||$.

Then $|| x^* ||_1 = \max (|| L_x ||, || R_x ||) = \max (|| L_x ||, || R_x ||) = || x ||_1$.

In view of Lemma 3.1 the result [10, p. 51] of Kelley and Vaught in question may be expressed as follows.

3.2. Theorem. Let $B$ be a Banach algebra with an identity and an involution $x \to x^*$. Then $B$ has a faithful*-representation if and only if $*$ is continuous and $P \cap (-P) = (0)$.

As it stands this criterion breaks down if $B$ has no identity. For let $B = C([0, 1])$ with the usual involution $x \to x^*$ and norm. Let $B_0$ be the algebra obtained from $B$ by keeping the norm and involution but defining all products to be zero. Then* is still continuous and $P \cap (-P) = (0)$. But $B_0$ has no faithful*-representation, for otherwise $B_0$ would be semi-simple [16, p. 626].

As in [4] we call the involution $x \to x^*$ in $B$ regular if, for $h$ s.a., $\rho(h) = 0$ implies $h = 0$. By [4, Lemma 2.15]. * is regular if and only if every
maximal commutative *-subalgebra of $B$ is semi-simple. Also every maximal commutative *-subalgebra of $B$ is closed [4, Lemma 2.13].

By a positive linear functional $f$ on $B$ we mean a linear functional such that $f(x^*x) \geq 0$, $x \in B$. The functional $f$ is not assumed to be continuous. If $B$ has an identity then [13, p. 115], $f(h)$ is real for $h$ s.a. and $f(x^*) = f(x^*)$. Trivial examples show this to be false, in general. However, from the positivity of $f$, $f(x^*y)$ and $f(y^*x)$ are complex conjugates which is the fact really needed for the introduction of the inner product in Theorem 3.4.

3.3 LEMMA. Let the involution on $B$ be regular. Then

(1) a positive linear $f$ satisfies the inequalities

$$f(y^*hy) \leq f(y^*y)\|h\|, \; y \in B, \; h \in H,$$

$$f(y^*x^*xy) \leq f(y^*y)\|x^*x\|, \; x, \; y \in B,$$

(2) if $B$ has an identity $e$, any $h \in H$, $\|e - h\| \leq 1$ has a s.a. square root and, moreover, any positive linear functional is continuous on $H$.

Suppose first that $B$ has an identity $e$, $\|e - h\| \leq 1$, s.a. In the course of the proof of [4, Theorem 2.16] it was shown that $h$ has a s.a. square root. Next do not assume that $B$ has an identity. Let $B_1$ be the Banach algebra obtained by adjoining an identity $e$ to $B$. Consider the power series $(1 - t)^{1/2} = 1 - t/2 - t^2/8 - \cdots$. Let $h \in B$, s.a. and $\|h\| \leq 1$. Then the expansion $-h/2 - h^2/8 - \cdots$ converges to an element $z \in B$. Let $B_0$ be a maximal abelian *-subalgebra of $B$ containing $h$. As noted above, $B_0$ is a semi-simple Banach algebra. The involution is continuous on $B_0$ ([16, Corollary 6.3]). Therefore $z$ is s.a. Also $(e + z)^2 = e - h$. Let $y \in B$ and set $k = y + zy$. Then $k^*k = (y^* + y^*z)(y + zy) = y^*(e + z)^2y = y^*y - y^*hy$. For any positive linear functional $f$ on $B$, $f(k^*k) \geq 0$ which yields (3.1). Formula (3.2) is a special case.

Suppose that $B$ has an identity $e$. If we set $y = e$ in (3.1) we obtain $|f(h)| \leq f(e)\|h\|$ which shows that $f$ is continuous on $H$.

3.4. THEOREM. $B$ has a faithful *-representation if and only if * is regular and $R \cap (-R) = (0)$.

Suppose that $B$ has a faithful *-representation $x \to T_x$ as operators on a Hilbert space $\mathcal{H}$. Let $h$ be s.a. and $\rho(h) = 0$. Then $\rho(T_h) = 0$. As $T_h$ is a s.a. operator on a Hilbert space, $T_h = 0$ and therefore $h = 0$. Thus the involution is regular. Let $x \in R \cap (-R)$ and let $f$ be a positive linear functional on $B$. Then clearly $f(y) \geq 0$, $y \in R_0$. From the definition of $R$ there exists $y \in R_0$ such that $tf(y) + (1 - t)f(x) \geq 0$, $0 < t \leq 1$. It follows that $f(x) \geq 0$ and hence $f(x) = 0$. Let $\xi \in \mathcal{H}$ and set $f(x) = (T_x\xi, \xi)$. Then $(T_x\xi, \xi) = 0$ for all $\xi \in \mathcal{H}$. Since $T_x$ is a s.a. operator we see that $T_x = 0$ and $x = 0$. 
Suppose now that $*$ is regular and $22 \Pi (-22) = (0)$. We show first that the regularity of the involution makes available a general representation procedure of Gelfand and Neumark [13].

Let $f$ be a positive linear functional on $B$. Let $I_f = \{ x | f(x^*x) = 0 \}$. $I_f$ is a left ideal of $B$. Let $\pi$ be the natural homomorphism of $B$ onto $B/ I_f$. Since $f(x^*y) = f(y^*x)$, $\delta_f = B/ I_f$ is a pre-Hilbert space if we define $(\pi(x), \pi(y)) = f(y^*x)$. As in [13, p. 120] we associate with $\pi$ an operator $A_\pi$ on $\delta_f$ defined by $A_\pi \pi(x) = \pi(yx)$. Formula (3.2) yields

$$
|| A_\pi \pi(x) ||^2 = f(x^*y^*yx) \leq || y^*y || || \pi(x) ||^2.
$$

Thus $A_\pi$ is a bounded operator with norm not exceeding $|| y^*y ||^{1/2}$. It may then be extended to $T_\pi$, a bounded operator on the completion $\delta_f$ of $\delta_f$. The mapping $x \to T_\pi$ is a $*$-representation of $B$ with kernel $\{ y \in B | yx \in I_f \}$ for all $x \in B \} = K$. Note that $K^* = K$.

Now take the direct sum $\oplus$ of the Hilbert spaces $\delta_f$ as $f$ ranges over all positive linear functionals on $B$ ([13, p. 95]). Since $|| T_\pi || \leq || y^*y ||^{1/2}$ by (3.3) and this estimate is independent of $f$, the direct sum ([13, p. 113]) $x \to T_\pi$ of the representations $x \to T_\pi$ yields a $*$-representation of $B$ as bounded operators on $\oplus$ with kernel $\{ y \in B | yx \in \cap I_f \}$ for all $x \in B \}$. If $B$ has an identity, the kernel is the reducing ideal of $B$ ([13, p. 130]), namely $\cap I_f$.

Suppose first that $B$ has an identity $e$. The set $R_0$ has the property that $x, y \in R_0, \lambda, \mu \geq 0$ imply $\lambda x + \mu y \in R_0$. By Lemma 3.3, $R_0 \supset \{ x \in H | || e - x || \leq 1 \}$. Thus $e$ is an interior point of $R_0$. By the theory of convex sets in normed linear spaces, $R$ is the closure in $H$ of $R_0$ and $R$ is a closed cone in $H$ ([11, p. 448]).

Let $f$ be a positive linear functional on $B$. By Lemma 3.2, $f$ is continuous on $H$. Also $f(w) \geq 0, w \in R$. Let $H'$ be the conjugate space of $H$ and $G = \{ g \in H' | g(w) \geq 0, w \in R \}$. It is easy to see ([10, p. 48]) that $G$, the dual cone of $R$, is the set of linear functionals on $H$ which are the restrictions to $H$ of positive linear functional on $B$. There is no loss generality in assuming that $|| e || = 1$. Let $x \in H$. By [10, Lemma 1.3],

$$
dist (-x, R) = \sup \{ g(x) | g \in G, g(e) \leq 1 \}. $$

We show that $R \cap (-R) = H \cap (\cap I_f)$. Let $y \in H, y \in \cap I_f$. For any fixed $f$, $T_\pi f = 0$ and $(T_\pi^* \xi, \xi) = 0, \xi \in \delta_f$. Then $(\pi(yx), \pi(x)) = 0$ for all $x \in B$ in the notation used above. Therefore $f(x^*yx) = 0, x \in B$. Setting $x = e$ we see that $f(y) = 0$. Then by the distance formula, $-y \in R$. Likewise $y \in R$. Suppose conversely that $y \in R \cap (-R)$. It is easy to see that for each $z \in B, z^*Rz \subset R_0$. Therefore $z^*Rz \subset R$. Hence $z^*yz \in R \cap (-R), z \in B$. From the distance formula, sup $\{ f(z^*yz) | f \text{ positive, } f(e) \leq 1 \} = 0 = \sup \{ f(-z^*yz) | f \text{ positive, } f(e) \leq 1 \}$. Hence $f(z^*yz) = 0$ for each positive linear functional. Then $(T_\pi^* \pi(z), \pi(z)) = 0$ for all $z$ whence $T_\pi^* = 0$. Therefore $T_\pi = 0$ and $y \in H \cap (\cap I_f)$.
This proves the theorem in case $B$ has an identity. Suppose that $B$ has no identity. Let $B_i$ be the algebra obtained by adjoining an identity $e$ to $B$. We extend the involution to $B_i$ by setting $(\lambda e + x)^* = \lambda e + x^*$. The involution on $B_i$ is regular [4, Lemma 2.14]. Let $R_i'$ and $R'$ be the sets $R_o$ and $R$ respectively computed for the algebra $B_i$. By the above it is sufficient to show that $R \cap (-R) = (0)$ implies $R' \cap (-R') = (0)$. Suppose that $R \cap (-R) = (0)$.

Let $x, y \in B$. Then $y^*(\lambda e + x)(\lambda e + x)y = (\lambda y + xy)^*(\lambda y + xy)$. This shows that $y^*R_i'y \subset R_i$ which implies $y^*R'y \subset R$. Note also that $B$ is semisimple [18, Lemma 3.5] which implies that $zB = (0)$, or $Bz = (0)$, $z \in B$, can hold only for $z = 0$.

Suppose that $\lambda e + x \in R' \cap (-R')$ where $x \in B$ and $\lambda$ is a scalar. We derive a contradiction from $\lambda \neq 0$. For every $y \in B$, $y^*(\lambda e + x)y \in R \cap (-R)$. Setting $u = -x/\lambda$ we have $y^*(e - u)y = 0$ or $y^*y = y^*wy$ for all $y \in B$. Then

$$h^2 = huh, \text{ h.s.a.}$$

Let $h_1$ and $h_2$ be s.a. Then $(h_1 + h_2)^2 = (h_1 + h_2)u(h_1 + h_2)$. From (3.4) we obtain

$$h_1h_2 + h_2h_1 = h_1uh_2 + h_2uh_1.$$

Also $(h_1 - ih_2)(h_1 + ih_2) = (h_1 - ih_2)u(h_1 + ih_2)$ From (3.4) we get

$$h_1h_2 - h_2h_1 = h_2uh_1 - h_1uh_2.$$

From (3.5) and (3.6) we see that $h_1h_2 = h_2h_1$. Consequently for $h_k$ s.a., $k = 1, 2, 3, 4$, we see that $(h_1 + ih_2)(h_2 + ih_3) = (h_1 + ih_2)u(h_3 + ih_4)$. In other words

$$zw = zuw, z, w \in B.$$ 

From (3.7) $(z - zw)w = 0$ for all $w \in B$ so that $z = zw$ for each $z$. Hence $u$ is a right identity for $B$. Likewise from $z(w - uw) = 0$ for all $z \in B$ we see that $u$ is an identity for $B$. But this is impossible since we are considering the case where $B$ has no identity.

We now have $x \in R' \cap (-R')$. Then $y^*xy = 0$ for all $y \in B$. Therefore $h_xh = 0, \text{ h.s.a.}$ Also for $h_k$ s.a., $k = 1, 2, (h_1 + h_2)x(h_1 + h_2) = 0$ so that $h_xh_1 + h_xh_2 = 0$. Also $(h_1 - ih_2)x(h_2 + ih_3) = 0$ so that $h_1xh_3 - h_2xh_1 = 0$. Therefore $h_xh_3 = 0$. It follows that $zw = 0$ for all $z, w \in B$. This implies that $x = 0$ and completes the proof.

4. Preliminary ring theory. Let $R$ be a semi-simple ring with minimal one-sided ideals. For a subset $A$ of $R$ let $\mathcal{A} = \{x \in R \mid xA = (0)\}$ and $\mathfrak{A}(A) = \{x \in R \mid Ax = (0)\}$. Consider a two-sided $I$ of $R$. If $x \in R(I), y \in R, z \in I$ then $zy \in I, x(yz) = 0$ so that $\mathfrak{A}(I)$ is a two-sided ideal of $R$. Therefore $\mathfrak{A}(I)I$ is an ideal. But $[\mathfrak{A}(I)I]^2 = (0)$. Thus, by semi-simplicity, $\mathfrak{A}(I)I = (0)$.
and \( \Re(I) \subseteq \mathfrak{L}(I) \). Likewise we have \( \mathfrak{L}(I) \subseteq \Re(I) \) and thus \( \Re(I) = \mathfrak{L}(I) \).

Let \( S \) be the socle \([5, \text{p. 64}]\) of \( R \). This is the algebraic sum of the minimal left (right) ideals of \( R \). \( S \) is a two-sided ideal. Therefore \( \mathfrak{L}(S) = \Re(S) \).

We call an idempotent \( e \) of \( R \) a minimal idempotent if \( e \) is a minimal right ideal.

4.1. Lemma. (a) Let \( I \) be a left (right) ideal of \( R \), \( I \neq (0) \). Then \( I \) contains no minimal left (right) ideal of \( R \) if and only if \( I \subseteq S^\perp \).

(b) \( R/S^\perp \) is semi-simple. If \( S_0 \) is the socle of \( R/S^\perp \) then \( S_0^\perp = (0) \).

Let \( I \neq (0) \) be a left ideal of \( R \). Suppose that \( I \subseteq S^\perp \). Then I cannot contain a minimal left ideal \( J \) of \( R \) for any such \( J \) would be contained in \( S \cap S^\perp \). Next suppose that \( I \nsubseteq S^\perp \). We must show that \( I \) contains a minimal left ideal of \( R \). There exists a minimal idempotent \( e \) such that \( e \) \( I \neq (0) \). Choose \( u \in I \) such that \( eu \neq 0 \). By semi-simplicity and the minimality of \( eR, eR = euR \). Thus there exists \( z \in R \) such that \( euz = e \). Since \( (euz)^2 = e \), we have \( j \neq 0 \) where \( j = zeu \). Note that \( j^2 = j \). As \( u \in I \) we have \( Rj \subseteq I \). To see that \( Rj \) is the desired minimal ideal it is sufficient to see that \( jRj \) is a division ring \([5, \text{p. 65}]\).

Note that \( jz = zeuz = ze \neq 0 \). Then \( Rze = Re \) so that there exists \( v \in R \) where \( vze = e \). Then \( vj = vzeu = eu \) and \( vjz = e \).

We assert that \( jx_j = jx_j \) if and only if \( eux_ze = eux_ze \). For if \( jx_j = jx_j \), multiply on the left by \( v \) and on the right by \( z \) and use the relations \( vj = eu \) and \( jz = ze \). If \( eux_ze = eux_ze \) multiply on the left by \( z \) and on the right by \( u \) and use \( zeu = j \).

Therefore the mapping \( \tau : \tau(jxj) = euxze \) is a well-defined one-to-one mapping of \( jRj \) into \( eRe \). The mapping is onto. For \( etu \in eRe \). Then \( etu = euzwvz = \tau(jzwvj) \). \( \tau \) is clearly additive. But also \( \tau((jx_j)(jy_j)) = \tau(jx_jy_j) = eux_jeuz = (euxze)(euyze) = \tau(jx_j)\tau(jy_j) \). Therefore \( \tau \) is a ring isomorphism of \( jRj \) onto \( eRe \). Since \( eRe \) is a division ring so is \( jRj \).

Let \( J \) be the radical of \( R/S^\perp \) and \( \pi \) be the natural homomorphism of \( R \) onto \( R/S^\perp \). Suppose that \( J \neq (0) \). Then \( \pi^{-1}(J) \supseteq S^\perp \) and \( \pi^{-1}(J) \neq S^\perp \). By (a), \( \pi^{-1}(J) \) contains a minimal idempotent \( e \) of \( R \). We then have \( \pi(e) \in J \), \( \pi(e) \neq 0 \). This is impossible since the radical of a ring contains no non-zero idempotents.

Let \( S_0 \) be the socle of \( R/S^\perp \) and \( e \) be a minimal idempotent of \( R \). Clearly \( \pi(e) \neq 0 \) and \( \pi \) is one-to-one on \( eRe \). Then \( \pi(e)\pi(R)\pi(e) \) is a division ring so that, since \( R/S^\perp \) is semi-simple, \( \pi(e) \in S_0 \). Let \( \pi(x) \in S_0^\perp \). Then \( \pi(ex) = 0 \) so that \( ex \in S^\perp \cap S = \). Hence \( x \in S^\perp \) and \( \pi(x) = 0 \).

The following result is due to Rickart \([17, \text{Lemma 2.1.}]\):

4.2. Lemma. Let \( A \) be any ring. Let \( x \to x^* \) be a mapping of \( A \) onto \( A \) such that \( x^{**} = x, (xy)^* = y^*x^* \) and \( xx^* = 0 \) implies \( x = 0 \). Then any
minimal right (left) ideal $I$ of $A$ can be written in the form $I=eA(I=Ae)$ where $e^2=e \neq 0$, $e^*=e$.

We improve this result by relaxing the conditions on $x \to x^*$ but at the expense of assuming the ring to be semi-simple.

4.3. Lemma. Let $R$ be semi-simple with minimal one-sided ideals. Let $x \to x^*$ be a mapping of $R$ onto $R$ satisfying $x^{**}=x$ and $(xy)^* = y^*x^*$. Then the following statements are equivalent.

1. Every minimal right ideal is generated by a s.a. idempotent.
2. Every minimal left ideal is generated by a s.a. idempotent.
3. $jj^* \neq 0$ for each minimal idempotent $j$ of $R$.
4. $xx^* = 0$ implies $x \in S^\perp$.

We say that the idempotent $e$ is s.a. if $e^*=e$. Note that $x \to x^*$ is one-to-one and $0^*=0$. As a preliminary we show that $j^*$ is a minimal idempotent if $j^*$ is a minimal idempotent. The ideal $I=jR$ is a minimal right ideal. Then $I^*=Rj^*$ is a left ideal $\neq (0)$. Suppose $I^* \supset K \neq (0)$, $I^* \neq K$ where $K$ is a left ideal of $R$. By semi-simplicity there exists $x \in K$ such that $x^2 \neq 0$. Then $I^* \supset Rx \neq (0)$, $I^* \neq Rx$. This implies that $I \supset x^*R \neq (0)$, $I \neq x^*R$. This is impossible. Therefore $I^*$ is a minimal left ideal and $j^*$ is a minimal idempotent. It is clear from this argument that (1) and (2) imply each other.

Assume (1). Let $j$ be a minimal idempotent, $I = jR$ a minimal left ideal. We can write $I = Re$ where $e$ is a s.a. idempotent. Then for some $v \in R$, $vj = e$. But $e = ee^* = vjj^*v$. Therefore $jj^* \neq 0$. Thus (1) implies (3).

Assume (3). Suppose that $xx^* = 0$, $x \neq 0$. Let $I =Rx$. Then $I \neq (0)$. Suppose that $I$ contains a minimal left ideal $Rj$ of $R$ where $j$ is a minimal idempotent. We can write $j = yx$, $y \in R$. Then $0 \neq jj^* = yxx^*y^* = 0$. This shows that $I$ contains no minimal left ideal of $R$. By Lemma 4.1, $I \subset S^\perp$. Then for any minimal idempotent $e$, $0 = e(ex)$ and $x \in S^\perp$. Thus (3) implies (4).

Assume (4). If $j$ is a minimal idempotent and $jj^* = 0$ then $j \in S^\perp$. But $j \in S$ and $S \cap S^\perp = (0)$. This shows that (4) implies (3).

Assume (3). Let $j$ be a minimal idempotent, $I = jR$. Since $jj^* \neq 0$, $jj^*R = I$. There exists $u \in R$, $jj^*u = j$. As noted above $j^*$ is a minimal idempotent. By (3), $0 \neq j^*j$. Then $0 \neq (u^*jj^*)(jj^*u) = u^*(jj^*)^yu$. Therefore $(jj^*)^2 \neq 0$. Set $h = jj^*$. Since $I$ is minimal, $I = hI$. As in the proof of [17, Lemma 2.1] there exists $u \in I$ such that $h = hu$. Set $e = uu^*$. As in that proof, $e$ is a s.a. idempotent and it remains only to check that $e \neq 0$ to obtain (2) from (3). If $e = 0$ then $0 = uu^* = huu^*h = h^2$ which is impossible.

5. Normed algebras with minimal ideals. We are concerned here with $^*$-representations of semi-simple normed algebras $B$ with an involution.
where $B$ has minimal one-sided ideals. $B$ may be incomplete.

5.1. Lemma. Let $B$ be a complex semi-simple normed algebra with minimal one-sided ideals. Let $e_1, e_2$ be minimal idempotents of $B$. Then the following statements are equivalent.

1. $e_1 Be_2 \neq (0)$.
2. $e_2 Be_1 \neq (0)$.
3. $e_1 Be_2$ is one-dimensional.
4. $e_2 Be_1$ is one-dimensional.

Suppose (1). There exists $u \in B$, $e_1 u e_2 \neq (0)$. Since $e_1 u e_2 B = e_1 B$, there exists $v \in B$ where $e_1 u e_2 v = e_1$. Then $e_2 v e_1 \neq 0$ and (1) implies (2). Let $E = \{ \lambda e_1 v e_2 | \lambda \text{ complex} \}$. Clearly $e_1 Be_2 \supseteq E$. Let $e_2 x e_1 = e_2 (e_1 u e_2 v) = (e_2 e_1 u e_2 v) e_2 v e_2$, a scalar multiple of $e_2$ by the Gelfand-Mazur Theorem. Thus (1) implies (4). The remainder of the argument is trivial.

For the remainder of § 5, $B$ denotes a semi-simple complex normed algebra with an involution and with minimal one-sided ideals.

5.2. Theorem. The following statements concerning $B$ are equivalent.

1. Every minimal one-sided ideal is generated by a s.a. idempotent.
2. There exists $\ast$-representation with kernel $S^\perp$.
3. There exists a representation with kernel contained in $S^\perp$.
4. $j - j^\ast$ is quasi-regular for every minimal idempotent $j$.
5. $jBj^\ast \neq (0)$ for every minimal idempotent $j$ and $xx^* = 0$ implies $x^* x \in S^\perp$, $x \in B$.

Suppose that (1) holds. Let $Q$ be the set of all s.a. minimal idempotents of $B$ and let $j \in Q$. By the Gelfand-Mazur Theorem, $jBj = \{ \lambda j | \lambda \text{ complex} \}$. Suppose $jx^* x j = \lambda j$. Taking adjoints, $\lambda = \overline{\lambda}$ so $\lambda$ is real. We show that $jx^* x j = - j$ is impossible. For suppose $jx^* x j = - j$. Now $j x j = \alpha j$ for some scalar $\alpha = a + bi$, where $a, b$ are real. Set $c = a + (a^2 + 1)^{1/2}$. By the use of $j x^* x j = - j$ one obtains $(j x^* - cj) (j x^* - cj)^* = 0$. From Lemma 4.3 we have $j x^* - cj = 0$. Then $(a - bi) j = j x^* j = cj$. It follows that $c = a$ and $b = 0$. This is impossible.

For $j \in Q$ we define the functional $f_j(x)$ on $B$ by the rule $f_j(x) j = j x j$. By the above, $f_j(x^* x) \geq 0$, $x \in B$, $x \in B$ and $f_j(x^*) = f_j(x)$.

The following inequality of Kaplansky [9, p. 55] is then available.

$$f_j(y^* x^* x y) \leq \nu(x^* x) f_j(y^* y), x, y \in B,$$

where $\nu(x^* x) = \lim ||(x^* x)^n||^{1/n}$. Let $I_j = \{ x | f_j(x^* x) = 0 \}$. Let $\pi$ be the natural homomorphism of $B$ onto $B/I_j$. The definition $(\pi(x), \pi(y)) = f_j(y^* x)$ makes $B/I_j$ a pre-Hilbert space. Let $\mathfrak{Q}$ be its completion. See the discussion of the Gelfand-Neumark procedure in § 3. To each $y \in B$ we correspond
the operator $A_j$ defined by $A_j^*[π(x)] = π(yx)$. Then

$$|| A_j^*[π(x)] ||^2 = f_j(x^*y^*yx) \leq ν(y^*y) || π(x) ||^2$$

by (5.1). Thus $A_j^*$ can be extended to a bounded linear operator $T_j^*$ on $ℱ_j$, and the mapping $y \rightarrow T_j^*$ is a $*$-representation of $B$.

Since $|| T_j^* || \leq ν(y^*y)^{1/2}$ and the estimate is independent of $j \in Q$ we can take the direct sum $ℱ_0$ of the Hilbert spaces $ℱ_j, j \in Q$ and the direct sum $x \rightarrow T_x$ of the representations $x \rightarrow T_x^*$. This gives a $*$-representation of $B$ with kernel $K$ where

$$K = \{ x \in B \mid xy \in \bigcap_{j \in Q} I_j, \text{ for all } y \in B \} .$$

We show that $K = S^\perp$.

It is clear that $S^* = S$ and therefore $(S^\perp)^* = S^\perp$. Using this and Lemma 4.3 we obtain the following chain of equivalences: $x \in \bigcap_{j \in Q} I_j \iff jx^*xj = 0$, all $j \in Q \iff jx^*x \in S^\perp$, all $j \in Q \iff jx^* = 0$, all $j \in Q \iff x^* \in S^\perp \iff x \in S^\perp$. Therefore $\bigcap_{j \in Q} I_j = S^\perp$. Thus $K = \{ x \mid xy \in S^\perp, \text{ all } y \in B \}$. If $x \in K$ then $xj \in S^\perp \cap S = (0)$ for all $j \in Q$ and $x \in S^\perp$. Clearly $S^\perp \subset K$. Therefore $K = S^\perp$. Hence (1) implies (2). Clearly (2) implies (3).

Assume (3) and let $φ$ be a $*$-representation whose kernel $\subset S^\perp$. Let $j$ be a minimal idempotent of $B$. Let $A$ be the subalgebra of $B$ generated by $j$ and $j^*$. By the Gelfand-Mazur Theorem, $jj^*j = \lambda j$ for some scalar $λ$. Thus $A$ is the linear space spanned by $j, j^*, j^*j$ and $j^*j$. $A$ is finite-dimensional and $A \subset S$. Since $S \cap S^\perp = (0)$, $φ$ is one-to-one on $A$. Note that $A = A^*$. Let $E$ be the $B^*$-algebra obtained by taking the closure in the operator algebra on the appropriate Hilbert space of $φ(B)$. Clearly $φ(A)$ is a closed $*$-subalgebra of $E$. The element $φ(j - j^*)$ is a skew element of $E$ and therefore quasi-regular in $E$. By [8, Theorem 4.2] its quasi-inverse in $E$ already lies in $ψ(A)$. As $φ$ is one-to-one on $A$, $j - j^*$ has a quasi-inverse in $A$. Thus (3) implies (4).

Assume (4). Let $j$ be a minimal idempotent of $B$. There exists $u \in B$ such that $j - j^* + u - (j - j^*)u = 0$. If $jj^* = 0$ then left multiplication by $j$ gives $j = 0$ which is impossible. Therefore $jj^* \neq 0$. By Lemma 4.3, we see that (4) implies (1). Clearly (1) implies (5) by Lemma 4.3. Assume (5). Let $j$ be a minimal idempotent of $B$. If $j^*j = 0$ then $0 = x^*j^*jx = (jx)^*(jx)$. Also $jxx^*j^* \in S^\perp \cap S = (0)$ for all $x \in B$. Since $jBj^* \neq (0)$, $jBj^*$ is one-dimensional by Lemma 5.1. Hence there exists $u \neq 0$ in $B$ and a linear functional $f(x)$ on $B$ such that $jxj^* = f(x)u$. Then $f(xx^*) = 0$ for all $x \in B$. Expanding $0 = f[(x + y)(x + y)^*] = f[(x + iy)(x + iy)^*]$ we see that $f(xy^*) = 0$ for all $x, y \in B$. Hence $f$ vanishes on $B^\perp$. Take any $z \in B$. We have $f(jz) = 0$ or $jzj^* = 0$. Thus $jBj^* = (0)$ which is impossible. Therefore $j^*j \neq 0$. By Lemma 4.3, (5) implies (1).

Algebras to which Theorem 5.2 can be applied most easily are those for
which $S^\perp = (0)$. Examples are semi-simple annihilator algebras studied by Bonsall and Goldie [3] and primitive algebras (Corollary 5.4).

5.3. **Corollary.** If $B$ is an Arens*-algebra with non-zero socle then $N \subset S^\perp$.

Let $x_0 \in N$, $sp(x_0x_0^*) \subset (-\infty, 0]$. Then we can write $x_0x_0^* = -h^2$ where $h$ is s.a. The ideal $S^\perp$ is closed and self-adjoint. Let $\pi$ be the natural homomorphism of $B$ onto $B/S^\perp$. An involution can be defined in $B/S^\perp$ by the rule $[\pi(x)]^* = \pi(x^*)$. Since $B$ is semi-simple, $B/S^\perp$ has non-zero socle. Let $\pi(x)$ be a minimal idempotent of $B/S^\perp$. Then $[\pi(x)]^* - \pi(x) = \pi(x^* - x)$ is quasi-regular in $B/S^\perp$ since $x^* - x$ is quasi-regular in $B$. By Theorem 5.2 and Lemma 4.1, $B/S^\perp$ has a faithful*-representation. Then, by Theorem 3.4, $\pi(x_0x_0^*) = 0 = \pi(h^2)$. Therefore $x_0x_0^* \in S^\perp$ and $(jx_0)(jx_0)^* = 0$ for each minimal idempotent $j$ of $B$. Therefore $jx_0 = 0$ for all such $j$ and $x_0 \in S^\perp$.

We call the involution $x \rightarrow x^*$ proper if $xx^* = 0$ implies $x = 0$. We call the involution quasi-proper if $xx^* = 0$ implies $x^*x = 0$. Not every involution is quasi-proper. For example let $B$ be all $2 \times 2$ matrices with the involution defined by

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^* = \begin{pmatrix}
\bar{a} & -\bar{c} \\
-\bar{b} & \bar{d}
\end{pmatrix}.
$$

To see that this is not quasi-proper choose $x$ as

$$
\begin{pmatrix}
1 & i \\
0 & 0
\end{pmatrix}.
$$

Every proper involution is quasi-proper but the converse is false. Consider, for example $B = C([0, 1])$ and set $x^*(t) = \bar{x}(1 - t)$.

5.4. **Corollary.** Let $B$ be primitive with non-zero socle. Then the following statements are equivalent.

(1) The involution* is proper.

(2) The involution* is quasi-proper.

(3) There exists a faithful*-representation of $B$.

Suppose that $S^\perp \neq (0)$. Then by [5, p. 75], $S \subset S^\perp$. Since $S \cap S^\perp = (0)$ this is impossible. Therefore $S^\perp = (0)$. Assume (2). Let $j$ be a minimal idempotent of $B$. Then $jBj^* \neq (0)$ (see the proof of [16, Theorem 4.4]) and, consequently (5) of Theorem 5.2 is satisfied. Then by Theorem 5.2, (2) implies (3); the remainder of the proof is obvious.

The equivalence of (1) and (3) was noted by Rickart [17, Theorem 3.5]. By Lemma 4.3 and Theorem 5.2 this equivalence of (1) and (3) holds for any $B$ for which $S^\perp = (0)$.

If $B$ is complete the following statements hold. (1) Any*-representation of $B$ is continuous [16, Theorem 6.2]. (2) If $B$ has a faithful*-representation then the involution is continuous [16, Lemma 5.3]. We show that
both these statements can be false for $B$ incomplete. Our discussion is based on work of Kakutani and Mackey [6, p. 56] (see also [7] for the complex case). Let $\mathfrak{X}$ be an infinite-dimensional complex Hilbert space, $(x, x)^{1/2} = \| x \|$. Let $\| x \|$ be any other norm on $\mathfrak{X}$ such that $\| x \| \leq \| x \|, x \in \mathfrak{X}$. Let $\mathfrak{X}_1 = \{ y \in \mathfrak{X} | (x, y) \text{ is continuous on } \mathfrak{X} \text{ in the norm } \| x \| \}$ and endow $\mathfrak{X}_1$ with the norm $\| x \|$. Then [6, p. 56] a linear functional $f(x)$ on $\mathfrak{X}_1$ has the form $f(x) = (x, y)$. Moreover $\mathfrak{X}_1$ is dense in $\mathfrak{X}$ in both norms. If there exists $c > 0$ such that $\| x \| \leq c \| x \|, x \in \mathfrak{X}_1$ then $\mathfrak{X} = \mathfrak{X}_1$ and $\mathfrak{X}_1$ is complete.

Let $\mathcal{G}(\mathfrak{X}_1)$ be the normed algebra of all bounded linear operators on $\mathfrak{X}_1$. As shown in [6, p. 56], $\mathcal{G}(\mathfrak{X}_1)$ has an involution $T \rightarrow T^*$ where $(T(x), y) = (x, T^*(y)), x, y \in \mathfrak{X}_1$. In these terms we show the following.

5.5. Theorem. The following statements are equivalent.
(1) $\mathfrak{X}_1$ is complete.
(2) The involution in $\mathcal{G}(\mathfrak{X}_1)$ is continuous.
(3) The faithful* -representation of Theorem 5.2 for $\mathcal{G}(\mathfrak{X}_1)$ is continuous.

As already noted (1) implies (2) and (3). Assume (2) and let $M$ be the norm of the involution. By [2] any minimal idempotent of $\mathcal{G}(\mathfrak{X}_1)$ is one-dimensional and the operator $J$ defined by the rule $J(x) = (x, u)u$ where $(u, u) = 1$ is a minimal idempotent. Since $(J(x), y) = (x, u)(u, y) = (x, J(y))$ we have $J = J^*$. The functional $f$ defined by $f(U)J = JUJ$ is a continuous positive linear functional on $\mathcal{G}(\mathfrak{X}_1)$. For $z \in \mathfrak{X}_1$ define the operator $W_z$ by the rule $W_z(x) = (x, u)z$. Then we can write the norm of $W_z$ as $C \| z \|$ where $C$ is independent of $z$. A simple computation gives $JW_z^*W_zJ = (z, z)J$. By formula (5.1), where $\| U \|$ denotes the norm in $\mathcal{G}(\mathfrak{X}_1)$,

\[
\| z \|^2 = (z, z) \leq \nu(W_z^*W_z) \leq \| W_z^*W_z \| \leq C^2M \| z \|^2.
\]

This shows that $\mathfrak{X}_1$ is complete.

Assume (3) and let $N$ be the norm of the faithful* -representation. Let

$I_f = \{ U \in \mathcal{G}(\mathfrak{X}_1) | f(U^*U) = 0 \}$, $\pi$ be the natural homomorphism of $\mathcal{G}(\mathfrak{X}_1)$ onto $\mathcal{G}(\mathfrak{X}_1)/I_f$ and $(\xi, \eta)_f$ be the inner product for the pre-Hilbert space $\mathcal{G}(\mathfrak{X}_1)/I_f$. Let $V \rightarrow T^f$ be the partial* -representation induced by $f$. Its norm cannot exceed $N$. Now $(\pi(J), \pi(J))_f = 1$ and

\[
N^2 \| U \|^2 \geq \| T^f(\pi(J)) \|^2 = (UJ, UJ)_f = f(JU^*UJ) = f(U^*U).
\]

Applying this formula to $U = W_z$ we obtain $N^2C^2 \| z \|^2 \geq (z, z)$ and again $\mathfrak{X}_1$ is complete.

A specific example is suggested in [6, p. 57]. Let $\mathfrak{X} = l^2, \| \{ x_n \} \| = \sup | x_n |$. An easy computation gives $\mathfrak{X}_1 = l^2 \cap l^1$ in the sup norm. Here the involution and* -representation are therefore not continuous.

6. Involutions on $\mathcal{G}(\mathfrak{S})$. Let $\mathfrak{S}$ be a Hilbert space and $\mathcal{G}(\mathfrak{S})$ the $B^*$-...
algebra of all bounded linear operators on $\mathcal{G}$. We determine in Theorem 6.2 all the involutions on $\mathcal{G}(\mathcal{G})$ for which there are faithful adjoint-preserving representations.

6.1. Lemma. Let $\Gamma$ be any involution on $\mathcal{G}(\mathcal{G})$. Then there exists an invertible s.a. element $U$ in $\mathcal{G}(\mathcal{G})$ such that $T^* = U^{-1}T^*U$ for all $T \in \mathcal{G}(\mathcal{G})$. Conversely any such mapping is an involution.

The mapping $T \mapsto T^{**}$, $T \in \mathcal{G}(\mathcal{G})$, is an automorphism of $\mathcal{G}(\mathcal{G})$. Thus there exists $V \in \mathcal{G}(\mathcal{G})$ where $T^{**} = VT^*V^{-1}$, $T \in \mathcal{G}(\mathcal{G})$. Set $U = V^*$. Then $T^* = U^{-1}T^*U$. Since $T^{**} = T$, $T = (U^{-1}T^*U)^* = U^{-1}U^*T^*(U^*)^{-1}U$. Thus $U^{-1}U^*$ lies in the center of $\mathcal{G}(\mathcal{G})$. Consequently $U = \lambda U^*$ for some scalar $\lambda$. Since $U^*U = |\lambda|^2 U^*U$ we see that $|\lambda| = 1$. Set $\lambda = \exp(i\theta)$ and $W = \exp(-i\theta/2)U$. Then $W^* = W$ and $T^* = W^{-1}T^*W$, $T \in \mathcal{G}(\mathcal{G})$. The remaining statement is easily verified.

6.2. Theorem. An involution $T \mapsto T^*$ on $\mathcal{G}(\mathcal{G})$ is proper if and only if it can be expressed in the form $T^* = U^{-1}T^*U$, $U \in \mathcal{G}(\mathcal{G})$ where $U$ is s.a. and $sp(U) \subset (0, \infty)$.

If $T \mapsto T^*$ is a proper involution then (see [7]) an inner product can be defined in $\mathcal{G}$ in terms of which $T^*$ is the adjoint of $T$. Hence the proper involutions are those for which there is an adjoint preserving faithful representation.

Let $W$ be a one-dimensional operator, $W(x) = (x, z)w$ with $w \neq 0, z \neq 0$. Then $W^*(x) = (x, w)z$. By Lemma 6.1 we can write $T^* = U^{-1}T^*U$, $T \in \mathcal{G}(\mathcal{G})$, where $U$ is s.a. Then $0 \neq W^*W = U^{-1}W^*UW$. Hence $0 \neq W^*UW$. But $W^*UW(x) = (x, z)W^*U(w) = (x, z)(U(w), w)z$. Therefore $(U(w), w) \neq 0$ for an arbitrary non-zero $w \in \mathcal{G}$. Hence $(U(w), w)$ has a constant sign and, by changing to $-U$ if necessary, we may suppose that $(U(w), w) \geq 0, w \in \mathcal{G}$. Then we can write $U = V^2$ where $V$ is s.a. in $\mathcal{G}(\mathcal{G})$.

Suppose conversely that $T^* = V^{-2}T^*V^2$, $T \in \mathcal{G}(\mathcal{G})$ where $V$ is s.a. Then $TT^* = (TV^{-1})(TV^{-1})^*V^2$. Thus $TT^* = 0$ implies that $TV^{-1} = 0$ and that $T = 0$.

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