A CLASS OF SMOOTH BUNDLES OVER A MANIFOLD

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1. Introduction. In this paper we illustrate certain constructions of importance in the geometry of smooth manifolds. First of all we prove that a homogeneous space $B$ of a connected Lie group $G$ can always be represented as a homogeneous space of a contractible Lie group $E$, necessarily of infinite dimension in general. In particular, that representation shows that the loop space of $B$ can be replaced effectively by a Lie group of infinite dimension. The construction is a special case of a general theory of differentiable structures in function spaces [4].

Secondly, we examine relations between the Lie algebra of $G$ and that of $E$ (this latter being a Banach-Lie algebra), in case $G$ is compact and semi-simple.

As an application we consider certain differentiable fibre bundles over a smooth (i.e., infinitely differentiable) manifold $X$ having infinite dimensional Lie structure groups. Particular attention is given to the bundles associated with maps of $X$ into a sphere; these bundles are important because they are in natural (Poincaré dual) correspondence with certain equivalence classes of normally framed submanifolds of $X$. Using a theory of smooth differential forms in function spaces, we give explicit integral representation formulas for the characteristic classes of these bundles. These formulas provide examples of a residue theory of differential forms with singularities [1]—and express those forms with singularities as forms without singularities in differentiable bundles over $X$.

2. The homogeneous spaces. (A) Let $G$ be a connected Lie group (of finite dimension!), and let $L(G)$ denote its Lie algebra, considered as the tangent space to $G$ at its neutral element $e$. If $K$ is a closed subgroup of $G$, we let $B$ denote the homogeneous space $G/K$ of left cosets of $K$. The coset map $\pi : G \to B$ is an analytic fibre bundle map [9, § 7].

We now construct an acyclic fibre bundle over $B$; our construction is a variant of Serre’s space of paths over $B$ based at a point [8, Ch. IV]. For this purpose we have chosen a special class of paths on $G$ suitable for our applications in §5. (These path spaces are also of importance in the calculus of variations.)

(B) Let $G$ be given a left invariant Riemann structure, determined by an inner product on $L(G)$. If $\mathcal{F}(G)$ denotes the tangent vector bundle of $G$ with projection map $q : \mathcal{F}(G) \to G$, then $\mathcal{F}(G)$ has induced Riemann structure. If $u, v$ are tangent vectors at a point

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If \( I \) is the unit interval \( \{ t \in I : 0 \leq t \leq 1 \} \), we say that a map \( x : I \rightarrow G \) is an admissible path on \( G \) if it satisfies the following conditions:

1. \( x(0) = e \), the neutral element of \( G \);
2. \( x \) is absolutely continuous in the metric of \( G \); then its tangent vector \( x'(t) \) exists for almost all \( t \in I \), and we require that
3. the tangent map \( x' : I \rightarrow \mathcal{F}(G) \) is square integrable; i.e., the Lebesgue integral

\[
\int_0^1 |x'(t)|^2_{x(t)} dt
\]
is finite. We observe that \( x(t) = q \circ x'(t) \) for each \( t \in I \) for which \( x'(t) \) exists.

Let \( E(G) \) denote the totality of admissible paths on \( G \). Using pointwise multiplication and metric defined analogously to (1), it is easily seen that \( E(G) \) is a topological (metrizable) group. As in the case of continuous path spaces [8, p. 481], \( E(G) \) is a contractible group with contraction \( h : E(G) \times I \rightarrow E(G) \) given by \( h(x, t)s = x(ts) \).

Let \( p : E(G) \rightarrow G \) be defined by \( p(x) = x(1) \). Then \( p \) is a continuous epimorphism whose kernel is the subgroup \( \Omega(G) = \{ x \in E(G) : x(1) = e \} \) of admissible loops on \( G \); thus we have an exact sequence

\[
0 \rightarrow \Omega(G) \rightarrow E(G) \xrightarrow{p} G \rightarrow 0
\]
of topological groups. If \( E(G, K) = \{ x \in E(G) : x(1) \in K \} \), then \( E(G, K) \) is a closed subgroup of \( E(G) \), and the composition \( \lambda = \pi \circ p : E(G) \rightarrow G \rightarrow B \) is a representation of \( B \) as a homogeneous space of \( E(G) \), with \( E(G, K) \) as fibre over \( b_0 = \pi(K) \in B \).

**Proposition.** \( \lambda : E(G) \rightarrow B \) is a principal \( E(G, K) \)-bundle.

To prove that it remains (by [9, p. 30]) to show that there is a local section of \( E(G) \) defined in a neighborhood of \( b_0 \); because \( \pi \) is a bundle map it suffices to find a neighborhood \( V \) of \( e \) in \( G \) and a section \( f \) of \( E(G) \) over \( V \). We use the Riemann structure of \( G \) to obtain a neighborhood \( V \) of \( e \) such that for any point \( m \in V \) there is a unique geodesic segment \( x_m : I \rightarrow V \) such that \( x_m(0) = e \) and \( x_m(1) = m \); then \( x_m \) is clearly an admissible path, and \( f(m) = x_m \) is a continuous map of \( V \) into \( E(G) \) such that \( p \circ f(m) = m \) for all \( m \in V \).

(C) The following result is an application of a general theory of function space manifolds [4].

**Theorem.** Let \( G \) be a connected Lie group, and \( E(G) \) the space of
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its admissible paths. Then $E(G)$ is an infinite dimensional Lie group modeled on a separable Hilbert space. The map $p : E(G) \rightarrow G$ is an analytic bundle epimorphism.

We recall the principal ideas of that construction. Given $x \in E = E(G)$, the tangent space to $E$ at $x$ is the separable Hilbert space $E(x)$ of maps $u : I \rightarrow \mathcal{T}(G)$ such that

1. $q \circ u(t) = x(t)$ for all $t \in I$,
2. $u(0) = 0$ (the zero in $L(G)$), and
3. the map $u$ is absolutely continuous with square integrable tangent vector field, and the norm $|u|_x$ induced from the inner product (3) below is finite. Thus $E(x)$ is considered as the space of admissible variations of the path $x$. The algebraic operations in $E(x)$ are defined pointwise; i.e., if $u, v \in E(x)$ and $a, b \in R$, then $(au + bv)t = au(t) + bv(t)$, where the right member is computed in the tangent space $G(x(t))$. A symmetric, bilinear form in $E(x)$ is defined by

$$(u, v)_x = \int_0^1 (u'(t), v'(t))_{x(t)} dt ;$$

this is an inner product, for if $(u, u)_x = 0$, then $|u(t)|_{x(t)} = 0$ for almost all $t \in I$, and the condition that $u$ is admissible then implies $u(t) = 0$ for all $t \in I$. We emphasize that each $E(x)$ is complete (by standard $L^2$ theory), a property that is used in the theory of differentiation in infinite dimensional linear spaces.

Using the natural correspondence (defined locally) between geodesic segments on $G$ emanating from a point $m$ and tangent vectors in $G(m)$, we can find a neighborhood $U_x$ (called a coordinate patch) of $x$ in $E(G)$ which is mapped homeomorphically (by a map $\phi_x$ called a coordinate system) onto a neighborhood of 0 in $E(x)$ [4, §3]. In overlapping coordinate patches $U_x$, $U_y$ we have a map

$\phi_{xy} : \phi_x(U_x \cap U_y) \longrightarrow \phi_y(U_x \cap U_y)$

defined by $\phi_{xy}(u) = \phi_y \circ \phi^{-1}_x(u)$, and this map is analytic in its domain of definition. (If $\phi$ is a map of an open subset $U$ of a Hilbert space $E$ into a Hilbert space $F$, then $\phi$ is analytic in $U$ if every $x \in U$ has a neighborhood in which $\phi$ can be expressed by the convergent power series

$\phi(x + v) = \phi(x) + \sum_{k=1}^{\infty} P^k_x(x, v)/k! ,$

where $P^k_x(x, v)$ denotes the $k$th iterated directional derivative of $\phi$ at $x$ in the direction $v$.) Easy modifications of standard Lie group theory show that the group operation in $E(G)$ is analytic and that $p : E(G) \rightarrow G$ is an analytic homomorphism.
**Corollary.** The fibration $\lambda : E(G) \to B$ is an analytic bundle map.

**(D) Remark.** The inner product (3) is easily seen to provide an analytic Riemann structure on $E(G)$. We note, however, that it is not left invariant on $E(G)$.

Suppose we let $G$ act on $E(G)$ by $T_g(x)t = gx(t)g^{-1}$ for all $t \in I$ and $x \in E(G)$. If $G$ is compact and semi-simple and if the inner product (3) is computed using the bi-invariant Riemann metric on $G$ (see our § 3A), then the Riemann structure on $E(G)$ is $G$-invariant.

3. The Lie algebra of certain path groups. (A) Suppose that $G$ is connected, compact, and semi-simple. Then its Killing form [7, §§ 6, 11] defines a bi-invariant Riemann structure on $G$ (essentially unique); furthermore, the inner product and the bracket in $L(G)$ are related by

\[
([x, y], z) = (x, [y, z])
\]

for all $x, y \in L(G)$. By taking a suitable real multiple of the Killing form we can suppose that the norm induced from the inner product and the bracket in $L(G)$ are related by

\[
|x, y| \leq |x| |y|
\]

for all $x, y \in L(G)$.

(B) If $e$ also denotes the neutral element of $E(G)$ (so that $e(t) = e$ for all $t \in I$), then the tangent space $E(e)$ consists of those admissible paths on $L(G)$ starting at 0; we introduce the bracket of $u$ and $v$ in $E(e)$ by

\[
[u, v]_t = [u(t), v(t)]
\]

for all $t \in I$.

We will call $E(e)$ the Lie algebra of $E(G)$, and henceforth will denote it by $L(E(G))$; note that $L(E(G) = E(L(G))$. Of course the exponential map $\exp : L(E(G)) \to E(G)$ is defined by $(\exp u)_t = \exp (u(t))$ for all $t \in I$.

If $|u|_e = (u, u)_e$ in the notation of § 2 (3), then the following result shows that the bracket (3) on $L(E(G))$ is continuous.

**Lemma.** For any $u, v \in L(E(G)$ we have

\[
| [u, v]_e | \leq 2 |u|_e |v|_e .
\]

**Proof.** First of all, we note that if $m_u = \max \{|u(t)| : t \in I\}$, then $m_u \leq |u|_e$. Namely, for any $t \in I$ we apply the Schwarz inequality to obtain

\[
2u(t) - u(1) = \int_0^1 \text{sgn } (t-s) u'(s) ds \leq \int_0^1 |\text{sgn } (t-s)| ds \int_0^1 |u'(s)|^2 ds .
\]
Thus
\[ m_u \leq \max \{|2u(t) - u(1)| : t \in I\} \leq |u|_e. \]

By (2) and the Schwarz inequality in \( L(G) \) we find that \(|[u, v]|_2^2\) is bounded by
\[
\int_0^1 \left\{ |u'(t)|^2 |v(t)|^2 + 2 |u'(t)||v(t)||u(t)||v'(t)| + |u(t)|^2 |v'(t)|^2 \right\} dt
\leq m_u^2 \int_0^1 |u'(t)|^2 dt + 2m_u m_v \int_0^1 |u'(t)||v(t)||v'(t)| dt + m_v^2 \int_0^1 |v'(t)|^2 dt
\leq 4 |u|_e^2 |v|_e^2.
\]
The inequality (4) follows.

**Remark.** Unlike the finite dimensional Hilbert-Lie algebra \( L(G) \), \( L(E(G)) \) does not satisfy a relation of the form (1). Thus the bracket in \( L(E(G)) \) respects its Banach space structure—i.e., \( L(E(G)) \) is a Banach-Lie algebra—rather than its structure as a Hilbert space.

(C) Let \( p_\ast : L(E(G)) \to L(G) \) be defined by \( p_\ast(u) = u(1) \); clearly \( p_\ast \) is a Lie algebra epimorphism, and the inequality
\[ |u(t_2) - u(t_1)| \leq |t_1 - t_2| |u|_e \]
for any \( t_1, t_2 \in I \) shows that \( |p_\ast(u)| \leq |u|_e \) for all \( u \in L(E(G)) \).

Our next result establishes an infinitesimal analogue of the analytic bundle over \( G \) given by Theorem 2C.

**Theorem.** If \( G \) is a connected, compact, semi-simple Lie group, then \( p_\ast \) is a continuous Lie epimorphism with kernel \( L(\Omega(G)) = \Omega(L(G)) \), the closed ideal of admissible loops on \( L(G) \); i.e.,
\[
0 \longrightarrow L(\Omega(G)) \longrightarrow L(E(G)) \overset{p_\ast}{\longrightarrow} L(G) \longrightarrow 0
\]
is an exact sequence of Banach-Lie algebras. Furthermore, as Hilbert spaces (but not as Lie algebras), \( p_\ast \) induces an orthogonal direct decomposition \( L(E(G)) \approx L(\Omega(G)) \oplus M \), where \( M \) is a vector space isomorphic to \( L(G) \).

**Proof.** The first statement follows from the algebraic properties of \( p_\ast \) and the fact that \( p_\ast \) is bounded, and therefore continuous. To prove the second, we define a map \( j : L(G) \to L(E(G)) \) by letting \( j(x) \) be the linear path \( j(x)t = tx \) for each \( x \in L(G) \); then \( j \) is a linear map of \( L(G) \) onto a subspace \( M \) of \( L(E(G)) \), and \( p_\ast \circ j \) is the identity; moreover, \( i \) is an isometry, because for any \( x, y \in L(G) \),
\[
(i(x), i(y)) = \int_0^1 (x, y) dt = (x, y).
\]
Note, however, that \( M \) is not a subalgebra of \( L(E(G)) \).

The subspaces \( L(\Omega(G)) \) and \( M \) are orthogonal complements in \( L(E(G)) \), for if \( x \in L(G) \) and \( v \in L(\Omega(G)) \), then

\[
(f(x), v) = \int_0^1 (x, v'(t)) \, dt = (x, v(1)) - (x, v(0)) = 0.
\]

**Corollary.** The group \( \Omega(G) \) of admissible loops on \( G \) forms a subgroup of \( E(G) \) whose codimension (as a submanifold of \( E(G) \)) equals the dimension of \( G \).

**Remark.** If \( K \) is a closed subgroup of \( G \) and if we set \( \lambda_* = \pi_* \circ p_* : L(E(G)) \to L(G) \to L(G)/L(K) \), then we have an exact sequence of vector spaces

\[
0 \to L(E(G, K)) \to L(E(G)) \xrightarrow{L_*} L(G)/L(K) \to 0.
\]

(D) **Problem.** Consider \( L(E(G)) \) as a Hilbert space, and form its topological exterior algebra \( C^*(L(E(G))) \), using the natural inner product on its \( p \)th exterior power. The inequality (4) implies that we can construct the Lie algebra cochain complex as in [7, §3] and that the differential operator in \( C^*(L(E(G))) \) is continuous. The elements \( \omega \in C^p(L(E(G))) \) determine left invariant differential \( p \)-forms on \( E(G) \)—an important property because a version of de Rham's Theorem is valid for \( E(G) \) (see §5A). What are the relations between the derived cohomology algebras \( H^*(L(E(G))), H^*(L(\Omega(G))), \) and \( H^*(L(G)) \approx H^*(G; \mathbb{R}) \)?

As a first step, because \( L(\Omega(G)) \) is a closed ideal in \( L(E(G)) \) we can appeal to our Theorem 3C and Theorem 4 of *Cohomology of Lie algebras*, G. Hochschild and J-P. Serre, Annals of Math. 57 (1953), 591–603, to obtain the

**Proposition.** The filtration of \( C^*(L(E(G))) \) by the ideal \( L(\Omega(G)) \) determines a spectral sequence such that

\[
E^{p,q}_r = H^p(L(G); H^q(L(\Omega(G)))
\]

and whose terminal algebra \( E_\infty \) is the graded algebra associated with \( H^*(L(E(G))) \), suitably filtered.

4. The bundles over a manifold. (A) Let \( B = G/K \) be the homogeneous space of §2A. Since \( E(G) \) is contractible, the fibre bundle \( \lambda : E(G) \to B \) can be interpreted as a universal bundle [9, §19] for the infinite dimensional Lie group \( E(G, K) \). In particular, by the Classification Theorem for principal bundles we have the

**Proposition.** If \( X \) is a paracompact smooth manifold of finite
dimension, then the isomorphism classes of smooth principal $E(G, K)$-bundles over $X$ are in natural one-to-one correspondence with the smooth homotopy classes of maps of $X$ into $B$.

In that statement we have made use of the fact that for maps of $X$ into $B$ their classification by homotopy equivalence coincides with classification by smooth homotopy equivalence.

**Remark.** There is a certain uniqueness theorem for universal bundles over $B$, which implies that for any other contractible bundle over $B$ with group $\Gamma$, the homotopy groups of $\Gamma$ are isomorphic to those of $E(G, K)$; see [6, p. 284]. Of course, it follows directly from the homotopy sequence of a bundle and the 5-lemma that the homotopy groups of $E(G, K)$ are isomorphic to those of the loop space of $B$.

(B) Suppose that $B$ is $(n - 1)$-connected and that the $n$th homotopy group $\pi_n(B)$ is infinite cyclic ($n > 1$); then the group $E(G, K)$ is $(n - 2)$-connected, and the connecting homomorphism of the homotopy sequence of the universal bundle of $B$ is an isomorphism of $\pi_n(B)$ onto $\pi_{n-1}(E(G, K))$.

Let $\mu: W \to X$ be an $E(G, K)$-bundle over $X$. Its characteristic class [9, p. 178] is the primary obstruction to the construction of a section of the bundle. The condition $n > 1$ insures that its structural group is 0-connected, whence the bundle $\mathcal{B}$ of local coefficients (used in defining characteristic classes in general) is simple [9, p. 153]. To orient the bundle is to choose one of the two isomorphism of $\mathcal{B}$ onto the product bundle $X \times \mathbb{Z}$. Thus the characteristic class of an oriented $E(G, K)$-bundle over $X$ is a cohomology class $w \in H^n(X, \mathbb{Z})$.

It is well known that such a characteristic class can be represented by a transgressive pair of cochains $(a^n, c^{n-1})$. (A transgressive pair in a bundle consists of a cochain of some sort $c$ on $W$ whose restriction to a fibre is a cocycle of $E(G, K)$, and such that its coboundary $dc = \mu^*a$ for some cocycle $a$ of $X$.) Furthermore, the restriction of $c^{n-1}$ to a fibre defines the generator of $H^{n-1}(E(G, K); \mathbb{Z}) \approx \mathbb{Z}$ which is the negative of that determined by the orientation of the bundle.

Let $w_o$ be the characteristic class of the universal oriented bundle $\lambda: E(G) \to B$. Suppose that $\mu: W \to X$ is induced by the smooth map $f: X \to B$, and let $g: W \to E(G)$ be a smooth bundle map covering $f$ [9, § 19]. If $(a_o, c_o)$ is a transgressive pair representing $w_o$, then $a = f^*a_o$, $c = g^*c_o$ is known to be a transgressive pair representing the characteristic class $w$ of $\mu: W \to X$ [2, § 18].

5. **Representations of the characteristic classes.** (A) Let $Y$ be any paracompact smooth manifold modeled on a Hilbert space $E$. A differential $r$-form $\gamma$ on $Y$ assigns to each point $y \in Y$ an alternating $r$-linear functional (with real values) on the tangent space $Y(y)$, which is continuous simultaneously in the $r$ variables, using the Hilbert space
topology in $Y(y)$. In terms of the differentiable structure on $Y$ we can
define the exterior algebra $\bigwedge^*(Y)$ of smooth differential forms on $Y$
and its derived cohomology algebra $H^*(\bigwedge^*(Y))$. It is known (an extension
of de Rham’s Theorem [4, § 4]) that there is a canonical isomorphism
of $H^*(\bigwedge^*(Y))$ onto $H^*(Y; R)$, the singular real cohomology algebra of $Y$.

We remark that this result uses the local Hilbert space structure
of $Y$ in two ways:

1. the square of the norm in $E$ is an analytic function on $E$,
which implies that there are sufficiently many smooth functions on $Y$;

2. there is a natural Hilbert space structure on the $r$th exterior
power of $E$; its completeness is used essentially in the differentiability
of differential forms.

We will now give examples of such forms which are transgressive
pairs on $E(G, K)$-bundles over $X$.

(B) We have seen in Theorem 2C that the group $E(G)$ of admis-
sible paths on a connected Lie group $G$ is itself a Lie group modeled
on a Hilbert space. Since $E(G)$ is contractible, the general existence
theorem quoted in (A) insures that any smooth closed $r$-form $\omega$ on
$E(G)$ is the exterior differential of a smooth $(r - 1)$-form $\xi$ (for $r > 0$). The
following result uses a standard homotopy construction to give an ex-
licit formula for $\xi$ in case $\omega$ is the $p^*$-image of a form on $G$.

**Proposition.** Given any smooth closed $r$-form $\omega$ on $G$ ($r > 0$),
consider the $(r - 1)$-form on $E(G)$ defined as follows: For any $x \in E(G)$
and $r - 1$ vectors $u_1, \ldots, u_{r-1}$ in the tangent space at $x$, set

\[ \xi(x) \cdot u_1 \vee \cdots \vee u_{r-1} = \int_0^1 \{\omega(x(t)) \cdot x'(t) \vee u_1(t) \vee \cdots \vee u_{r-1}(t)\} \, dt, \]

where $x'(t)$ denotes the tangent vector to $x$ at $x(t)$, and the bracket in
the right member (involving the exterior product $\vee$) is computed in the
tangent space $G(x(t))$. Then $\xi$ is a smooth $(r - 1)$-form on $E(G)$ and
$\partial \xi = p^* \omega$.

**Proof.** The contraction $h : I \times E(G) \to E(G)$ given by $h(t, x)s = x(ts)$
is simultaneously continuous in the arguments $(t, x)$, and is a smooth
function of $x$ for each $t \in I$. Furthermore, for each $x \in E(G)$ the dif-
ferential $h_*(t, x)$ is a square integrable function of $t$; in particular, if $e_i$
denotes the unit vector of $I$, then $(h_*(t, x) \cdot e_i)s = sx'(ts)$ for almost all
$x \in I$.

Because the homomorphism $p$ is analytic, the induced form $\omega^* = p^* \omega$
is a smooth closed $r$-form on $E(G)$ for which

\[ \xi(x) = (k \omega^*)x = \int_0^1 h^* \omega^*(t, x) \wedge e_i dt \]
exists (as a Lebesgue integral, where the integrand in the right member involves the interior product with $e_i$). The explicit formula (3) for $\xi(x)$ below shows that $\xi(x)$ is actually an $(r - 1)$-covector and that $\xi$ is smooth. Standard reasoning about homotopy operators for differential forms leads to the identity $\omega^* = d\omega^* + k\omega^*$, and because $d\omega = 0$, we have $d\xi^* = \omega^*$. Consider the composite map $q = p \circ h : I \times E(G) \to B$. It is easily checked that $q_*(t, x)e_i = x'(t)$ for almost all $t \in I$, and for any $u$ in the tangent space at $x$ (interpreted as the vector $0 \oplus u$ in the tangent space of $I \times E(G)$ at $(t, x)$) we have $q_*(t, x)u = u(t)$. If we take vectors $u_1, \cdots, u_{r-1}$ as in the hypotheses,

$$\xi(x) \cdot u_1 \vee \cdots \vee u_{r-1} = \int_0^1 h^* \circ p^* \omega(t, x) \cdot e_1 \vee u_1 \vee \cdots \vee u_{r-1} \, dt$$

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The explicit formula (3) for $\xi(x)$ below shows that $\xi(x)$ is actually an $(r - 1)$-covector and that $\xi$ is smooth. Standard reasoning about homotopy operators for differential forms leads to the identity $\omega^* = d\omega^* + k\omega^*$, and because $d\omega = 0$, we have $d\xi^* = \omega^*$.

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Suppose that bundle is induced by a smooth map \( f \) of \( X \) into \( B \), and let \( g \) be a smooth bundle map covering \( f \):

\[
\begin{array}{ccc}
W & \longrightarrow & E(G) \\
\mu & \downarrow & \lambda \\
X & \longrightarrow & B
\end{array}
\]

If \((\omega_0, \xi_0)\) is a transgressive pair of forms representing the characteristic class \( w_0 \) of \( \lambda : E(G) \to B \) as in (B), then \( \omega = f^*\omega_0, \xi = g^*\xi_0 \) is a transgressive pair representing \( w \) (§ 4B).

**DEFINITION.** An admissible partial section of the bundle \( \mu : W \to X \) is a smooth section \( \phi \) defined over \( X - e(\phi) \), where \( e(\phi) \) is a smooth polyhedral subset of \( X \) with \( \dim e(\phi) \leq \dim X - n \). Admissible partial sections exist because \( E(G, K) \) is \((n - 2)\)-connected. (For example, we can take a smooth locally finite simplicial subdivision \( L \) of \( X \) and let \( L^* \) be a dual subdivision; then standard obstruction theory provides a smooth section over a neighborhood of the \((n - 1)\)-skeleton \( L^{(n-1)} \) of \( L \) which can be smoothly extended over \( X - L^{(m-n)}_* \), where \( m = \dim X \).

The following result is an example of the general representation theorem of [1, § 4]; note that the present pair \((\omega, \phi^*\xi)\) satisfies the conditions of Corollary 5B of [1]. We will use freely the concepts and results of that paper. As usual in constructing integral formulas for characteristic classes, our method of proof follows that of the Gauss-Bonnet Theorem as given by Chern [3, § 2]: We first obtain a transgressive pair of forms representing the class; we then appeal to Stokes’ Formula to localize and interpret the residue (i.e., the right member of (4) below.

**THEOREM.** In the above notation, the characteristic class \( w \) of the oriented bundle \( \mu : W \to X \) is represented by

\[
(4) \quad w \cdot c = \int_c \omega - \int_{e(\phi)} \phi^*\xi
\]

for any admissible partial section \( \phi \), where \( c \) is any smooth integral \( n \)-chain on \( X \) whose boundary does not intersect \( e(\phi) \).

**Proof.** First of all, \((\omega, \phi^*\xi)\) is an \((R, n)\)-pair on \( X \) because \( \phi \) is admissible, and in \( X - e(\phi) \) we have \( d(\phi^*\xi) = \phi^*d\xi = (\mu \circ \phi)^*\omega = \omega \). Secondary, to verify (4) it suffices to do so for the \( n \)-simplexes of a simplicial subdivision \( L \) of \( X \) (by Corollary 5A of [1]), provided that \( e(\phi) \) lies on the \((m - n)\)-skeleton of the dual \( L_* \). Furthermore, in considering its obstruction cocycle we will suppose that \( \phi \) is defined only on \( L^{(n-1)} \), and then make below a (piecewise smooth) extension to \( L^{(n)} - e, \)
where $e$ is a discrete set of points; such an alteration will not change the obstruction class.

Let $b_\sigma$ be the barycenter of the oriented $n$-simplex $\sigma$, and let $\sigma_t$ be that simplex radially contracted toward $b_\sigma$ by the ratio $1:(1-t)$, using an admissible coordinate system on $X$ containing $\sigma$. Let $h$ be a smooth covering homotopy of that contraction. For any $t < 1$ and $x$ in $\partial\sigma_t$ let $r(x)$ be the radial projection $x$ on $\partial\sigma$; setting $\phi(x) = h(t, \phi(r(x)))$ defines an extension of $\phi$ over $\sigma - b_\sigma$.

Applying Stokes' Formula to the chain $\tau_t = \sigma - \sigma_t$, we obtain

$$-\int_{\partial\sigma_t} \phi^* \xi = \int_{\tau_t} \omega - \int_{\partial\sigma} \phi^* \xi.$$  \hspace{1cm} (5)

As $t \to 1$ the right member approaches the right member of (4) with $c = \sigma$, because $\omega$ is defined on all $\sigma$. To complete the proof of the theorem we will show that as $t \to 1$ the left member determines the obstruction cocycle.

Since $-\xi$ defines the generator of $\mu^{-1}(b_\sigma)$ by § 4B, we see that (writing $w$ for the obstruction cocycle)

$$w \cdot \sigma = -\int_{\partial\sigma} \phi^* \xi.$$  

On the other hand, the homotopy $h$ satisfies a Lipschitz condition locally on $\mu^{-1}(\sigma)$ (relative to any metric on $W$), whence there is a number $M$ independent of $t$ such that $t < 1$ implies

$$\left| \int_{\phi(\partial\sigma)} \xi - \int_{\phi(\partial\sigma_t)} \xi \right| \leq M|1 - t|.$$  

Using the transformation of integral formula, we find that

$$\left| w \cdot \sigma + \int_{\partial\sigma_t} \phi^* \xi \right| = \left| \int_{\partial\sigma} \phi^* \xi - \int_{\partial\sigma_t} \phi^* \xi \right| \leq M|1 - t|.$$  

This shows that as $t \to 1$ the left member of (5) approaches $w \cdot \sigma$, and formula (4) follows.

6. Spherical maps of a manifold. (A) As an example of the preceding constructions let $G = SO(n + 1)$, the rotation group in its usual matrix representation in numerical space $R_{n+1}$. Let $K = SO(n)$, considered as the subgroup of $G$ which acts trivially on the $(n + 1)$th axis of $R_{n+1}$. The unit sphere $S_n$ in $R_{n+1}$ is then naturally identified with the homogeneous space $G/K$, and the coset map $\pi : SO(n + 1) \to S_n$ represents $SO(n + 1)$ as the principal $SO(n)$-bundle of orthonormal $n$-frames on $S_n$ [9, § 7]. We will suppose that $S_n$ has its usual Riemann structure and is oriented by the coordinate axes in $R_{n+1}$. Henceforth we denote the infinite dimensional Lie group $E(SO(n + 1), SO(n))$ by $\Lambda_n$. 

Let $\omega_{ij}$ (1 < $i < j < n + 1$) be a base of Maurer-Cartan forms for the conjugate space of $L(SO(n + 1))$; if we let $k(n)$ denote the reciprocal of the volume of $S^n$, then the exterior polynomial (the Kronecker Index form) on $SO(n + 1)$ given by

$$\omega^*_0 = k(n)\omega_{1,n+1} \vee \cdots \vee \omega_{n,n+1}$$

is known to be $S^n$-basic (i.e., there is a unique $SO(n + 1)$-invariant $n$-form $\omega_0$ on $S^n$ such that $\pi^*\omega_0 = \omega^*_0$), and thereby represents the harmonic generator of $H^n(S^n; \mathbb{Z})$.

Suppose $n$ is even; then a crucial step in the derivation of the Gauss-Bonnet Theorem [3] for $S^n$ establishes that $\omega_0$ is part of a transgressive pair in the principal frame bundle of $S^n$. If $n$ is odd, then $\omega_0$ does not generally have that property. However, for all $n > 1$ Proposition 5B gives an explicit transgressive pair in the oriented universal bundle of $S^n$, determined entirely by the Kronecker Index form.

(B) If $X$ is a compact, oriented, smooth Riemann manifold of dimension $n + m$, then the isomorphism classes of smooth principal $\Lambda_n$-bundles over $X$ play an important role in its geometry, primarily because of the following construction: Let $V$ be a closed, oriented, $m$-dimensional regularly imbedded submanifold of $X$; suppose that $V$ admits a smooth normal $n$-frame in $X$, and let $\phi$ be such a frame field; we will call the pair $(V, \phi)$ a normally framed submanifold of $X$. These have been studied by Kervaire [5, § 1] and Thom [10, Ch. II, 4]. It is known that certain equivalence classes of normally framed $m$-submanifolds of $X$ are in natural one-to-one correspondence with the homotopy classes of maps of $X$ into $S^n$ [5, § 1]. Combining with the Classification Theorem for $\Lambda_n$-bundles, we have the

**Proposition.** If $X$ is a compact, oriented, smooth Riemann $(n + m)$-manifold, then there is a natural one-to-one correspondence between equivalence classes of normally framed $m$-submanifolds of $X$ and isomorphism classes of smooth $\Lambda_n$-bundles over $X$.

Let $(V, \phi)$ be a normally framed $m$-submanifold, and let $i : V \rightarrow X$ be the inclusion map; then since $V$ is closed and oriented (the orientation on $X$ and the frame field $\phi$ determine an orientation of $V$) we have a distinguished generator $v_0 \in H_m(V, \mathbb{Z})$, which determines a definite homology class $i_*(v_0) = v \in H_m(X, \mathbb{Z})$; Furthermore, $v$ depends only on the equivalence class of $(V, \phi)$. On the other hand, applying a theorem of Thom [10, Théorème II.2], we obtain the

**Proposition.** In the correspondence of the above proposition, the homology class of a normally framed submanifold is the Poincaré dual of the characteristic class of the oriented $\Lambda_n$-bundle associated with it.
Let $X$ be a smooth manifold of finite dimension. In the study of differential forms with singularities [1] it is important (e.g., in working with exterior products of such forms) to know when a closed $(Z, r)$-pair is cohomologous to a pair defined in terms of a transgressive pair (as in Theorem 5C). For example, it is well known that the isomorphism classes of $SO(2)$-bundles over $X$ are (by their characteristic classes) in natural one-to-one correspondence with the elements of $H^2(X; Z)$. An easy construction shows that every 2-dimensional integral cohomology class of $X$ can be represented by a transgressive pair in a canonically defined $SO(2)$-bundle over $X$.

A cohomology class $u \in H^n(X; Z)$ is said to be spherical if there is a map $f: X \rightarrow S_n$ such that $u = f^*(s)$ for some $s \in H^n(S_n; Z)$. The representation theorem [1, § 4] of cohomology classes by forms with singularities together with our Theorem 5C gives a transgressive integral representation formula for every spherical class of $X$ in a $\Lambda_n$-bundle. That bundle is uniquely defined by the homotopy class of $f: X \rightarrow S_n$, but is not generally determined by $u$.

**Example.** Suppose that $X$ has dimension $n$. The Hopf Classification Theorem then implies that the isomorphism classes of smooth $\Lambda_n$-bundles over $X$ are in natural one-to-one correspondence with the elements of $H^n(X; Z)$, the correspondence assigning to each isomorphism class its characteristic class. Theorem 5C gives a transgressive integral representation formula for each element $v$ of $H^n(X; Z)$ in a bundle canonically associated with $v$. Of course that fact is significant only for compact manifolds, because $H^n(X; Z) = 0$ if $X$ is open. On the other hand, it is particularly useful for non-orientable compact manifolds, because then $H^n(X; Z)$ has torsion, in which case the singularity of a $(Z, n)$-pair representing $v$ plays an essential role.

If $X$ is orientable and if its Euler characteristic $\chi(X) \neq 0$, then the Gauss-Bonnet Theorem provides a transgressive integral formula for the elements of $H^n(X; Z)$ in a finite dimensional bundle over $X$. In general (and for lower dimensional spherical classes) it appears necessary to use infinite dimensional smooth bundles to obtain such a formula.

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