NOTE ON ALDER’S POLYNOMIALS

L. CARLITZ
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1. Alder's polynomial $G_{M,t}(x)$ may be defined by means of

\[ 1 + \sum_{s=1}^{\infty} (-1)^s k^s x^{\frac{1}{2} s \binom{2M+1}{s-1}} (1 - k x^{2s}) (kx)^{s-1} (x)^s = \prod_{n=1}^{\infty} \left( 1 - k x^n \right) \sum_{t=0}^{\infty} k^t G_{M,t}(x) (x)_t, \]

where $M$ is a fixed integer $\geq 2$ and

\[(a)_t = (1 - a)(1 - ax) \cdots (1 - ax^{t-1}), \quad (a)_0 = 1.\]

Alder [1] obtained the identities

\[ \prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-M})(1 - x^{(2M+1)n-M-1})(1 - x^{(2M+1)n})}{1 - x^n} = \sum_{t=1}^{\infty} \frac{G_{M,t}(x)}{(x)_t}, \]

\[ \prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-1})(1 - x^{(2M+1)n-2M})(1 - x^{(2M+1)n})}{1 - x^n} = \sum_{t=0}^{\infty} x^t \frac{G_{M,t}(x)}{(x)_t}, \]

thus generalizing the well-known Rogers-Ramanujan identities. Singh [2, 3] has further generalized (2), (3); he showed that

\[ \prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-s})(1 - x^{(2M+1)n-2M-1+s})(1 - x^{(2M+1)n})}{1 - x^n} = \sum_{t=0}^{\infty} A_s(x, t) G_{M,t}(x) (x)_t, \]

where the $A_s(x, t)$ are polynomials in $x$.

In a recent paper [4] Singh has proved that

\[ G_{M,t}(x) = x^t \quad \quad (t \leq M - 1). \]

In the present note we give another proof of (4) and indeed obtain the explicit formula

\[ G_{M,t}(x) = \sum_{s \leq t, s \geq 0} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-M s}} x^{\frac{1}{2} s(s-1) + st}(1 - x^s + x^{t-M s+s}) \]

valid for all $t$.

2. Since

\[(1 - k x^{2s})(kx)^{s-1} = (kx)_s + k x^s (1 - x^s)(kx)_{s-1},\]

the left member of (1) is equal to

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\[
1 + \sum_{s=1}^{\infty} (-1)^s k^{M+s} x^{\frac{1}{2} s \left(\frac{2M+1}{2} s - 1\right)} \left\{ \frac{(kx)^s}{(x)^s} + kx^s \frac{(kx)^{s-1}}{(x)^{s-1}} \right\} \\
= \sum_{s=0}^{\infty} (-1)^s k^{M+s} x^{\frac{1}{2} s \left(\frac{2M+1}{2} s - 1\right)} \left(\frac{kx}{x}\right)^s \\
- \sum_{s=0}^{\infty} (-1)^s k^{M+(s+1)} x^{\frac{1}{2} s \left(\frac{2M+1}{2} (s+1) + (s+1) + (s+1) - 1\right)} \left(\frac{kx}{x}\right)^s \\
= \sum_{s=0}^{\infty} (-1)^s k^{M+s} x^{\frac{1}{2} s \left(\frac{2M+1}{2} s - 1\right)} \left(\frac{kx}{x}\right)^s \{1 - k^{M+(s+1)} x^{(s+1) (2s+1)}\}.
\]

Thus (1) becomes

\[
\sum_{t=0}^{\infty} \frac{k^t G_{M,t}(x)}{(x)_t} = \sum_{s=0}^{\infty} (-1)^s k^{M+s} x^{\frac{1}{2} s \left(\frac{2M+1}{2} s - 1\right)} \cdot \frac{1 - k^{M+1} x^{(M+1) (2s+1)}}{(x)_s} \prod_{j=1}^{\infty} (1 - k x^{s+j})^{-1}
\]

\[
(6)
\]

\[
= \sum_{s=0}^{\infty} (-1)^s k^{M+s} x^{\frac{1}{2} s \left(\frac{2M+1}{2} s - 1\right)} \cdot \frac{1 - k^{M+1} x^{(M+1) (2s+1)}}{(x)_s} \sum_{j=0}^{\infty} \left(\frac{k^j x^{s+j}}{(x)_j}\right).
\]

For \(t < M\), it is clear that the coefficient of \(k^t\) on the right is simply \(x^t/(x)_t\). This proves Singh's result (4).

For \(t = M\) we get

\[
\frac{G_{M,M}(x)}{(x)_M} = - \frac{x^M}{1 - x} + \frac{x^M}{(x)_M},
\]

so that

\[
G_{M,M}(x) = x^M - x^M \frac{(x)_M}{1 - x},
\]

which also was found by Singh.

For \(t = M + 1\), similarly, we have

\[
\frac{G_{M,M+1}(x)}{(x)_{M+1}} = \frac{x^{M+1}}{(x)_{M+1}} - x^{M+1} - \frac{x^{M+2}}{(1 - x)^2},
\]

so that

\[
G_{M,M+1}(x) = x^{M+1} \left\{1 - (x)_{M+1} - x \frac{(x)_{M+1}}{(1 - x)^2}\right\}
\]

\[
= x^{M+1} \left\{1 - (1 + x^3)(x^3)_{M-1}\right\}.
\]

also due to Singh.

3. For arbitrary \(t \geq M + 1\), it follows from (6) that

\[
G_{M,t}(x) = \sum_{M \leq s \leq t} (-1)^s \frac{(x)_s}{(x)_s (x)_{t-M-s}} x^{\frac{1}{2} s \left(\frac{2M+1}{2} (s+1) - 1\right)}
\]

\[
- \sum_{M(s+1) \leq t} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-M(s+1)-1}} x^{s},
\]

...
where
\[ e_s = \frac{1}{2}s \{(2M + 1)s - 1\} + (s + 1)\{t - M(s + 1) - 1\}(M + 1)(2s + 1). \]

This simplifies to
\[
G_{M,t}(x) = x^t \sum_{M s \leq t} \frac{(-1)^s}{(x)_l(x)_{t-M s}} x^{\frac{1}{2}s(s-1) + s(t-M)} + \sum_{0 < M s < t} (-1)^s \frac{(x)_l}{(x)_{l-1}(x)_{t-M s-1}} x^{\frac{1}{2}s(s-1) + st},
\]
or if we prefer
\[
G_{M,t}(x) = \sum_{M s \leq t \geq 0} (-1)^s \frac{(x)_l}{(x)_{t-M s}} x^{\frac{1}{2}s(s-1) + st}(1 - x^s + x^{t-M s + s}).
\]

For example (9) reduces to
\[
G_{M,t}(x) = x^t\left\{1 - \frac{(x)_l}{(x)_{t-M}}(1 - x + x^{t-M+1})\right\}
\]
for \(M + 1 \leq t \leq 2M - 1\). When \(t = M + 1\), it is easily verified that (9) reduces to (7). Singh [4] conjectured the truth of (10) for \(t \leq 2(M - 1)\).

**References**


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