UNIONS OF CELL PAIRS IN $E^3$

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In [4] it is shown that there are pairs of cells of all dimensions possible in euclidean 3-space, $E^3$, which are tame separately, but which have a wild set as their union. Such pairs can be constructed when the individual cells intersect in a single point. The present paper gives conditions that unions of some such pairs be tame sets as well as a number of other results.

**Lemma 1.** Let $D_1$ be a disk which is polyhedral and which lies on the boundary, $\partial T$, of a tetrahedron $T$ in $E^3$. If $D_2$ is a disk in $E^3$ which has a polygonal boundary and is locally polyhedral mod $\partial D_2$ while $D_2 \cap T = D_2 \cap D_1 = \partial D_2 \cap \partial D_1 = J$, an arc, then $D_1 \cup D_2$ is a tame disk.

**Proof.** Let $P_1$ and $P_2$ be polyhedral disks in $\partial T$, $P_1 \cap P_2 = \square$ and $(P_1 \cup P_2) \cap D_1 = \square$. Then $\partial T \setminus (P_1 \cup P_2)$ is a polyhedral annulus, $A_1$. If $Q$ is a polyhedral disk in $D_1 \setminus \partial D_1$, then $\overline{D_1 \setminus Q}$ is an annulus $A_2$ which is locally polyhedral mod $\partial D_2$. By applying Lemma 5.1 of [8] to $A_1$ and $A_2$ one obtains a space homeomorphism $h$ carrying $E^3$ onto $E^3$ while $h(D_1 \cup D_2)$ is a polyhedral set. This completes the proof of Lemma 1.

**Lemma 2.** Let $D_1$ be the disk of Lemma 1 while $D_2$ is a tame disk in $E^3$ such that $D_2 \cap T = D_2 \cap D_1 = \partial D_2 \cap \partial D_1 = J$, an arc. Then $\partial T \cup \partial D_2$ is tame.

**Proof.** By Theorem 2 of [3] $\partial D_1 \cup \partial D_2$ is locally tame and hence tame by [1] or [8]. Let $a$ be a point of $\partial J$ and $J'$ be an interval of $\partial D_1$ having $a$ as an end point and $J' \cap \partial D_2 = a$. We choose a polygonal disk $M$ on $\partial T$ with $(J' \cap J')$ in its interior while $\partial D_1 \cap M = J'$. By a swelling of $M$ toward the component of $E^3 \setminus \partial T$ which meets $\partial D_2$ we obtain a disk $M'$ which is locally polyhedral mod $\partial M$ and $M' \cap \partial T = \partial M = \partial M'$. The sphere $S = M' \cup (\partial T \setminus M)$ is tame by [8] and $S$ is pierced at $a$ by a tame arc lying on $\partial (D_1 \cup D_2)$. Hence by [7] $\partial D_1 \cup S$ is locally tame at $a$. We select an arc $P$ in $(S \setminus M') \cup a$ which is locally polyhedral except at the point $a$. There is an arc $A$ on $\partial D_2$ which lies in the exterior of $S$ except for its end point $a$. The arc $A \cup P$ is tame since $S \cup \partial D_2$ is tame. Let the arc $P$ be swollen into a 3-cell $C^3$ with $P$ in its interior such that $C^3$ is locally polyhedral mod $a$, $C^3 \cap S$ is a disk while $C^3 \cap M = a$. Then $\partial C^3$ is pierced at $a$ by $A \cup P$ and so $A \cup P \cup \partial C^3$ is tame by [7]. Evidently there is an arc $P'$ on $\partial C^3$ so

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Theorem 1. Let \( D_1 \) and \( D_2 \) be two tame disks in \( E^3 \) such that \( D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = \gamma \), an arc. Then \( D_1 \cup D_2 \) is a tame disk.

Proof. Since \( D_1 \) is tame there is a homeomorphism \( h_1 \) of \( E^3 \) onto \( E^3 \) such that \( h_1(D_1) \) is a plane triangle. The disk \( h_1(D_1) \) is to be swollen so that a 3-cell \( e^3 \) is formed such that

1. \( h_1(D_1) \subset \partial e^3 \),
2. \( e^3 \) is tame,
3. \( e^3 \cap h_1(D_1) = h_1(\gamma) \).

That such a cell \( e^3 \) exists follows from Lemma 5.1 of [5] and Theorem 9.3 of [8].

There is a homeomorphism \( h_2 \) of \( E^3 \) onto \( E^3 \) which carries \( \partial e^3 \) and \( h_1(D_1) \) onto the boundary of a tetrahedron and a polyhedral disk, respectively. By Lemma 2 \( h_2(e^3) \cup h_2h_1(\partial D_1) \) is a tame set. By Theorem 2 of [6] we can insist that \( h_2h_1(\partial D_1) \) be locally polyhedral mod \( h_2h_1(\partial D_1) \), while \( h_2h_1(\partial D_1) \) is polygonal. Hence by Lemma 1 \( h_2h_1(D_1 \cup D_2) \) is tame and so \( D_1 \cup D_2 \) is tame.

The following result gives a characterization of tame 1-dimensional complexes in \( E^3 \). By a 1\( _{v} \)-star we mean a homeomorphic image of a 1-dimensional simplicial complex \( K \) with a vertex \( x \) whose star is \( K \) and \( x \) is the common end point of the \( n \) segments meeting only in \( x \).

Theorem 2. If \( N \) is a 1\( _{v} \)-star in \( E^3 \) such that \( (n - 1) \) of the branches of \( N \) lie on a disk \( D \) which meets the remaining branch \( J \) at \( x \) only and if each arc in \( N \) is tame, then \( N \) is tame.

Proof. By [2] we may assume that \( D \) is locally polyhedral mod \( N \). An application of the method in Theorem 1 of [3] makes it possible to select a subset \( D' \) of \( D \) which is a disk consisting of \( (n - 1) \) tame disks which contain arcs with \( x \) as an end point of all branches of \( N \) except \( J \). An argument almost identical with that of Theorem 2 of [3] suffices to show that \( J \cup D' \) is tame and hence \( N \) is tame by [1] or [8].

Corollary 1. Let \( G \) be a graph in \( E^3 \) such that the star of each vertex of \( G \) meets the conditions of Theorem 2, then \( G \) is tame. The conditions are evidently necessary as well.

Corollary 2. Let \( D \) be a tame disk and \( J \) a tame arc in \( E^3 \). If \( D \cap J = \partial D \cap J = \gamma \), an end point of \( J \), and if \( \partial D \cup J \) is tame, then \( D \cup J \) is tame.

Proof. Since \( D \) is tame there is a space homeomorphism \( h \) which
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carries $D$ onto a face of a tetrahedron $T$, $[h(J) \setminus h(p)] \subset E^n \setminus T$. Let $P$ be a segment on $h(\partial D)$ with $h(p)$ as an end point. We enclose $P$ in a polyhedral disk $M$ in $\partial T$ such that $P$ spans $M$ and $h(\partial D) \cap M = P$. We swell $M$ as in Lemma 2 to obtain a tame disk $M'$ such that $\partial M' = \partial M$, and $M' \setminus \partial M' \subset E^n \setminus T$. Then $h(J) \cup h(\partial D)$ contains a tame arc which pierces the tame sphere $[8] S = M' \cup (\partial T \setminus M) \setminus h(p)$ and so $S \cup h(J)$ is tame by [7]. The construction of an arc $P'$ as in Lemma 2 completes the proof.

In Example 1.4 of [4] an arc $A$ which is the union of two tame arcs is shown. Although $A$ has an open 3-cell complement in compactified $E^n$, it is nevertheless wild. A similar example can be obtained from Example 1.4 of two tame disks which meet at a point on the boundary of each and which have a wild union. In this connection we give the following result.

**Theorem 3.** Let $D_1$ and $D_2$ be disks in $E^3$ such that each arc in $D_1$ and $D_2$ is tame and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = J$, an arc. Then $D_1 \cup D_2$ is a disk such that each arc in $D_1 \cup D_2$ is tame.

**Proof.** Let $J'$ be an arc in $D_1 \cup D_2$. If $\partial J'$ does not lie in $\partial D_1 \cup \partial D_2$ we extend $J'$ so that this is the case, obtaining $J'' \supset J'$, $\partial J'' \subset \partial D_1 \cup \partial D_2$ and $J'' \subset D_1 \cup D_2$. By [2] there is a disk $D$ such that $\partial D = \partial (D_1 \cup D_2)$, $J \cup J'' \subset D$ and $D$ is locally polyhedral mod $J \cup J'' \cup \partial D$. The arc $J$ in $D$ is the intersection of two disks in $D$, $D'_1$ and $D'_2$, such that $D'_1 \cup D'_2 = D$. Consider any point $x$ of $J''$ in $D'_1 \setminus \partial D'_1$. In [3] a method is given for enclosing $x$ in the interior of a tame subdisk of $D'_1$. Hence $D'_1$ is locally tame at each of its interior points and $\partial D'_1$ is tame. By [8] $D'_1$ is tame. A similar argument can be applied to $D'_2$. Hence $D'_1 \cup D'_2$ is a tame disk by Theorem 2. Then $J''$ is tame and so $J'$ is tame. Since $J'$ was arbitrarily chosen $D_1 \cup D_2$ is a disk in which each arc is tame.

**Corollary 1.** Let $L_1$ and $L_2$ be tame disks which intersect in a single point on the boundary of each. If $L_1 \cup L_2$ lies on a disk in which each arc is tame, then $L_1 \cup L_2$ is tame.

**Proof.** Let $L_1 \cup L_2$ lie on a disk $D$ such that each arc in $D$ is tame. By Theorem 2 $\partial L_1 \cup \partial L_2$ is tame. There is a disk $D'$ in $D$ with a tame boundary such that $D' \cap (L_1 \cup L_2) \subset \partial L_1 \cup \partial L_2$ while $D' \cup L_1 \cup L_2$ is a disk. Then by [2] there is a disk $D''$ such that $\partial D'' = \partial D'$, $D''$ is locally polyhedral mod $\partial D''$ and $\partial D'' \cap (L_1 \cup L_2) = \partial D' \cap (L_1 \cup L_2)$. Now $D''$ is tame by [8] and so $D'' \cup L_1 \cup L_2$ is tame by Theorem 2. It follows that $L_1 \cup L_2$ is tame.

**References**

7. ———, *Affine structures in 3-manifolds, VII. Disk which are pierced by intervals*, Ann. of Math. *58* (1953), 403-408.
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