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**AN ELEMENTARY PROOF OF THE PRIME NUMBER
THEOREM WITH REMAINDER TERM**

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Introduction. In this paper, the prime number theorem in the form $\psi(x) \equiv \sum_{p^m \leq x} \log p = x + o(x \cdot \log^{-1/6+\varepsilon} x)$, for every $\varepsilon > 0$, is established via a proof that in the well-known formula

$$(1) \quad \rho(x) \equiv \sum_{p^m \leq x} \frac{\log p}{p^m} = \log x + O(1) \equiv \log x + a_x,$$

$a_x = -A_0 + o(\log^{-1/6+\varepsilon} x)$. (A_0 is Euler's constant.)

Throughout the paper, p and q stand for prime numbers, k, m, n, t , and others are positive integers, and x, y , and z are positive real numbers.

Some well-known formulas, used in the proof, are

$$(2) \quad \sum_{n \leq x} \frac{\log^k n}{n} = \frac{1}{k+1} \cdot \log^{k+1} x + A_k + O\left(\frac{\log^k x}{x}\right), \quad \text{for } k = 0, 1, \dots$$

$$(2') \quad \sum_{y < n \leq z} \frac{1}{n} \cdot \log^k(n/y) = \frac{1}{k+1} \cdot \log^{k+1}(z/y) + O\left(\frac{1}{y} \cdot \log^k(z/y)\right),$$

for $k = 0, 1, \dots$

$$(3) \quad \sum_{n \leq x} \log^k(x/n) = O(x), \quad \text{for } k = 1, 2, \dots$$

$$(4) \quad \sum_{p^m \leq x} \log p \cdot \log^k(x/p^m) = O(x), \quad \text{for } k = 0, 1, \dots$$

$$(5) \quad \sum_{n \leq x} \mu(n)/n = O(1) \quad (\mu(n) \text{ is Moebius' function.})$$

Two other formulas, used prominently, are

$$(6) \quad \sigma(x) \equiv \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot \log(x/p^m) = \frac{1}{2} \cdot \log^2 x - A_0 \cdot \log x + g_x \quad (g_x = O(1))$$

$$(7) \quad \tau(x) \equiv \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot \log^2(x/p^m) = \frac{1}{3} \cdot \log^3 x - A_0 \cdot \log^2 x$$

$+ (2 \cdot A_0^2 + 4 \cdot A_1) \log x + O(1)$.

With the help of (1), (2), and (4), (6) can be proved easily:

$$\sigma(x) = \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot \left(\sum_{n \leq x/p^m} 1/n - A_0 + O(p^m/x) \right), \quad \text{or, with } k = n \cdot p^m,$$

$$\sigma(x) = \sum_{k \leq x} \frac{1}{k} \cdot \sum_{p^m | k} \log p - A_0 \cdot \log x + O(1)$$

$$= \sum_{k \leq x} \frac{\log k}{k} - A_0 \log x + O(1) = \frac{1}{2} \cdot \log^2 x - A_0 \log x + O(1).$$

Also, again with $k = n \cdot p^m$,

$$\begin{aligned} & \sum_{k \leq x} \frac{\log k}{k} \cdot \log \frac{x}{k} \\ &= \sum_{k \leq x} \frac{1}{k} \cdot \log \frac{x}{k} \cdot \sum_{p^m/k} \log p = \sum_{p^m \leq x} \log p \cdot \sum_{n \leq x/p^m} \frac{1}{n \cdot p^m} \cdot \log \left(\frac{x}{n \cdot p^m} \right) \\ &= \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot \left\{ \log \left(\frac{x}{p^m} \right) \cdot \sum_{n \leq x/p^m} \frac{1}{n} - \sum_{n \leq x/p^m} \frac{\log n}{n} \right\} \\ &= \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot \left\{ \log^2(x/p^m) + A_0 \log(x/p^m) - \frac{1}{2} \log^2(x/p^m) - A_1 \right\} + O(1) \\ & \hspace{15em} \text{(by (2) and (4))} \\ &= \frac{1}{2} \cdot \tau(x) + A_0 \cdot \sigma(x) - A_1 \cdot \rho(x) + O(1) . \end{aligned}$$

(7) follows now by (1), (2), and (6).

The proof now proceeds in the following steps: in part I, certain asymptotic formulas for a_n (see (1)) and g_n (see (6)) are derived; they suggest that “on the average,” a_n is $-A_0$, and g_n is $A_0^2 + 2A_1$. In part II, formulas for a_n and g_n are derived which are of the type of Selberg’s asymptotic formula for $\psi(x)$; part III contains the final proof.

PART I

First, the following five formulas will be derived; K_1, K_2, \dots , are constants, independent of x .

- (8) $\sum_{n \leq x} \frac{1}{n} \cdot a_n = -A_0 \log x + g_x + K_2 + O\left(\frac{\log x}{x}\right)$
- (9) $\sum_{n \leq x} \frac{1}{n} \cdot a_{x/n} = -A_0 \log x + K_3 + O\left(\frac{\log x}{x}\right)$
- (10) $\sum_{p \leq x} \frac{\log p}{p^m} \cdot a_{p^m} = -A_0 \log x + g_x + \frac{1}{2} a_x^2 + K_4 + O\left(\frac{\log x}{x}\right)$
- (11) $\sum_{n \leq x} \frac{1}{n} \cdot g_n = (A_0^2 + 2 \cdot A_1) \cdot \log x + O(1)$
- (12) $\sum_{n \leq x} \frac{1}{n} \cdot g_{x/n} = (A_0^2 + 2 \cdot A_1) \cdot \log x + K_5 + O\left(\frac{\log^2 x}{x}\right)$.

Proofs.

$$\begin{aligned} \sigma(x) &= \sum_{n \leq x} \log \frac{x}{n} (\rho(n) - \rho(n - 1)) = \sum_{n \leq x} \rho(n) \cdot \log \frac{n + 1}{n} + O\left(\frac{\log x}{x}\right) \\ &= \sum_{n \leq x} \frac{\rho(n)}{n} + K_1 + O\left(\frac{\log x}{x}\right) \\ &= \sum_{n \leq x} \frac{\log n}{x} + \sum_{n \leq x} \frac{1}{n} a_n + K_1 + O\left(\frac{\log x}{x}\right) . \end{aligned}$$

(8) follows now from (6) and (2).

Also

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n} \cdot a_{x/n} &= \sum_{n \leq x} \frac{1}{n} \cdot \left(\sum_{p^m \leq x/n} \frac{\log p}{p^m} - \log \frac{x}{n} \right) \\ &= \sum_{k \leq x} \frac{1}{k} \sum_{p^m/k} \log p - \sum_{n \leq x} \frac{1}{n} \log \frac{x}{n} \quad (k = n \cdot p^m) \\ &= \sum_{k \leq x} \frac{\log k}{k} - \sum_{n \leq x} \frac{1}{n} \log \frac{x}{n}. \quad \text{which proves (9) by (2).} \end{aligned}$$

And

$$\begin{aligned} \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot a_{p^m} &= \sum_{p^m \leq x} \frac{\log p}{p^m} \left(\sum_{q^t \leq p^m} \frac{\log q}{q^t} - \log(p^m) \right) \\ &= \frac{1}{2} \left(\sum_{p^m \leq x} \frac{\log p}{p^m} \right)^2 + \frac{1}{2} \sum_{p^m \leq x} \frac{\log^2 p}{p^{2m}} - \log x \cdot \sum_{p^m \leq x} \frac{\log p}{p^m} + \sum_{p^m \leq x} \frac{\log p}{p^m} \log \frac{x}{p^m}. \end{aligned}$$

Thus, by (1), (2) and (6),

$$\begin{aligned} \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot a_{p^m} &= \frac{1}{2} (\log x + a_x)^2 + K_4 + O\left(\frac{\log x}{x}\right) - \log x \cdot (\log x + a_x) \\ &\quad + \frac{1}{2} \log^2 x - A_0 \log x + g_x, \quad \text{which proves (10).} \end{aligned}$$

In the next proof, use is made of the easily established fact that

$$\rho(n) \cdot \log \frac{n+1}{n} = \sigma(n+1) - \sigma(n).$$

$$\begin{aligned} \tau(x) &= \sum_{n \leq x} \log^2 \left(\frac{x}{n} \right) (\rho(n) - \rho(n-1)) \\ &= \sum_{n \leq x} \rho(n) \left(\log^2 \left(\frac{x}{n} \right) - \log^2 \left(\frac{x}{n+1} \right) \right) + O(1) \\ &= \sum_{n \leq x} \rho(n) \log \frac{n+1}{n} \cdot \log \frac{x^2}{n(n+1)} + O(1) \\ &= \sum_{n \leq x} (\sigma(n+1) - \sigma(n)) \cdot \log \frac{x^2}{n(n+1)} + O(1) \\ &= \sum_{n \leq x} \sigma(n) \cdot \log \frac{n+1}{n-1} + O(1) = \sum_{n \leq x} \sigma(n) \cdot \frac{2}{n} + O(1) \\ &= \sum_{n \leq x} \frac{\log^2 n}{n} - 2 \cdot A_0 \cdot \sum_{n \leq x} \frac{\log n}{n} + 2 \cdot \sum_{n \leq x} \frac{1}{n} \cdot g_n + O(1) \quad (\text{by (6)}). \end{aligned}$$

This proves (11), with the help of (2) and (7).

Finally

$$\sum_{n \leq x} \frac{1}{n} \cdot g_{x/n} = \sum_{n \leq x} \frac{1}{n} \cdot \left(\sum_{p^m \leq x/n} \frac{\log p}{p^m} \cdot \log \frac{x}{n \cdot p^m} - \frac{1}{2} \log^2 \left(\frac{x}{n} \right) + A_0 \log \frac{x}{n} \right),$$

or, with $k = n \cdot p^m$,

$$\begin{aligned} & \sum_{n \leq x} \frac{1}{n} \cdot g_{x/n} \\ &= \sum_{k \leq x} \frac{1}{k} \cdot \log \frac{x}{k} \cdot \sum_{p^m/k} \log p - \frac{1}{2} \cdot \sum_{n \leq x} \frac{1}{n} \cdot \log^2 \left(\frac{x}{n} \right) + A_0 \sum_{n \leq x} \frac{1}{n} \cdot \log \frac{x}{n} \\ &= \sum_{k \leq x} \frac{1}{k} \cdot \log \frac{x}{k} \cdot \log k - \frac{1}{2} \cdot \sum_{n \leq x} \frac{1}{n} \cdot \log^2 \left(\frac{x}{n} \right) + A_0 \sum_{n \leq x} \frac{1}{n} \cdot \log \frac{x}{n} . \end{aligned}$$

(12) now follows by (2).

Formulas (8) through (12) suggest setting

$$(13) \quad b_x \equiv a_x + A_0, \quad h_x \equiv g_x - (A_0^2 + 2A_1).$$

In terms of b_x and h_x , the five formulas read

$$(8') \quad \sum_{n \leq x} \frac{1}{n} \cdot b_n = h_x + K_6 + O\left(\frac{\log x}{x}\right)$$

$$(9') \quad \sum_{n \leq x} \frac{1}{n} \cdot b_{x/n} = K_7 + O\left(\frac{\log x}{x}\right)$$

$$\begin{aligned} (10') \quad \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot b_{p^m} &= A_0 \cdot (-A_0 + b_x) + A_0^2 + 2 \cdot A_1 + h_x + K_4 \\ &\quad + \frac{1}{2} \cdot (-A_0 + b_x)^2 + O\left(\frac{\log x}{x}\right) \\ &= h_x + \frac{1}{2} \cdot b_x^2 + K_8 + O\left(\frac{\log x}{x}\right) \end{aligned}$$

$$(11') \quad \sum_{n \leq x} \frac{1}{n} \cdot h_n = O(1)$$

$$(12') \quad \sum_{n \leq x} \frac{1}{n} \cdot h_{x/n} = K_9 + O\left(\frac{\log^2 x}{x}\right).$$

Next, it will be shown that

$$(14) \quad \sum_{n \leq x} \frac{1}{n} \cdot b_n^2 = \sum_{n \leq x} \frac{1}{n} \cdot b_{x/n}^2 + O(1),$$

and

$$(15) \quad \sum_{n \leq x} \frac{1}{n} \cdot b_n \cdot b_{x/n} = \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot h_{x/p^m} + O(1).$$

For a proof of (14), we know, by (10'), that

$$\frac{1}{n} \cdot b_n^2 = \frac{2}{n} \cdot \sum_{p^m \leq n} \frac{\log p}{p^m} \cdot b_{p^m} - \frac{2}{n} \cdot h_n - \frac{2}{n} \cdot K_8 + O\left(\frac{\log n}{n^2}\right),$$

and

$$\frac{1}{n} \cdot b_{x/n}^2 = \frac{2}{n} \cdot \sum_{p^m \leq x/n} \frac{\log p}{p^m} \cdot b_{p^m} - \frac{2}{n} \cdot h_{x/n} - \frac{2}{n} \cdot K_8 + O\left(\frac{1}{x} \cdot \log \frac{x}{n}\right).$$

Thus, by (3), (11') and (12'),

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n} \cdot (b_n^2 - b_{x/n}^2) &= 2 \cdot \sum_{n \leq x} \frac{1}{n} \left(\sum_{p^m \leq n} \frac{\log p}{p^m} \cdot b_{p^m} - \sum_{p^m \leq x/n} \frac{\log p}{p^m} \cdot b_{p^m} \right) + O(1) \\ &= 2 \cdot \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot b_{p^m} \left(\sum_{p^m \leq n \leq x} \frac{1}{n} - \sum_{n \leq x/p^m} \frac{1}{n} \right) + O(1) \\ &= 2 \cdot \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot b_{p^m} \cdot \left(\log(x/p^m) + O(1/p^m) - \log(x/p^m) \right. \\ &\qquad \qquad \qquad \left. - A_0 - O(p^m/x) \right) + O(1) \\ &= O(1), \text{ by (10') and (4). This proves (14).} \end{aligned}$$

Also

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n} \cdot b_n \cdot b_{x/n} &= \sum_{n \leq x} \frac{1}{n} \cdot b_n \cdot \left(\sum_{p^m \leq x/n} \frac{\log p}{p^m} - \log \frac{x}{n} + A_0 \right) \\ &= \sum_{n \leq x} \frac{1}{n} \cdot b_n \cdot \left(\sum_{p^m \leq x/n} \frac{\log p}{p^m} - \sum_{t \leq x/n} \frac{1}{t} + 2 \cdot A_0 + O\left(\frac{n}{x}\right) \right) \\ &= \sum_{p^m \leq x} \frac{\log p}{p^m} \sum_{n \leq x/p^m} \frac{1}{n} b_n - \sum_{t \leq x} \frac{1}{t} \sum_{n \leq x/t} \frac{1}{n} b_n + O(1), \text{ by (8')} \\ &= \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot h_{x/p^m} + K_6 \log x - \sum_{t \leq x} \frac{1}{t} h_{x/t} - K_6 \log x + O(1) \\ &\qquad \qquad \qquad \text{(by (8'), (1) and (4))} \\ &= \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot h_{x/p^m} + O(1), \text{ by (12')}. \end{aligned}$$

From (14) and (15) it follows that

$$\sum_{n \leq x} \frac{1}{n} \cdot (b_n \pm b_{x/n})^2 = 2 \cdot \sum_{n \leq x} \frac{1}{n} \cdot b_n^2 \pm 2 \cdot \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot h_{x/p^m} + O(1),$$

and therefore

$$(16) \quad \sum_{n \leq x} \frac{1}{n} \cdot b_n^2 \geq \left| \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot h_{x/p^m} \right| + O(1).$$

PART II

In the following, we shall employ the inversion formula

$$G(x) = \sum_{n \leq x} g\left(\frac{x}{n}\right) \quad \text{for all } x > 0 \Rightarrow g(x) = \sum_{n \leq x} \mu(n) \cdot G\left(\frac{x}{n}\right),$$

as well as

$$(17) \quad \sum_{n \leq x} \frac{\mu(n)}{n} \log \frac{x}{n} = O(1).$$

For a proof of (17), we make use of the fact that $\sum_{n \leq x} x/n = x \cdot \log x + A_0 x + O(1)$; thus, by the inversion formula,

$$x = \sum_{n \leq x} \mu(n) \cdot \frac{x}{n} \log \frac{x}{n} + A_0 \cdot \sum_{n \leq x} \mu(n) \cdot \frac{x}{n} + O(x) .$$

(17) follows now by (5).

If $f(x)$ is defined for $x > 0$, then

$$\begin{aligned} & \sum_{n \leq x} \left\{ \frac{x}{n} \cdot \log \frac{x}{n} \cdot f\left(\frac{x}{n}\right) + \frac{x}{n} \cdot \sum_{p^m \leq x/n} \frac{\log p}{p^m} \cdot f\left(\frac{x}{n \cdot p^m}\right) \right\} \\ &= \sum_{n \leq x} \frac{x}{n} \cdot \log \frac{x}{n} \cdot f\left(\frac{x}{n}\right) + \sum_{k \leq x} \frac{x}{k} \cdot f\left(\frac{x}{k}\right) \cdot \sum_{p^m/k} \log p \quad (k = n \cdot p^m) \\ &= \sum_{n \leq x} \frac{x}{n} \cdot \log \frac{x}{n} \cdot f\left(\frac{x}{n}\right) + \sum_{k \leq x} \frac{x}{k} \cdot f\left(\frac{x}{k}\right) \cdot \log k = x \cdot \log x \cdot \sum_{n \leq x} \frac{1}{n} \cdot f\left(\frac{x}{n}\right) . \end{aligned}$$

Thus, if we set

$$F(x) \equiv x \cdot \log x \cdot \sum_{n \leq x} \frac{1}{n} \cdot f\left(\frac{x}{n}\right) ,$$

then, by the inversion formula,

$$x \cdot \log x \cdot f(x) + x \cdot \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot f(x/p^m) = \sum_{n \leq x} \mu(n) \cdot F\left(\frac{x}{n}\right) .^1$$

In particular, if

$$\sum_{n \leq x} \frac{1}{n} \cdot f\left(\frac{x}{n}\right) = K + O\left(\frac{\log^k x}{x}\right) ,$$

then

$$\sum_{n \leq x} \mu(n) \cdot F\left(\frac{x}{n}\right) = K \cdot \sum_{n \leq x} \mu(n) \cdot \frac{x}{n} \cdot \log\left(\frac{x}{n}\right) + O\left(\sum_{n \leq x} \log^{k+1}\left(\frac{x}{n}\right)\right) = O(x) ,$$

by (17) and (3), and thus

$$(18) \quad f(x) \cdot \log x + \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot f(x/p^m) = O(1) ,$$

$$\text{if} \quad \sum_{n \leq x} \frac{1}{n} \cdot f\left(\frac{x}{n}\right) = K + O\left(\frac{\log^k x}{x}\right) .$$

(Selberg's asymptotic formula for $\psi(x)$ corresponds to $f(x) \equiv \psi(x)/x - 1$.) By (9') and (12'), $f(x) \equiv b_x$ and $f(x) \equiv h_x$ both satisfy the condition of (18), and thus

$$(19) \quad b_x \cdot \log x + \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot b_{x/p^m} = O(1)$$

$$(20) \quad h_x \cdot \log x + \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot h_{x/p^m} = O(1) .$$

¹ Compare K. Iseki and T. Tatzuwa, "On Selberg's elementary proof of the prime number theorem." Proc. Jap. Acad. 27, 340-342 (1951).

From (16) and (20) it follows that

$$(21) \quad \sum_{n \leq x} \frac{1}{n} \cdot b_n^2 \geq |h_x| \cdot \log x + O(1).$$

If we add to (19)

$$(\log x - A_0) \cdot \log x + \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot (\log(x/p^m) - A_0),$$

which by (1) and (6) is equal to $3/2 \cdot \log^2 x - 3 \cdot A_0 \cdot \log x + O(1)$, we obtain

$$\rho(x) \cdot \log x + \sum_{p^m \leq x} \frac{\log p}{p^m} \cdot \rho(x/p^m) = \frac{3}{2} \cdot \log^2 x - 3 \cdot A_0 \cdot \log x + O(1).$$

If $0 < c < 1$, and $c \cdot x < y < x$, then it follows from the last equation that

$$\begin{aligned} \rho(x) \cdot \log x - \rho(y) \cdot \log y &\leq \frac{3}{2} \cdot (\log^2 x - \log^2 y) + O(1) \\ &= \frac{3}{2} \cdot \log \frac{x}{y} \cdot (\log x + \log y) + O(1), \end{aligned}$$

$$\log x \cdot (\rho(x) - \rho(y)) + \log \frac{x}{y} \cdot \rho(y) \leq \frac{3}{2} \cdot \log \frac{x}{y} \cdot (\log x + \log y) + O(1),$$

or, since $\rho(y) = \log y + O(1)$,

$$\begin{aligned} \log x \cdot (\rho(x) - \rho(y)) &\leq \log \frac{x}{y} \cdot \left(\frac{3}{2} \cdot \log x + \frac{1}{2} \cdot \log y \right) + O(1) \\ &< 2 \cdot \log \frac{x}{y} \cdot \log x + O(1). \end{aligned}$$

Thus

$$\rho(x) - \rho(y) < 2 \cdot \log \frac{x}{y} + O\left(\frac{1}{\log x}\right),$$

and, since $\rho(x) = \log x - A_0 + b_x$, it follows that $b_x - b_y < \log x/y + O(1/\log x)$. Also obviously $b_x - b_y \geq -\log x/y$, because $\rho(x)$ is non-decreasing. Thus we obtain

$$(22) \quad |b_x - b_y| \leq \log \frac{x}{y} + O\left(\frac{1}{\log x}\right) \quad \text{if } c \cdot x < y < x, \quad 0 < c < 1.$$

PART III

Let $B \geq 1$ be an upper bound of $|b_n|$.

Since $b_n - b_{n-1}$ is either $-\log[n/(n-1)]$, or $\log p/n - \log[n/(n-1)]$, it cannot happen that $b_n = b_{n-1} = 0$.

Let the integers $r_1, r_2, \dots, r_t, \dots$ be the indices n for which the b_n change signs. Precisely:

$$(23) \quad \begin{cases} r_1 = 1; n = r_t \text{ if } b_n \cdot b_{n+1} \leq 0, \text{ and } b_{n+1} \neq 0; \\ \text{if } r_t < v \leq w < r_{t+1} \text{ then } b_v \cdot b_w > 0; \text{ and} \\ |b_{r_t}| < (\log r_t)/r_t \text{ for } t > 1. \end{cases}$$

Let $\{s_k\}$ be a sequence of integers, determined as follows: every r_t is an s_k ; if $\log(r_{t+1}/r_t) < 7 \cdot B$, and $r_t = s_k$, then $r_{t+1} = s_{k+1}$; if $\log(r_{t+1}/r_t) \geq 7 \cdot B$, enough integers s_{k+v} are inserted between $r_t = s_k$ and $r_{t+1} = s_{k+m}$ such that $3 \cdot B \leq \log(s_{k+v+1}/s_{k+v}) < 7 \cdot B$, for $v = 0, 1, \dots, m - 1$. If there is a last $r_{t_0} = s_{k_0}$, a sequence $\{s_{k_0+v}\}$ is formed such that $3 \cdot B \leq \log(s_{k_0+v+1}/s_{k_0+v}) < 7 \cdot B$. Thus the s_k form a sequence with the following properties:

$$(24) \quad \begin{cases} s_1 = 1; \log(s_{k+1}/s_k) < 7 \cdot B; \text{ for } k > 1, \text{ either} \\ \log(s_{k+1}/s_k) \geq 3 \cdot B, \text{ or } |b_{s_k}| \text{ and } |b_{s_{k+1}}| \text{ are both} \\ \text{less than } \frac{\log s_k}{s_k}; b_v \cdot b_w > 0 \text{ for } s_k < v \leq w < s_{k+1}. \end{cases}$$

Assume now that α ($0 < \alpha < 1/2$) is such that

$$(25) \quad \text{not } h_x = O(\log^{-\alpha} x).$$

Then $|h_x| \cdot \log^\alpha x$ is unbounded. Let x be large, and such that $|h_x| \cdot \log^\alpha x \geq |h_y| \cdot \log^\alpha y$ for all $y \leq x$. Let c and d be positive integers such that

$$(26) \quad s_{c-1} < \log x \leq s_c, \quad \text{and} \quad s_d \leq x < s_{d+1}.$$

It will be shown that

$$\frac{1}{2} \cdot (1 - \alpha - o(1)) \cdot S(x) \leq |h_x| \cdot \log x \leq \frac{1}{3} \cdot (1 + o(1)) \cdot S(x),$$

where

$$(27) \quad S(x) \equiv \sum_{k=c+1}^d |h_{s_k} - h_{s_{k-1}}| \cdot \log(s_k/s_{k-1}).$$

From this it will follow that $\alpha \geq 1/3$.

Clearly

$$\begin{aligned} |h_x| \cdot \log x &= |h_x| \cdot \log^\alpha x \cdot \left\{ \log^{1-\alpha} x - \log^{1-\alpha} s_d + \sum_{k=2}^d (\log^{1-\alpha} s_k - \log^{1-\alpha} s_{k-1}) \right\} \\ &\geq \frac{1}{2} \cdot \sum_{k=c+1}^d (|h_{s_k}| \cdot \log^\alpha s_k + |h_{s_{k-1}}| \cdot \log^\alpha s_{k-1}) \cdot (\log^{1-\alpha} s_k - \log^{1-\alpha} s_{k-1}) \\ &\geq \frac{1}{2} \cdot \sum_{k=c+1}^d |h_{s_k} - h_{s_{k-1}}| \cdot \log^\alpha s_{k-1} \cdot (\log^{1-\alpha} s_k - \log^{1-\alpha} s_{k-1}). \end{aligned}$$

If $y < z$, it is easily shown by the mean value theorem that

$$y^\alpha \cdot (z^{1-\alpha} - y^{1-\alpha}) > (1-\alpha) \cdot \frac{y}{z} \cdot (z-y) > \left(1-\alpha - \frac{z-y}{z}\right)(z-y).$$

With $y = \log s_{k-1}$, $z = \log s_k$, and from the fact that $s_k > \log x$, $\log(s_k/s_{k-1}) < 7 \cdot B$, it follows by (27) that

$$(28) \quad |h_x| \cdot \log x > \frac{1}{2} \cdot \left(1 - \alpha - \frac{7 \cdot B}{\log \log x}\right) \cdot S(x).$$

For the next estimate, we need the following lemma.

LEMMA. Let v and w be positive integers such that

- (1) $\log \frac{w}{v} = O(1)$;
- (2) $b_n > 0$ for $v \leq n \leq w$;
- (3) $b_v < \frac{\log v}{v}$

Then

$$\sum_{v \leq n \leq w} \frac{1}{n} \cdot b_n^2 \leq \frac{2}{3} \cdot \log \frac{w}{v} \cdot \sum_{v \leq n \leq w} \frac{1}{n} b_n + O\left(\frac{\log(w/v)}{\log v}\right).$$

Proof. If $b_n \leq 1/3 \cdot \log w/v$ for every n in $[v, w]$, the lemma is obviously correct. Otherwise, let n_1 be such that

$$b_{n_1} \geq \frac{1}{3} \cdot \log \frac{w}{v}, \quad b_n < \frac{1}{3} \cdot \log \frac{w}{v} \quad \text{for } v \leq n < n_1.$$

If $\log(n_1/v) > 1/3 \log(w/v)$, let z ($v \leq z < n_1$) be such that $\log(n_1/z) = 1/3 \log(w/v)$; otherwise, let $z = v$. Thus by (22), in every case, $\log(n_1/z) = 1/3 \log(w/v) + O(1/\log v)$. Clearly $b_n - 2/3 \cdot \log w/v < 0$ for $v \leq n \leq z$. Thus

$$\begin{aligned} T &\equiv \sum_{v \leq n \leq w} \frac{1}{n} \cdot b_n^2 - \frac{2}{3} \cdot \log \frac{w}{v} \cdot \sum_{v \leq n \leq w} \frac{1}{n} \cdot b_n \\ &\leq \sum_{z \leq n \leq w} \frac{1}{n} \cdot b_n^2 - \frac{2}{3} \cdot \log \frac{w}{v} \cdot \sum_{z \leq n \leq w} \frac{1}{n} \cdot b_n, \end{aligned}$$

$$T \leq \sum_{z \leq n \leq w} \frac{1}{n} \cdot \left(b_n - \frac{1}{3} \cdot \log \frac{w}{v}\right)^2 - \frac{1}{9} \cdot \log^2(w/v) \cdot \log(w/z) + O\left(\frac{\log(w/v)}{v}\right).$$

By (22),

$$\begin{aligned} \left|b_n - \frac{1}{3} \log(w/v)\right| &= |b_n - b_{n_1}| + O\left(\frac{\log v}{v}\right) \\ &\leq |\log(n_1/n)| + O\left(\frac{1}{\log v}\right) = |\log(n_1/z) - \log(n/z)| + O\left(\frac{1}{\log v}\right), \end{aligned}$$

and thus

$$\left| b_n - \frac{1}{3} \cdot \log \frac{w}{v} \right| \leq \left| \log \frac{n}{z} - \frac{1}{3} \log \frac{w}{v} \right| + O\left(\frac{1}{\log v}\right).$$

Thus

$$\begin{aligned} T &\leq \sum_{z \leq n \leq w} \frac{1}{n} \cdot \left(\log \frac{n}{z} - \frac{1}{3} \cdot \log \frac{w}{v} \right)^2 \\ &\quad - \frac{1}{9} \cdot \log^2(w/v) \cdot \log(w/z) + O\left(\frac{\log(w/v)}{\log v}\right) \\ &= \sum_{z \leq n \leq w} \frac{1}{n} \cdot \log^2(n/z) - \frac{2}{3} \cdot \log \frac{w}{v} \cdot \sum_{z \leq n \leq w} \frac{1}{n} \cdot \log \frac{n}{z} + O\left(\frac{\log(w/v)}{\log v}\right) \\ &= \frac{1}{3} \cdot \log^3(w/z) - \frac{2}{3} \cdot \log^2(w/v) \cdot \frac{1}{2} \cdot \log^2(w/z) + O\left(\frac{\log(w/v)}{\log v}\right), \end{aligned}$$

by (2'), and thus $T \leq O(\log(w/v)/\log v)$. This completes the proof of the lemma.

COROLLARY 1. *If condition (3) is replaced by $b_w < \log w/w$, the conclusion still holds; if $b_n < 0$ in $v \leq n \leq w$, the conclusion holds if b_n is replaced by $|b_n|$.*

COROLLARY 2. *If instead of (3) it is known that $b_v < \log v/v$ and $b_w < \log w/w$ then*

$$\sum_{v \leq n \leq w} \frac{1}{n} \cdot b_n^2 \leq \frac{1}{3} \cdot \log \frac{w}{v} \cdot \sum_{v \leq n \leq w} \frac{1}{n} \cdot |b_n| + O\left(\frac{\log(w/v)}{\log v}\right).$$

For a proof, we split $[v, w]$ into two intervals by a division point at $(v \cdot w)^{1/2}$, and apply the lemma separately to each subinterval.

COROLLARY 3.

$$(29) \quad \sum_{s_{k-1} < n \leq s_k} \frac{1}{n} \cdot b_n^2 \leq \frac{1}{3} \cdot \log(s_k/s_{k-1}) \cdot \sum_{s_{k-1} < n \leq s_k} \frac{1}{n} \cdot |b_n| + O\left(\frac{\log(s_k/s_{k-1})}{\log s_k}\right).$$

Proof. If $\log(s_k/s_{k-1}) < 3 \cdot B$, this follows from (24) and Corollary 2; if $\log(s_k/s_{k-1}) \geq 3B$, it is obvious, since $|b_n| \leq B$.

By (26), $\sum_{m \leq s_c} 1/n \cdot b_n^2 = O(\log \log x)$, and $\sum_{s_d < n \leq s_x} 1/n \cdot b_n^2 = O(1)$; also

$$\sum_{k=c+1}^d \frac{\log(s_k/s_{k-1})}{\log s_k} \leq \sum_{k=c+1}^d \log\left(\frac{\log s_k}{\log s_{k-1}}\right) \leq \log \log x.$$

It follows from (29) that

$$\sum_{n \leq x} \frac{1}{n} \cdot b_n^2 \leq \frac{1}{3} \cdot \sum_{k=c+1}^d \log(s_k/s_{k-1}) \cdot \sum_{s_{k-1} < n \leq s_k} \frac{1}{n} \cdot |b_n| + O(\log \log x).$$

By (8') $\sum_{s_{k-1} < n \leq s_k} 1/n \cdot |b_n| = |h_{s_k} - h_{s_{k-1}}| + O(\log s_k/s_k)$, and thus, by (21) and (27),

$$(30) \quad |h_x| \cdot \log x \leq \frac{1}{3} \cdot S(x) + O(\log \log x).$$

It follows from (28) and (30) that

$$\left[\frac{1}{3} - \frac{1}{2} \cdot \left(1 - \alpha - \frac{7 \cdot B}{\log \log x} \right) \right] \cdot S(x) \geq O(\log \log x),$$

and since by (25) and (30) $S(x) \geq K \cdot \log^{1/2} x$, this implies that $\alpha \geq 1/3$. Thus $h_x = o(\log^{-1/3+\varepsilon} x)$, for every $\varepsilon > 0$, and therefore, by (8'),

$$(31) \quad \sum_{s_{k-1} < n \leq s_k} \frac{1}{n} \cdot |b_n| = o(\log^{-1/3+\varepsilon} s_k).$$

In order to find a bound for $|b_x|$, we consider now a particular interval $I_k = (s_{k-1}, s_k]$; let us assume that $b_n > 0$ in I_k . Let $n_2 \in I_k$ be such that $b_{n_2} \geq b_n$ for every $n \in I_k$. Let n_1 ($s_{k-1} \leq n_1 < n_2$) be such that

$$b_{n_1} \leq \frac{1}{2} \cdot b_{n_2} < b_{n_1+1}.$$

Then

$$\sum_{n \in I_k} \frac{1}{n} \cdot b_n > \sum_{n=n_1+1}^{n_2} \frac{1}{n} \cdot b_n > \frac{1}{2} \cdot b_{n_2} \cdot \log(n_2/n_1) - O(1/s_k).$$

But by (22),

$$\log(n_2/n_1) \geq b_{n_2} - b_{n_1} - O\left(\frac{1}{\log s_k}\right) \geq \frac{1}{2} \cdot b_{n_2} - O\left(\frac{1}{\log s_k}\right).$$

Thus

$$\sum_{n \in I_k} \frac{1}{n} \cdot b_n > \frac{1}{4} \cdot b_{n_2}^2 - O\left(\frac{1}{\log s_k}\right).$$

It follows from (31) that $b_{n_2}^2 = o(\log^{-1/3+\varepsilon} n_2)$, and thus

$$(32) \quad b_x = o(\log^{-1/6+\varepsilon} x).$$

Finally,

$$\begin{aligned} \psi(x) &= \sum_{n \leq x} n \cdot (\rho(n) - \rho(n-1)) = [x] \cdot \rho([x]) - \sum_{n \leq x-1} \rho(n) \\ &= x \cdot (\log x - A_0 + b_x) - \sum_{n \leq x} (\log n - A_0 + b_n) + O(\log x) \\ &= x \cdot \log x - A_0 \cdot x + b_x \cdot x - x \cdot \log x + x + A_0 \cdot x - \sum_{n \leq x} b_n + O(\log x) \\ &= x + o(x \cdot \log^{-1/6+\varepsilon} x) + o\left(\sum_{n \leq x} \log^{-1/6+\varepsilon} n\right), \quad \text{by (32).} \end{aligned}$$

The last sum is easily seen to be $o(x \cdot \log^{-1/6+\varepsilon} x)$, and thus

$$(33) \quad \psi(x) = x + o(x \cdot \log^{-1/6+\varepsilon} x).$$

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