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# NOTE ON ALDER'S POLYNOMIALS

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1. Alder's polynomial $G_{M,t}(x)$ may be defined by means of

$$1 + \sum_{s=1}^{\infty} (-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}} (1 - kx^{2s}) \frac{(kx)_{s-1}}{(x)_s} \tag{1}$$
$$= \prod_{n=1}^{\infty} (1 - kx^n) \sum_{t=0}^{\infty} \frac{k^t G_{M,t}(x)}{(x)_t} ,$$

where $M$ is a fixed integer $\geq 2$ and

$$(a)_t = (1-a)(1-ax) \cdots (1-ax^{t-1}), \ (a)_0 = 1 .$$

Alder [1] obtained the identities

$$\prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-M})(1-x^{(2M+1)n-M-1})(1-x^{(2M+1)n})}{1-x^n} = \sum_{t=1}^{\infty} \frac{G_{M,t}(x)}{(x)_t} , \tag{2}$$

$$\prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-1})(1-x^{(2M+1)n-2M})(1-x^{(2M+1)n})}{1-x^n} = \sum_{t=0}^{\infty} \frac{x^t G_{M,t}(x)}{(x)_t} \tag{3}$$

thus generalizing the well-known Rogers-Ramanujan identities. Singh [2, 3] has further generalized (2), (3); he showed that

$$\prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-s})(1-x^{(2M+1)n-2M-1+s})(1-x^{(2M+1)n})}{1-x^n} = \sum_{t=0}^{\infty} \frac{A_s(x,\ t) G_{m,t}(x)}{(x)_t} ,$$

where the $A_s(x, t)$ are polynomials in $x$.

In a recent paper [4] Singh has proved that

$$G_{M,t}(x) = x^t \qquad (t \leq M-1) . \tag{4}$$

In the present note we give another proof of (4) and indeed obtain the explicit formula

$$G_{M,t}(x) = \sum_{\substack{Ms \leq t \\ s \geq 0}} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-Ms}} x^{\frac{1}{2}s(s-1)+st} (1 - x^s + x^{t-Ms+s}) \tag{5}$$

valid for all $t$.

2. Since

$$(1-kx^{2s})(kx)_{s-1} = (kx)_s + kx^s(1-x^s)(kx)_{s-1} ,$$

the left member of (1) is equal to

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$$\begin{aligned}1 &+ \sum_{s=1}^{\infty}(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}}\left\{\frac{(kx)_s}{(x)_s} + kx^s\frac{(kx)_{s-1}}{(x)_{s-1}}\right\}\\ &= \sum_{s=0}^{\infty}(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s=1\}}\frac{(kx)_s}{(x)_s}\\ &\quad - \sum_{s=0}^{\infty}(-1)^s k^{M(s+1)+1} x^{\frac{1}{2}(s+1)\{(2M+1)(s+1)-1\}+(s+1)}\frac{(kx)_s}{(x)_s}\\ &= \sum_{s=0}^{\infty}(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}}\frac{(kx)_s}{(x)_s}\{1 - k^{M+1}x^{(M+1)(2s+1)}\}\ .\end{aligned}$$

Thus (1) becomes

$$\begin{aligned}\sum_{t=0}^{\infty}\frac{k^t G_{M,t}(x)}{(x)_t} &= \sum_{s=0}^{\infty}(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}}\cdot\frac{1-k^{M1}x^{(M+1)(2s+1)}}{(x)_s}\prod_{j=1}^{\infty}(1-kx^{s+j})^{-1}\\ &= \sum_{s=0}^{\infty}(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}}\cdot\frac{1-k^{M+1}x^{(M+1)(2s+1)}}{(x)_s}\sum_{j=0}^{\infty}\frac{k^j x^{sj+j}}{(x)_j}\ .\end{aligned}\tag{6}$$

For $t < M$, it is clear that the coefficient of $k^t$ on the right is simply $x^t/(x)_t$. This proves Singh's result (4).

For $t = M$ we get

$$\frac{G_{M,M}(x)}{(x)_M} = -\frac{x^M}{1-x} + \frac{x^M}{(x)_M}\ ,$$

so that

$$G_{M,M}(x) = x^M - x^M\frac{(x)_M}{1-x}\ ,$$

which also was found by Singh.

For $t = M + 1$, similarly, we have

$$\frac{G_{M,M+1}(x)}{(x)_{M+1}} = \frac{x^{M+1}}{(x)_{M+1}} - x^{M+1} - \frac{x^{M+2}}{(1-x)^2}\ ,$$

so that

$$\begin{aligned}G_{M,M+1}(x) &= x^{M+1}\left\{1 - (x)_{M+1} - x\frac{(x)_{M+1}}{(1-x)^2}\right\}\\ &= x^{M+1}\{1 - (1+x^3)(x^3)_{M-1}\}\ .\end{aligned}\tag{7}$$

also due to Singh.

3. For arbitrary $t \geq M + 1$, it follows from (6) that

$$\begin{aligned}G_{M,t}(x) &= \sum_{Ms\leq t}(-1)^s\frac{(x)_t}{(x)_s(x)_{t-Ms}}x^{\frac{1}{2}s\{(2M+1)s-1\}(s+1)(t-Ms)}\\ &\quad - \sum_{M(s+1)\leq t}(-1)^s\frac{(x)_t}{(x)_s(x)_{t-M(s+1)-1}}x^{e_s}\ ,\end{aligned}$$

where

$$e_s = \tfrac{1}{2}s\{(2M+1)s-1\} + (s+1)\{t - M(s+1) - 1\}(M+1)(2s+1)\,.$$

This simplifies to

$$(8)\qquad \begin{aligned} G_{M,t}(x) &= x^t \sum_{Ms\leq t} (-1)^s \frac{(x)_t}{(x)_s(x)_{t-Ms}} x^{\frac{1}{2}s(s-1)+s(t-M)} \\ &+ \sum_{0<Ms<t} (-1)^s \frac{(x)_t}{(x)_{s-1}(x)_{t-Ms-1}} x^{\frac{1}{2}s(s-1)+st}\,, \end{aligned}$$

or if we prefer

$$(9)\qquad G_{M,t}(x) = \sum_{\substack{Ms\leq t\\ s\geq 0}} (-1)^s \frac{(x)_t}{(x)_s(x)_{t-Ms}} x^{\frac{1}{2}s(s-1)+st}(1 - x^s + x^{t-Ms+s})\,.$$

For example (9) reduces to

$$(10)\qquad G_{M,t}(x) = x^t\left\{1 - \frac{(x)_t}{(x)_1(x)_{t-M}}(1 - x + x^{t-M+1})\right\}$$

for $M+1 \leq t \leq 2M-1$. When $t = M+1$, it is easily verified that (9) reduces to (7). Singh [4] conjectured the truth of (10) for $t \leq 2(M-1)$.

## Refefences


1. H. L. Alder, *Generalizations of the Rogers-Ramanujan identities*, Pacific J. Math. **4** (1954), 161–168.
2. V. N. Singh, *Certain generalized hypergeometric identities of the Rogers-Ramanjan type*, Pacific J. Math. **7** (1957), 1011–1014.
3. ———, *Certain generalized hypergeometric identities of the Rogers-Ramanujan type* (II), Pacific J. Math. **7** (1957), 1691–1699.
4. ———, *A note on the computation of Alder's polynomials*, Pacific J. Math. **9** (1959), 271–275.


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