COMPUTATIONS OF THE MULTIPLICITY FUNCTION

SHAUL FOGUEL
1. Introduction. Let $H$ be a separable Hilbert space. The following two problems will be studied:

1. Given a bounded normal operator $A$, of multiplicity $m$, what are the conditions, on the bounded measurable function $f$, so that the multiplicity of $S = f(A)$ is $n$, $n < \infty$?

2. How to compute the multiplicity of a normal operator that commutes with a given normal operator, of finite multiplicity?

NOTATION. Let $S$ be a normal operator of multiplicity $n$, $n < \infty$. There exist a Borel measure $\mu$ and $n$ Borel sets in the complex plane $e_1 \supset e_2 \supset \cdots \supset e_n$, such that, up to unitary equivalence,

$$H = \sum_{i=1}^{n} L_2(\mu, e_i)$$

This is the Multiplicity Theorem. (See Theorem X. 5.10) of [1]. The operator $S$ has uniform multiplicity if $e_1 = e_2 = \cdots = e_n$.

The resolution of the identity, of a normal operator $A$, will be denoted by $E(A; \alpha)$. The Boolean algebra of projections, generated by $E(A; \alpha)$ will be denoted by $\mathcal{E}_A$. Let $E(\alpha)$ stand for $E(S; \alpha)$ and $\mathcal{E}$ for $\mathcal{E}_S$. Throughout this note all operators are assumed to be bounded.

We shall use the following results from [2]:

Let $S$ be a normal operator of multiplicity $n$, and $B$ a normal operator that commutes with $S$. Let $H$ and $S$ be represented by 1.1.

**Theorem A.** There exist $k$ Borel measurable bounded complex functions $y_1(\lambda)$, $\cdots$, $y_k(\lambda)$ and $k$ matrices of Borel measurable bounded complex functions $\varepsilon_1(\lambda)$, $\cdots$, $\varepsilon_k(\lambda)$ such that:

For a fixed $\lambda$ the matrices $\varepsilon(\lambda)$ are disjoint self adjoint projections whose sum is the identity and

$$B \begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = \left( \sum_{i=1}^{k} y_i \varepsilon_i(\lambda) \right) \begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix}.$$
Equivalently, if the self-adjoint projections $E_i$, are defined by

$$E_i (\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix}) = \varepsilon_i(\lambda) (\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix})$$

then

$$\begin{cases} B = \sum_{i=1}^{k} y_i(S) E_i \\ E(B; \alpha) = \sum_{i=1}^{k} E(y_i^{-1}(\alpha)) E_i . \end{cases} \tag{1.3}$$

REMARK. In the above decomposition the numbers $y_i(\lambda)$ for a fixed $\lambda$ are different eigenvalues of a certain matrix. Thus for each $\lambda$ there is an integer $k' \leq k$ such that

$$y_i(\lambda) \neq y_j(\lambda) \quad i \neq j \quad i, j \leq k', \quad \varepsilon_i(\lambda) \neq 0 \quad i \leq k',$$

and

$$y_{k'+1}(\lambda) = \cdots = y_k(\lambda) = 0 ,$$

$$\varepsilon_{k+1}(\lambda) = \cdots = \varepsilon_{k+1}(\lambda) = 0 .$$

This is essential for the proof of Lemma 2.1. Also the matrices $\varepsilon_i(\lambda)$ are $n \times n$ matrices.

**Theorem B.** The number $n$ is the largest integer such that there exists a nilpotent operator, commuting with $S$, of order $n$. See [2] Theorem 3.1 and its corollary.

2. The multiplicity of a function of an operator. The main result in this section is:

**Theorem 2.1.** Let $A$ be a normal operator of multiplicity $m$, $m < \infty$, and $f$ a bounded measurable function. The operator $S = f(A)$ has finite multiplicity, if and only if, there exist $k$ disjoint Borel sets $\beta_1, \cdots, \beta_k$ and $k$ bounded measurable functions $z_i(\lambda), \cdots, z_k(\lambda)$ such that:

a. $\sigma(A) = \bigcup_{i=1}^{k} \beta_i$.

b. if $\lambda \in \beta_i$, then $z_i(f(\lambda)) = \lambda$ almost everywhere, with respect to $E(A; \alpha)$.

**Proof of sufficiency of conditions a and b.** Let $S_i$ and $A_i$ be the restrictions of $S$ and $A$ to $E(A; \beta_i)H$. Then
\[ S_i = \int_{\beta_i} f(\lambda)E(A; d\lambda) \]

hence

\[ z_i(S_i) = A_i. \]

Now, it follows from Theorem B that

\[ \mu(A_i) \geq \mu(S_i) \quad (\mu(T) = \text{multiplicity of } T) \]

But the multiplicity function is subadditive:

\[ \mu(S) \leq \sum_{i=1}^{k} \mu(S_i). \]

To see this we have to observe that \( \mu(S) \) is the smallest number \( n \) such that there exists a set of \( n \) elements, \( \{x_1, \cdots, x_n\}, x_i \in H \) and span \( \{E(\alpha)x_i, \alpha \in \text{Borel set}\} = H. \) (\( n \) generating elements.)

Thus

\[ \mu(A) \leq \sum_{i=1}^{k} \mu(S_i) \leq \sum_{i=1}^{k} \mu(A_i) \leq mk < \infty . \]

In order to prove necessity we need the following:

**Lemma 2.1.** Let \( S = f(A) \) have finite multiplicity \( n \) and let

\[ A = \sum_{i=1}^{k} z_i(S)E_i \]

be the representation 1.3 then \( E_i \in \Phi_A. \)

**Proof.** For every Borel set \( \alpha \) \( E(\alpha) \in \Phi_A \) because \( S = f(A) \). Let \( E(\alpha) \) be maximal with respect to the property that \( E(\alpha)E_i \in \Phi_A \). Such a maximal projection exists by Zorn’s Lemma. Now if \( E(\sigma(S) - \alpha) \neq 0 \) there exists, by the proof of 3.2 in [2] a set \( \beta \) such that:

\[ \beta \subseteq \sigma(S) - \alpha \quad E(\beta) \neq 0 \]

and for some Borel set \( \gamma \)

\[ E(\beta)E_i = E(\beta)E(A; \gamma) \in \Phi_A. \]

This contradicts the maximality of \( \alpha \), hence \( E(\alpha) = I. \)

**Proof of necessity of conditions a and b.** Let \( S \) have finite multiplicity \( n. \) By Lemma 2.1 there exist \( n \) sets \( \beta_i \) such that \( E(A; \beta_i) = E_i. \) Thus
\[ E(A; \beta_i)E(A; \beta_j) = 0 \text{ if } i \neq j \]

and

\[ \sum_{i=1}^{k} E(A; \beta_i) = I. \]

Therefore the sets \( \beta_i \) can be chosen to be disjoint and satisfy condition a. Also

\[
A = \sum_{i=1}^{k} z_i(S)E_i = \sum_{i=1}^{k} z_i(f(A))E(A; \beta_i) = \sum_{i=1}^{k} \int_{\beta} z_i(f(\lambda))E(A; d\lambda).
\]

Hence, if \( \beta \subset \beta_i \) then

\[
E(A; \beta)A = \int_{\beta} \lambda E(A; d\lambda) = \int_{\beta} z_i(f(\lambda))E(A; d\lambda)
\]
or: on the set \( \beta_i\lambda = z_i(f(\lambda)) \) almost everywhere with respect to the measure \( E(A; \alpha) \).

**DEFINITION.** The function \( f \) will be said to have \( k \) repetitions, with respect to the measure \( E(A; \alpha) \), if conditions a and b of Theorem 2.1 are satisfied.

In the rest of this section we compute \( m\mu S \). It is enough to consider the case where the operator \( A \) has uniform multiplicity \( m \): otherwise \( A \) can be written as direct sum of operators of uniform multiplicity and one has to study each component of \( A \) separately.

The following Theorem is needed:

**THEOREM 2.2** Let \( H \) be the direct sum of the orthogonal subspaces \( H_1, \ldots, H_k \). Let \( S_i \) be a normal operator, on \( H_i \), of uniform multiplicity \( m_i \) and \( S \) be the direct sum of \( S_i \).

If

\[
E(S; \alpha) = 0 \text{ whenever } E(S_i; \alpha) = 0 \text{ for some } i
\]

then

\[
m\mu S = \sum_{i=1}^{k} m_i.
\]

**Proof.** It is enough to prove that \( m\mu S \geq \sum_{i=1}^{k} m_i \). Let \( \sigma = \sigma(S_i) = \cdots = \sigma(S_k) = \sigma(S) \). By the Spectral Multiplicity Theorem each operator \( S_i \) can be described as follows: There exists a measure \( \mu_i \) on \( \sigma \) and \( H_i \) is the direct sum of \( m_i \) spaces \( L_2(\mu_i) \). The operator \( S_i \) is given by

\[
S_i \begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_{m_i}(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda f_1(\lambda) \\ \vdots \\ \lambda f_{m_i}(\lambda) \end{pmatrix}.
\]
Now, the measures $\mu_i$ are equivalent, by the condition of the Theorem. Thus there exist functions $\varphi_i, \varphi_i \in L(\mu_{i+1}) \ 1 \leq i \leq k - 1$ such that

$$\mu_i(e) = \int_e \varphi_i(\lambda) d\mu_{i+1}$$

for every Borel set $e$. (Radon Nikodym Theorem, see [3], p. 128). Let us define an operator on $H$:

If $x \in H_i$,

$$x = \begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_{m_i-1}(\lambda) \\ 0 \end{pmatrix}$$

then

$$Mx \in H_i, \quad Mx = \begin{pmatrix} 0 \\ f_1(\lambda) \\ \vdots \\ f_{m_i-1}(\lambda) \end{pmatrix}.$$ 

If $x \in H_i$,

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_{m_i}(\lambda) \end{pmatrix}$$

then

$$Mx \in H_{i+1}, \quad Mx = \begin{pmatrix} \sqrt{\varphi_i(\lambda)} f_{m_i}(\lambda) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$ 

Where $H_{k+1}$ is the zero space.

It is easy to see that $M$ is a bounded operator and

$$\sum_{i=1}^k m_i = 0$$

but

$$\sum_{i=1}^k m_{i-1} \neq 0.$$
Also \( MS = SM \), hence \( \mu S \geq \sum_{i=1}^{k} m_i \).

**Remark.** It was proved in Theorem 2.1 that if a function \( f \) has \( k \) repetitions then

\[
\mu_f(A) \leq kmu A.
\]

However the number of repetitions of a function is not uniquely defined. In order to compute \( \mu_f(A) \) we have to find the minimal number of repetitions. This is what the next Theorem does.

**Theorem 2.3.** Let \( A \) be a normal operator of uniform multiplicity \( m \). Let \( f \) be a bounded measurable function which has \( k \) repetitions with respect to the measure \( E(A; \alpha) \). A necessary and sufficient condition that \( \mu S = mk \), where \( S = f(A) \), is:

There exists a Borel set \( \alpha_0 \)

\[
(2.1) \quad E(A; f^{-1}(\alpha_0)) \neq 0
\]

and

\[
E(A; f^{-1}(\alpha)) = 0 \quad \text{whenever} \quad E(A; f^{-1}(\alpha) \cap \beta_i) = 0 \quad \text{for some} \quad i \quad \text{and} \quad \alpha \subseteq \alpha_0.
\]

**Proof.** Assume condition 2.1. We may restrict \( A \) and \( S \) to \( E(A; f^{-1}(\alpha_0))H \). Let

\[
H_i = E(A; f^{-1}(\alpha_0) \cap \beta_i)H,
\]

and \( A_i, S_i \) the restriction of \( A, S \) to \( H_i \). Now

\[
f(A_i) = S_i \quad z_i(S_i) = A_i
\]

(See Theorem 2.1.). Thus the operators \( S_i \) have uniform multiplicity \( m \) because the operators \( A_i \) do. It follows from Theorem 2.2 that the multiplicity of \( S \) restricted to \( E(A; f^{-1}(\alpha_0))H \) is \( mk \). But \( \mu S \leq mk \), hence \( \mu S = mk \).

(Note that on \( \alpha_0 \) the operator \( S \) has uniform multiplicity \( mk \)). Conversely, let us assume that for each Borel set \( \alpha_0 \) with \( E(A; f^{-1}(\alpha_0)) \neq 0 \), there exists a subset \( \alpha \) such that \( E(A; f^{-1}(\alpha)) \neq 0 \) but \( E(A; f^{-1}(\alpha) \cap \beta_i) = 0 \) for some \( i \). Let \( E(A; f^{-1}(\alpha_i)) \) be maximal with respect to the property

\[
E(A; f^{-1}(\alpha_i))E(A; \beta_i) = 0
\]

Let \( E(A; f^{-1}(\alpha_i)) \) be maximal, with respect to the property

\[
\alpha_2 \cap \alpha_1 = \varnothing \quad \text{and} \quad E(A; f^{-1}(\alpha_2))E(A; \beta_2) = 0
\]

and choose inductively \( \alpha_3 \cdots \alpha_n, \alpha_i \cap \alpha_j = \varnothing \)
There exist such maximal projections by Zorn's Lemma. Now if $E(A ; \bigcup_{i=1}^{k} f^{-1}(\alpha_i)) \neq I$ there will be a set $\alpha$ and an integer $j$ such that

$$\alpha \cap \left( \bigcup_{i=1}^{k} \alpha_i \right) = 0; \quad E(A ; f^{-1}(\alpha) \cap \beta_j) = 0$$

Thus $\alpha_j$ will not be maximal. Let

$$\tilde{\beta}_j = \beta_j \cup (f^{-1}(\alpha_j) \cap \beta_j), \quad j \geq 2.$$ 

Then $\bigcup_{j=2}^{m} \tilde{\beta}_j = \sigma(A)$ and on $\tilde{\beta}_j$ the function $f$ possesses a bounded measurable inverse. Thus $f$ has $k - 1$ repetitions and $\mu S \leq m(k - 1)$.

3. The multiplicity of a matrix of functions. Let $S$ be a normal operator of uniform multiplicity $n$. Let $B$ be a normal operator and $BS = SB$. The operator $B$ is represented as the matrix of functions $\sum_{i=1}^{k} y_i(\lambda) \varepsilon_i(\lambda)$ and also $B = \sum_{i=1}^{k} y_i(S)E_i$ (Equation 1.2 and 1.3). Let us denote by $B_i$ and $S_i$ the restrictions of $B$ and $S$, respectively, to $E_i H = H_i$.

**Theorem 3.1.** The operator $B$ has finite multiplicity, if and only if, the functions $y_i$ have $j_i (j_i < \infty)$ repetitions with respect to the spectral measure of $S_i$.

Also

$$\max_i \mu B_i \leq \sum_{i=1}^{k} \mu B_i \leq \sum_{i=1}^{k} j_i \mu S_i.$$ 

**Proof.** From the definition of multiplicity, as the smallest number of generating elements, it follows that

$$\max_i \mu B_i \leq \mu B \leq \sum_{i=1}^{k} \mu B_i.$$ 

Now, $B_i = y_i(S_i)$, hence the rest of the Theorem follows from Theorem 2.1. The problem of this section is reduced to the following

$$H = \sum_{i=1}^{k} E_i H \text{ where } E_i E_j = 0 \text{ if } i \neq j$$

and $B_i =$ restriction $B$ to $E_i H$, where the multiplicity of $B_i$ is known. Now by decomposing each operator $B_i$ into sum of operators of uniform multiplicity we will have $H = \sum_{i=1}^{k} H_i$, where the spaces $H_i$ are mutually orthogonal, and $C_i =$ restriction of $B$ to $H_i$ is an operator of uniform multiplicity. We shall show how to compute $\mu B$ from $\mu C_i$ by reducing this case to the one studied in Theorem 2.2.
Denote the projection on $H_i$ by $F_i$. Let $E(B; \alpha_i)$ be the maximal projection such that

$$E(C_i; \alpha_i) = E(B; \alpha_i)F_i = 0.$$  

Such a projection exists by Zorn's Lemma. Finally let $\beta_i = \sigma(B) - \alpha_i$. On $\beta_i$ the spectral measure of $C_i$ can vanish only when the spectral measure of $B$ vanishes. Now $E(B; \bigcup_{i=1}^m \beta_i) = I$ because $\sum_{i=1}^m F_i = I$.

The set $\sigma(B)$ can be decomposed into disjoint sets $\gamma_j$ such that

a. Each $\gamma_j$ is a subset of one of the sets $\beta_{j_0}$.

b. If $\gamma_j \cap \beta_i \neq \emptyset$ then $\gamma_j \subseteq \beta_i$.

Assuming, for a moment, that this decomposition is given then

$$muB = \max_j mu(B \text{ restricted to } E(B; \gamma_j)H).$$  

But the multiplicity of $B$ restricted to $E(B; \gamma_j)H$ is

$$\sum_{i \mid \gamma_j \subseteq \beta_i} mu(C_i \text{ restricted to } E(B; \gamma_j)H_i)$$  

by Theorem 2.2.

We shall show how to choose the sets $\gamma_i$ by an induction argument on the number $m$. Let $\gamma_1 = \beta_1 - \bigcup_{i \geq 2} \beta_i \beta_1$. This set (which might be void) satisfies conditions a and b. The rest of $\sigma(B)$ is

$$\left(\bigcup_{i \geq 2} \beta_i \beta_1\right) \cup \left(\bigcup_{i \geq 2} (\beta_i - \beta_i)\right)$$

In both sets there are only $m - 1$ subsets and by induction there exists a decomposition.

**BIBLIOGRAPHY**


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