

Pacific Journal of Mathematics

NOTE ON ALDER'S POLYNOMIALS

L. CARLITZ

NOTE ON ALDER'S POLYNOMIALS

L. CARLITZ

1. Alder's polynomial $G_{M,t}(x)$ may be defined by means of

$$(1) \quad 1 + \sum_{s=1}^{\infty} (-1)^s k^M s x^{\frac{1}{2}s(2M+1)s-1} (1 - kx^{2s}) \frac{(kx)_{s-1}}{(x)_s} \\ = \prod_{n=1}^{\infty} (1 - kx^n) \sum_{t=0}^{\infty} \frac{k^t G_{M,t}(x)}{(x)_t},$$

where M is a fixed integer ≥ 2 and

$$(a)_t = (1 - a)(1 - ax) \cdots (1 - ax^{t-1}), \quad (a)_0 = 1.$$

Alder [1] obtained the identities

$$(2) \quad \prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-M})(1 - x^{(2M+1)n-M-1})(1 - x^{(2M+1)n})}{1 - x^n} = \sum_{t=1}^{\infty} \frac{G_{M,t}(x)}{(x)_t},$$

$$(3) \quad \prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-1})(1 - x^{(2M+1)n-2M})(1 - x^{(2M+1)n})}{1 - x^n} = \sum_{t=0}^{\infty} \frac{x^t G_{M,t}(x)}{(x)_t}$$

thus generalizing the well-known Rogers-Ramanujan identities. Singh [2, 3] has further generalized (2), (3); he showed that

$$\prod_{n=1}^{\infty} \frac{(1 - x^{(2M+1)n-s})(1 - x^{(2M+1)n-2M-1+s})(1 - x^{(2M+1)n})}{1 - x^n} = \sum_{t=0}^{\infty} \frac{A_s(x, t) G_{M,t}(x)}{(x)_t},$$

where the $A_s(x, t)$ are polynomials in x .

In a recent paper [4] Singh has proved that

$$(4) \quad G_{M,t}(x) = x^t \quad (t \leq M - 1).$$

In the present note we give another proof of (4) and indeed obtain the explicit formula

$$(5) \quad G_{M,t}(x) = \sum_{\substack{Ms \leq t \\ s \geq 0}} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-Ms}} x^{\frac{1}{2}s(s-1)+st} (1 - x^s + x^{t-Ms+s})$$

valid for all t .

2. Since

$$(1 - kx^{2s})(kx)_{s-1} = (kx)_s + kx^s(1 - x^s)(kx)_{s-1},$$

the left member of (1) is equal to

$$\begin{aligned}
& 1 + \sum_{s=1}^{\infty} (-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}} \left\{ \frac{(kx)_s}{(x)_s} + kx^s \frac{(kx)_{s-1}}{(x)_{s-1}} \right\} \\
&= \sum_{s=0}^{\infty} (-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}} \frac{(kx)_s}{(x)_s} \\
&\quad - \sum_{s=0}^{\infty} (-1)^s k^{M(s+1)+1} x^{\frac{1}{2}(s+1)\{(2M+1)(s+1)-1\}+(s+1)} \frac{(kx)_s}{(x)_s} \\
&= \sum_{s=0}^{\infty} (-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}} \frac{(kx)_s}{(x)_s} \{1 - k^{M+1} x^{(M+1)(2s+1)}\}.
\end{aligned}$$

Thus (1) becomes

$$\begin{aligned}
(6) \quad \sum_{t=0}^{\infty} \frac{k^t G_{M,t}(x)}{(x)_t} &= \sum_{s=0}^{\infty} (-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}} \cdot \frac{1 - k^{M+1} x^{(M+1)(2s+1)}}{(x)_s} \prod_{j=1}^{\infty} (1 - kx^{s+j})^{-1} \\
&= \sum_{s=0}^{\infty} (-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}} \cdot \frac{1 - k^{M+1} x^{(M+1)(2s+1)}}{(x)_s} \sum_{j=0}^{\infty} \frac{k^j x^{s+j}}{(x)_j}.
\end{aligned}$$

For $t < M$, it is clear that the coefficient of k^t on the right is simply $x^t/(x)_t$. This proves Singh's result (4).

For $t = M$ we get

$$\frac{G_{M,M}(x)}{(x)_M} = -\frac{x^M}{1-x} + \frac{x^M}{(x)_M},$$

so that

$$G_{M,M}(x) = x^M - x^M \frac{(x)_M}{1-x},$$

which also was found by Singh.

For $t = M + 1$, similarly, we have

$$\frac{G_{M,M+1}(x)}{(x)_{M+1}} = \frac{x^{M+1}}{(x)_{M+1}} - x^{M+1} - \frac{x^{M+2}}{(1-x)^2},$$

so that

$$\begin{aligned}
(7) \quad G_{M,M+1}(x) &= x^{M+1} \left\{ 1 - (x)_{M+1} - x \frac{(x)_{M+1}}{(1-x)^2} \right\} \\
&= x^{M+1} \{1 - (1+x^3)(x^3)_{M-1}\}.
\end{aligned}$$

also due to Singh.

3. For arbitrary $t \geq M + 1$, it follows from (6) that

$$\begin{aligned}
G_{M,t}(x) &= \sum_{Ms \leq t} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-Ms}} x^{\frac{1}{2}s\{(2M+1)s-1\}+(s+1)(t-Ms)} \\
&\quad - \sum_{M(s+1) \leq t} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-M(s+1)-1}} x^{e_s},
\end{aligned}$$

where

$$e_s = \frac{1}{2}s\{(2M+1)s-1\} + (s+1)\{t-M(s+1)-1\}(M+1)(2s+1).$$

This simplifies to

$$(8) \quad G_{M,t}(x) = x^t \sum_{Ms \leq t} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-Ms}} x^{\frac{1}{2}s(s-1) + s(t-M)} \\ + \sum_{0 < Ms < t} (-1)^s \frac{(x)_t}{(x)_{s-1} (x)_{t-Ms-1}} x^{\frac{1}{2}s(s-1) + st},$$

or if we prefer

$$(9) \quad G_{M,t}(x) = \sum_{\substack{Ms \leq t \\ s \geq 0}} (-1)^s \frac{(x)_t}{(x)_s (x)_{t-Ms}} x^{\frac{1}{2}s(s-1) + st} (1 - x^s + x^{t-Ms+s}).$$

For example (9) reduces to

$$(10) \quad G_{M,t}(x) = x^t \left\{ 1 - \frac{(x)_t}{(x)_1 (x)_{t-M}} (1 - x + x^{t-M+1}) \right\}$$

for $M+1 \leq t \leq 2M-1$. When $t = M+1$, it is easily verified that (9) reduces to (7). Singh [4] conjectured the truth of (10) for $t \leq 2(M-1)$.

REFERENCES

1. H. L. Alder, *Generalizations of the Rogers-Ramanujan identities*, Pacific J. Math. **4** (1954), 161-168.
2. V. N. Singh, *Certain generalized hypergeometric identities of the Rogers-Ramanujan type*, Pacific J. Math. **7** (1957), 1011-1014.
3. ———, *Certain generalized hypergeometric identities of the Rogers-Ramanujan type (II)*, Pacific J. Math. **7** (1957), 1691-1699.
4. ———, *A note on the computation of Alder's polynomials*, Pacific J. Math. **9** (1959), 271-275.

DUKE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG

Stanford University
Stanford, California

F. H. BROWNELL

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

L. J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

E. HEWITT

A. HORN

L. NACHBIN

M. OHTSUKA

H. L. ROYDEN

M. M. SCHIFFER

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Maynard G. Arsove, <i>The Paley-Wiener theorem in metric linear spaces</i>	365
Robert (Yisrael) John Aumann, <i>Acceptable points in games of perfect information</i>	381
A. V. Balakrishnan, <i>Fractional powers of closed operators and the semigroups generated by them</i>	419
Dallas O. Banks, <i>Bounds for the eigenvalues of some vibrating systems</i>	439
Billy Joe Boyer, <i>On the summability of derived Fourier series</i>	475
Robert Breusch, <i>An elementary proof of the prime number theorem with remainder term</i>	487
Edward David Callender, Jr., <i>Hölder continuity of n-dimensional quasi-conformal mappings</i>	499
L. Carlitz, <i>Note on Alder's polynomials</i>	517
P. H. Doyle, III, <i>Unions of cell pairs in E^3</i>	521
James Eells, Jr., <i>A class of smooth bundles over a manifold</i>	525
Shaul Foguel, <i>Computations of the multiplicity function</i>	539
James G. Glimm and Richard Vincent Kadison, <i>Unitary operators in C^*-algebras</i>	547
Hugh Gordon, <i>Measure defined by abstract L_p spaces</i>	557
Robert Clarke James, <i>Separable conjugate spaces</i>	563
William Elliott Jenner, <i>On non-associative algebras associated with bilinear forms</i>	573
Harold H. Johnson, <i>Terminating prolongation procedures</i>	577
John W. Milnor and Edwin Spanier, <i>Two remarks on fiber homotopy type</i>	585
Donald Alan Norton, <i>A note on associativity</i>	591
Ronald John Nunke, <i>On the extensions of a torsion module</i>	597
Joseph J. Rotman, <i>Mixed modules over valuations rings</i>	607
A. Sade, <i>Théorie des systèmes demosiens de groupoi des</i>	625
Wolfgang M. Schmidt, <i>On normal numbers</i>	661
Berthold Schweizer, Abe Sklar and Edward Oakley Thorp, <i>The metrization of statistical metric spaces</i>	673
John P. Shanahan, <i>On uniqueness questions for hyperbolic differential equations</i>	677
A. H. Stone, <i>Sequences of coverings</i>	689
Edward Oakley Thorp, <i>Projections onto the subspace of compact operators</i>	693
L. Bruce Treybig, <i>Concerning certain locally peripherally separable spaces</i>	697
Milo Wesley Weaver, <i>On the commutativity of a correspondence and a permutation</i>	705
David Van Vranken Wend, <i>On the zeros of solutions of some linear complex differential equations</i>	713
Fred Boyer Wright, Jr., <i>Polarity and duality</i>	723