SEPARABLE CONJUGATE SPACES

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A Banach space $B$ is reflexive if the natural isometric mapping of $B$ into the second conjugate space $B^{**}$ covers all of $B^{**}$. All conjugate spaces of a reflexive separable space $B$ are separable. The nonreflexive space $l^{(1)}$ is separable and its first conjugate space is $(m)$, which is non-separable. The space $(c_0)$ is separable, its first conjugate space is $l^{(1)}$, and its second conjugate space is $(m)$. An example is known of a nonreflexive Banach space whose conjugate spaces are all separable [4]. This space is pseudo-reflexive in the sense that its natural image in the second conjugate space has a finite-dimensional complement. The structure of such spaces has been studied carefully [2].

The main purpose of this paper is to show that the sequence started by $l^{(1)}$ and $(c_0)$ can be extended to give a sequence $\{B_n\}$ of separable Banach spaces such that, for each $n$, the $n$th conjugate space of $B_n$ is its first nonseparable conjugate space. The principal tool used is a theorem which states a sufficient condition on a space $T$ for the existence of a space $B$ with

$$B^{**} = \pi(B) + T,$$

where $\pi(B)$ is the natural image of $B$ in $B^{**}$. The following definition and notation will be used.

A basis for a Banach space $B$ is a sequence $\{u_i\}$ such that, for each $x$ of $B$, there is a unique sequence of numbers $\{a_i\}$ for which $\lim_{n \to \infty} \| x - \sum_{i=1}^{n} a_i u_i \| = 0$. A sequence $\{u_i\}$ is a basis for its closed linear span if and only if there is a number $\varepsilon > 0$ such that

$$\left\| \sum_{i=1}^{n+p} c_i x_i \right\| \geq \varepsilon \left\| \sum_{i=1}^{n} c_i x_i \right\|$$

for any numbers $\{c_i\}$ and positive integers $n$ and $p$ [1, page 111]. If $\varepsilon$ can be $+1$, the basis is an orthogonal basis. It will be useful to classify bases as follows:

**Type $\alpha$.** If $\{a_i\}$ is a sequence of numbers for which $\sup_n \| \sum_{i=1}^{n} a_i u_i \| < \infty$, then $\sum_{i=1}^{n} a_i u_i$ converges.

**Type $\beta$.** If $f$ is a linear functional defined on $B$ and $\| f \|_n$ is the norm of $f$ on the closed linear span of $\{u_i \mid i \geq n\}$, then $\lim_{n \to \infty} \| f \|_n = 0$.

There are Banach spaces which have bases which are neither of type $\alpha$ nor of type $\beta$, while a basis is of both types if and only if the space

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is reflexive [3; Theorem 1].

The symbols \( C, (m), l^\omega \), and \( (c_0) \) are used in the usual sense [1; pages 11, 12, 181]. The set of all \( r + t \) with \( r \in R \) and \( t \in T \) is denoted by \( R + T \). A space \( R \) is said to be embedded in a space \( S \) if \( R \) is mapped isomorphically and isometrically on a subspace of \( S \); for \( x \in R \), the image of \( x \) is indicated by \( x^{(s)} \). In particular, \( x^{(o)} \) is a continuous function defined on \([0, 1]\) and the value of \( x^{(o)}(t) \) at \( t \) is denoted by \( x^{(o)}(t) \). If \( w = (w_1, w_2, \ldots) \) is a sequence of numbers, then \( \overline{w} \) is the sequence obtained by replacing \( w_i \) by 0 if \( i > n \). A block of \( w \) is a sequence \( \overline{w} \) obtained from \( w \) by replacing \( w_i \) by 0 if \( i \leq m \) or \( i > n \). Two blocks \( \overline{w}_1 \) and \( \overline{w}_2 \) are said to overlap if the intervals \( (m_1, n_1] \) and \( (m_2, n_2] \) overlap.

**Lemma 1.** Let \( T \) be a Banach space with an orthogonal basis \( \{u_i\} \). Then \( T \) can be embedded in \( (m) \) in such a way that:

(i) if \( x = \sum a_iu_i \), then the first \( 2N \) coordinates of \( x^{(m)} \) are zero if and only if \( a_i = 0 \) for \( i \leq N \);

(ii) if \( \{a_i\} \) and \( \{x_i^{(m)}\} \) are related by \( x = \sum a_iu_i \) and \( x^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots) \), then \( a_1, \ldots, a_N \) are each continuous functions of \( x_1^{(m)}, x_2^{(m)} \), \( a_{2N} \) is continuous in \( x_1^{(m)} \), while \( a_{2N+1} \) is continuous in \( x_2^{(m)} \);

(iii) if \( x^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots) \), then \( ||x^{(m)}|| = \lim \sup ||x_k^{(m)}|| \).

**Proof.** Let \( T \) be embedded in the space \( C \). Let \( \{t_i\} \) be a sequence of numbers in the interval \([0, 1]\) for which the sequence \( \{t_{2i-1}\} \), \( i = 1, 2, \ldots \), is dense in \([0, 1]\) and, for each \( i \), \( u_i^{(o)}(t_{2i}) \neq 0 \). If \( x = \sum a_iu_i \), let \( x^{(m)} \) be the sequence \( (x_1^{(m)}, x_2^{(m)}, \ldots) \) for which

\[
x_{2k-1}^{(m)} = \sum_{i=1}^{k} a_iu_i^{(o)}(t_{2k-1}), \quad x_{2k}^{(m)} = \sum_{i=1}^{k} a_iu_i^{(o)}(t_{2k}).
\]

Then for any \( t \in [0, 1] \),

\[
\left| \sum_{i=1}^{k} a_iu_i^{(o)}(t) \right| \leq \left| \sum_{i=1}^{k} a_iu_i^{(o)} \right| = \left| \sum_{i=1}^{k} a_iu_i \right| \leq ||x||.
\]

Hence \( ||x^{(m)}|| \leq ||x|| \). But if \( \varepsilon > 0 \) and \( N \) is chosen so that \( ||x - \sum_{i=1}^{k} a_iu_i|| < \varepsilon \) if \( k > N \), then it follows from \( \{t_{2k-1}\} \) being dense in \([0, 1]\) that

\[
||x^{(m)}|| \geq \sup_{k>N} \left| \sum_{i=1}^{k} a_iu_i^{(o)}(t_{2k-1}) \right| \geq ||x|| - \varepsilon.
\]

Hence \( ||x|| = ||x^{(m)}|| \) and \( T \) and its image in \((m)\) are isometric. But if \( x = \sum_{i=1}^{N+1} a_iu_i \), then \( x_{2k-1}^{(m)} = x_{2k}^{(m)} = 0 \) if \( k \leq N \). If \( a_i^m = 0 \) for \( i \leq 2N \), then the equations \( x_{2k}^{(m)} = \sum a_iu_i^{(o)}(t_{2k}) = 0 \), \( k \leq N \), successively imply \( 0 = a_1 = a_2 = \cdots = a_N \), since \( u_k^{(o)}(t_{2k}) \neq 0 \). The conclusion (ii) follows from this system of equations and the continuity of \( \sum a_iu_i \) in \( a_1, \ldots, a_N \), while (iii) follows from \( \{t_{2k-1}\} \) being dense in \([0, 1]\).
LEMMA 2. Let $T$ be a Banach space with an orthogonal basis $\{u_i\}$ and let $T$ be embedded in $(m)$ as described in Lemma 1. Then the following are equivalent:

(i) the basis $\{u_i\}$ is of type $\alpha$;

(ii) if $w \in (m)$, then $w = v + t$, with $v$ an element of $(m)$ which has all coordinates zero after the $M$th $(M \geq 0)$ and $t$ the image of an element of $T$, provided there is a sequence of elements $\{y_k\}$ of $T$ for which $\sup \|y_k\| < \infty$ and

$$\lim_{k \to \infty} y_{k,i}^m = w_i \text{ for } i > M,$$

where $w = (w_1, w_2, \cdots)$ and $y_k^m = (y_{k,1}^m, y_{k,2}^m, \cdots)$.

Proof. Assume the basis $\{u_i\}$ is of type $\alpha$ and let $w = (w_1, w_2, \cdots)$ and $\{y_k\}$ satisfy the hypotheses of (ii). Since $\|y_k\|$ is bounded, there is a subsequence $\{z_k\}$ of $\{y_k\}$ such that

$$\lim_{k \to \infty} z_{k,i}^m = v_i$$

exists for $i \leq M$. Let $v = (w_1 - v_1, \cdots, w_M - v_M, 0, 0, \cdots)$. Also let $z_k = \sum_i a_i^k u_i$ for each $k$. It now follows from (ii) of Lemma 1 that $\lim_{k \to \infty} a_i^k = a_i$ exists for each $i$. Since the basis is orthogonal, $\| \sum_i a_i^k u_i \| \leq \sup \| z_k \|$. Since $\{u_i\}$ is a basis of type $\alpha$, it then follows that $\sum_i a_i^k u_i$ is convergent. Also, $w - v = t$ is the $(m)$-image of $\sum_i a_i^k u_i$. This follows from the fact that the numbers $a_i$, $i \leq N$, continuously determine the first $2N$ coordinates of the $(m)$-image of $\sum_i a_i^k u_i$, while $z_k = \sum_i a_i^k u_i$, $\lim_{k \to \infty} a_i^k = a_i$, and $\lim_{k \to \infty} z_{k,i}^m$ exists and is the $i$th coordinate of $w - v$.

Now assume (ii) and let $\| \sum_i a_i^k u_i \|$ be a bounded function of $n$. Let $w = (w_1, w_2, \cdots)$ be the element of $(m)$ whose first $2N$ coordinates are determined by $a_1, \cdots, a_N$. Take $M = 0$ and $y_k$ to be the $(m)$-image of $\sum_i a_i u_i$. It then follows from (ii) that $w$ is the $(m)$-image of some element of $T$, which can only be $\sum_i a_i u_i$.

THEOREM 1. Let $T$ be a Banach space which has an orthogonal basis of type $\alpha$. Then there is a Banach space $B$ which has a basis of type $\beta$ and for which

$$B^{**} = \pi(B) + T$$

where $\pi(B)$ is the natural image of $B$ in $B^{**}$, $T$ and $T_1$ are isometric, and $\| r + t \| \geq \| t \|$ if $r \in \pi(B)$ and $t \in T_1$.

Proof. Let $T_1$ be the embedding of $T$ in $(m)$ as described in Lemma 1. The norm of $(m)$ will be denoted by $\| \|$. For elements $w$ of $(m)$ which have only a finite number of nonzero coordinates, let

(1) $\theta(w) = \inf \| t \|$ for $w$ a block of $t$, where $t$ is either a member
of $T_1$ or has only one nonzero coordinate (note that $\theta(w)$ is defined only for elements $w$ which are blocks of at least one $t \in T_1$ or which have only one nonzero coordinate);

(2) $h(w) = \left( \inf \sum [\theta(b_i)]^2 \right)^{1/2}$, where $w = \sum b_i$, each $b_i$ is a block of $w$, and no two blocks overlap.

(3) $|||x||| = \inf \sum h(w_j)$ for $x = \sum w_j$.

In the above, all sums have a finite number of terms. The triangular inequality for $||| |||$ is a direct consequence of (3). Also, $||| x ||| \geq ||| x |||$, since $\theta(w) \geq ||| w |||$ and $h(w) \geq ||| w |||$. Let $B$ be the completion of the space of sequences with a finite number of nonzero coordinates, using the norm $||| |||$. The sequence of elements $\{u_i\}$ for which $u_i$ has all coordinates 0 except the $i$th, which is 1, is an orthogonal basis for $B$.

The basis $\{u_i\}$ is of type $\beta$. For suppose there is a linear functional $f$ for which $\lim_{n \to \infty} ||| f \alpha ||| = K \neq 0$ and choose $N$ so that $||| f \alpha ||| \leq 7/6K$. Then there are two elements $x = \sum a_i u_i$, $y = \sum a_i u_i$, for which $N < n_1 < n_2 < n_3$, $||| x \alpha ||| = ||| y \alpha ||| = 1$, $f(x) > 7/8K$ and $f(y) > 7/8K$. Then

$$\frac{7}{4} K < f(x) + f(y) \leq \left( \frac{7}{6} K \right) ||| x + y \alpha |||$$

Since $\theta$ and $h$ are both monotone decreasing as a block has coordinates at the ends replaced by zeros, there exists $\{x_j\}$ and $\{y_j\}$ such that $x = \sum x_j$, $y = \sum y_j$, $\sum h(x_j) < ||| x \alpha ||| + \varepsilon$, and $\sum h(y_j) < ||| y \alpha ||| + \varepsilon$, where each $x_j$ has zero coordinates outside the index interval $[n_1, n_2]$ and each $y_j$ has zero coordinates outside the index interval $[n_3, n_4]$. Now replace the sets $\{x_j\}$ and $\{y_j\}$ by $\{\tilde{x}_j\}$ and $\{\tilde{y}_j\}$ defined as follows: if $h(x_i)$ is the smallest of all the numbers $h(x_j)$ and $h(y_j)$, then let $\tilde{x}_i = x_i$ and $\tilde{y}_i = [h(x_i)/h(y_i)]y_i$ (for some $r$) and replace $y_i$ by $[1 - h(x_i)/h(y_i)]y_i$. The analogous process is used if $h$ takes on its minimum at one of the $y_i$'s. This process creates two new elements and eliminates one old one at each step, until all of the $x_j$'s or all of the $y_j$'s are eliminated. If only $x_j$'s remain, say $x_{p_j}$'s, then $\sum h(x_{p_j}) < \varepsilon$, and similarly $\sum h(y_{p_j}) < \varepsilon$ if only $y_j$'s remain. Also

$$\sum h(\tilde{x}_j) - \varepsilon = \sum h(\tilde{y}_j) - \varepsilon < ||| x \alpha ||| = ||| y \alpha ||| = 1$$

and $h(\tilde{x}_j) = h(\tilde{y}_j)$ for each $j$. For each $j$, there are nonoverlapping blocks $\{\tilde{x}_{ji}\}$ and $\{\tilde{y}_{ji}\}$ such that

$$h(\tilde{x}_j) = h(\tilde{y}_j) = \left( \sum_i [\theta(\tilde{x}_{ji})]^2 \right)^{1/2} = \left( \sum_i [\theta(\tilde{y}_{ji})]^2 \right)^{1/2} .$$

Then
\[ h(\bar{x}_j + \bar{y}_j) \leq \left\{ \sum \left[ \theta(\bar{x}_{ji}) \right]^2 + \sum \left[ \theta(\bar{y}_{ji}) \right]^2 \right\}^{1/2} = \sqrt{2} h(\bar{x}_j). \]

Hence
\[ \|x + y\| \leq \sum h(\bar{x}_j + \bar{y}_j) + \epsilon \leq \sqrt{2} \sum h(\bar{x}_j) + \epsilon \leq \sqrt{2} + \epsilon. \]

Since \( \|x + y\| > 3/2 \), this is contradictory if \( \sqrt{2} + \epsilon < 3/2 \). It has therefore been shown that \( \{u_i\} \) is a basis of type \( \beta \).

Since \( \{u_i\} \) is an orthogonal basis of type \( \beta \) for \( B \), it follows that \( B^{**} \) consists of all sequences \( F = (F_1, F_2, \cdots) \) for which
\[ \|F\| = \lim_{n \to \infty} \| (F_1, \cdots, F_n, 0, 0, \cdots) \| \]
exists [4; page 174]. Note first that if \( t = (t_1, \cdots) \in T_1 \), then
\[ \| (t_1, \cdots, t_n, 0, 0, \cdots) \| = \| (t_1, \cdots, t_n, 0, 0, \cdots) \| \]
and \( \lim_{n \to \infty} \| (t_1, \cdots, t_n, 0, 0, \cdots) \| = \| t \| = \| t \| \). Thus \( T_1 \subset B^{**} \). Also, the natural mapping of \( B \) into \( B^{**} \) is merely the mapping of a sequence in \( B \) onto the identical sequence in \( B^{**} \). It then follows that \( \| r + t \| \geq \| t \| \) if \( r \in \pi(B) \) and \( t \in T_1 \), since \( r \) can be approximated by a sequence with a finite number of nonzero coordinates but (Lemma 1) \( \| t \| = \lim \sup |t_i| \).

Now suppose that \( F = (F_1, F_2, \cdots) \) is a sequence for which \( \lim_{n \to \infty} \| ^n F \| \) exists; i.e., \( F \in B^{**} \). It will be shown that there is an element \( v \) of \( \pi(B) + T_1 \) for which \( \| F - v \| \leq 15/16 \| F \| \). Successive application of this would then establish that \( F \in \pi(B) + T_1 \). For each \( n \), there are \( ^n w_j \) and blocks \( b_{j,i}^n \), which are either blocks of elements of \( T_1 \) or have only one nonzero coordinate, such that
\[ \| ^n F \| = \sum h(^n w_j), \quad ^n F = \sum _j ^n w_j, \quad \text{and} \quad h(^n w_j) = \left\{ \sum \left[ \theta(b_{j,i}^n) \right]^2 \right\}^{1/2}, \]
where each \( ^n w_j \) and each \( b_{j,i}^n \) have all coordinates zero after the \( n \)th. This follows by a limit argument, using the facts (1) that there are only a finite number \( K_n \) of ways of choosing division points for nonoverlapping blocks from the integers \( 1, 2, \cdots, n \) and (2) that it follows from Lemma 1 and the orthogonality of the basis for \( T \) that \( \theta(b_{j,i}^n) \), for a block \( b_{j,i}^N \), which has zero coordinates beyond the \( 2N \)th coordinate, can be evaluated by using only members of the span of the first \( N \) basis elements of \( T \).

If \( m < n \) and \( ^m w_j^m \) is obtained from \( ^n w_j \) by replacing coordinates after the \( m \)th by zeros, then
\[ \| ^m F \| \leq \sum h(^m w_j^m) \leq \| ^n F \| \leq \| F \|. \]

If \( ^m w_j^1 \) and \( ^m w_j^2 \) are of the ‘same type’ in the sense that they are divided into blocks by using the same division points, then it follows by using these same division points for \( ^m w_j^1 + ^m w_j^2 \) that
\[ h(mw_{i_1}^n + mw_{i_2}^n) \leq h(mw_{i_1}^n) + h(mw_{i_2}^n). \]

For each \( n > m \), let \( mw_j^n \) be the sum of all \( mw_j^m \) of the "same type" as \( mw_j^n \). A limit argument gives a sequence of integers \( \{n_i\} \) such that \( \lim mw_j^{n_i} = mw_j \) exists for each "type". If \( m < n \), then there exist \( \tilde{b}_{j,i}^n \) such that

\[ ||mF|| \leq \sum_j h(m\tilde{w}_j) \leq \sum_k h(n\tilde{w}_k) \leq ||F||, \]

\[ h(m\tilde{w}_j) = \left\{ \sum_i [\theta(b_{j,i}^m)]^{1/2} \right\} mF = \sum m\tilde{w}_j, \]

and \( m\tilde{w}_j \) is equal to the sum of all \( m\tilde{w}_j^n \) which are of the same type as \( m\tilde{w}_j \) and are obtained from \( n\tilde{w}_j \) by replacing all coordinates after the \( m \)th by zeros. The points used to divide \( m\tilde{w}_j \) into the blocks \( \tilde{b}_{j,i}^n \) will be called the division points of \( m\tilde{w}_j \).

Choose \( M \) so that \( \|mF\| > 15/16 \|F\| \). Note that if \( m\tilde{w}_j \) is of a particular type and \( n > m \), then \( m\tilde{w}_j \) is the sum of one or more elements obtained from \( n\tilde{w}_k \)'s by replacing coordinates after the \( m \)th by zeros. For \( n > m \geq M \), let \( nt \) be the sum of all \( n\tilde{w}_k \)'s which have no division points between \( M \) and \( n \) and let \( nt^n \) be obtained from \( nt \) by replacing coordinates after the \( m \)th by zeros. Let \( \{n_i\} \) be chosen so that

\[ \lim_{i \to \infty} nt^{n_i} = nt \]

exists for each \( m \geq M \). Let \( \tilde{t} \) be defined so as to have the same first \( m \) coordinates as \( nt \). Then any finite block of \( \tilde{t} \) whose first \( M \) coordinates are zero is also approximately a block of an element of \( T_1 \) and these elements of \( T_1 \) are of bounded norm. It then follows from Lemma 2 that there is an element \( v_0 \), with a finite number of nonzero coordinates, such that \( v_0 + \tilde{t} \in T_1 \). Thus

\[ \tilde{t} \in \pi(B) + T_1. \]

First assume that \( ||\tilde{t}|| > 1/8 \|F\| \) and choose \( N \) so that

\[ ||nt^n|| > 1/8 \|F\| \text{ if } n > N. \]

For \( n > N \), choose \( p > n \) so that

\[ ||nt^n - nt^p|| < \frac{1}{32} \|F\|. \]

Since \( \|nF\| \leq \sum_j h(n\tilde{w}_j) \), discarding all \( n\tilde{w}_j \) without division points between \( M \) and \( p \) gives

\[ ||nF - nt^p|| \leq \sum_j h(n\tilde{w}_j) - ||nt^p|| \leq \|F\| - ||nt^p||. \]
Hence \( ||\overline{t}|| < ||F|| - ||\overline{t}|| + 1/16||F|| < 15/16||F||.\) Since \( n \) was an arbitrary integer with \( n > N \), it follows that

\[
||F - \overline{t}|| \leq \frac{15}{16}||F||.
\]

Now assume that \( ||\overline{t}|| \leq 1/8||F||.\) Then \( ||\overline{t}|| \leq 1/8||F||\) for all \( n \). Choose \( q \) so that

\[
||\overline{t} - \overline{t}^q|| < \frac{1}{16}||F||.
\]

For each \( \overline{w}_j \) which has a division point between \( M \) and \( q \), let \( \overline{w}_j^q \) be obtained from \( \overline{w}_j \) by replacing all coordinates after the last such division point by zeros. Let

\[
u = \sum_{j} \overline{w}_j^q.
\]

Choose \( n > q \). Then \( nF = \sum n\overline{w}_j \), and

\[
||nF|| \leq \sum h(n\overline{w}_j) \leq \sum h(\overline{w}_j^q) + ||nt^q|| < \frac{3}{16}||F||.
\]

Since \( ||nF|| > 15/16||F|| \), we have \( \sum h(\overline{w}_j^q) > 3/4||F||.\) Now consider \( F - u. \) Since \( ||nF|| \leq \sum h(n\overline{w}_j), \) where \( h(n\overline{w}_j) = \{\sum_{i} [\theta(b_{n,i})]^2\}^{1/2}, \) we have

\[
n(F - u) = \sum n\overline{w}_j - \sum \overline{w}_j^q = \sum n\overline{w}_j^q,
\]

where \( n\overline{w}_j^q \) is obtained from \( n\overline{w}_j \) by replacing all coordinates before the last division point between \( M \) and \( q \) by zeros (if there is no such point, then \( n\overline{w}_j = n\overline{w}_j \)). The following trivial facts will be used: If \( A \) and \( B \) are nonnegative and

if \( \sqrt{3}A < B, \) then \( \sqrt{A^2 + B^2} > 2A; \)

if \( \sqrt{3}A \geq B, \) then \( B < \sqrt{A^2 + B^2} - \frac{1}{4}A. \)

Each \( n\overline{w}_j \) which has a division point between \( M \) and \( q \) makes a contribution to some \( \overline{w}_j. \) For such an \( n\overline{w}_j, \) let

\[
h(n\overline{w}_j) = [\sum_{r}(A_{r})^2 + \sum_{s}(B_{s})^2]^{1/2},
\]

where the \( A_{r}'s \) and \( B_{s}'s \) are, respectively, the values of \( \theta(b_{n,i}) \) for \( b_{n,i} \) a block of some \( \overline{w}_j^q \) and \( b_{n,i} \) not a block of any \( \overline{w}_j^q. \) Then
where the sum is over all $n \omega_j$ which make a contribution to $u_j$. Let $\sum (A_r)^2$ be of class (1) or of class (2) according as

\[ \sqrt{3} \left[ \sum (A_r)^2 \right]^{1/2} < \left[ \sum (B_j)^2 \right]^{1/2} \text{ or } \sqrt{3} \left[ \sum (A_r)^2 \right]^{1/2} \geq \left[ \sum (B_j)^2 \right]^{1/2}. \]

Since $\sum h(u_j^2) > 3/4 || F ||$, the sum of all terms of class (1) is not larger than $1/2 || F ||$ (otherwise we would have $\sum h(n \omega_j) > || F ||$) and the sum of all terms of class (2) is greater than $1/4 || F ||$. But for a term of class (2),

\[ \left[ \sum (B_j)^2 \right]^{1/2} < h(n \omega_j) - \frac{1}{4} \left[ \sum (A_r)^2 \right]^{1/2}. \]

Adding these inequalities for each $n \omega_j$ and discarding each $\sum (A_r)^2$ which is of class (1) gives

\[ \sum h(n \omega_j) < \sum h(n \omega_j) - \frac{1}{16} || F || \text{ and } || n(F - u) || < \frac{15}{16} || F ||. \]

Since $n$ was an arbitrary integer with $n > q$, it follows that

\[ || F - u || \leq \frac{15}{16} || F ||. \]

The importance of the assumption in Theorem 1 that $T_1$ have a basis of type $\alpha$ is made clear by the fact that the theorem breaks down if $T_1$ has a subspace isomorphic with $(c_0)$. In fact, in this case there can not be a separable space $B$ with

\[ B^{**} = \pi(B) + T_1 \]

and $T_1$ separable, whether or not $B$ and $T_1$ have bases. This follows from the fact that if a conjugate space $R^*$ contains a subspace isomorphic with $(c_0)$, then $R^*$ contains a subspace isomorphic with $(m)$ and is not separable. To establish this fact, suppose that $\{F_n\}$ are continuous linear functionals defined on some Banach space $B$ and that the closed linear span of $\{F_n\}$ is isomorphic with $(c_0)$, the correspondence being

\[ \sum a_i F_i \leftrightarrow (a_1, a_2, \ldots). \]

For any bounded sequence $w = (w_1, w_2, \ldots)$, define $F_w$ by

\[ F_w(f) = \lim_{n} \left( \sum_{i=1}^{n} w_i F_i \right)(f), \]

for each $f$ of $B$. This limit exists, since if it did not there would exist
\[ \varepsilon > 0 \text{ and } G_1 = \sum_{i=1}^{n_1} w_i F_i, \quad G_2 = \sum_{i=2}^{n_3} w_i F_i, \ldots, \text{ with } 1 \leq n_1 < n_2 \leq n_3 < n_4 \leq \cdots, \]

such that \( G_i(f) > \varepsilon \). Then correct choice of signs would give

\[ \sum_{i=1}^{n_i} \pm G_i(f) > n\varepsilon, \]

which contradicts the boundedness of \( \| \sum_{i=1}^{n_i} \pm G_i \| \). Clearly the correspondence with \((c_0)\) is thus extended to a bicontinuous correspondence with \((m)\).

**Theorem 2.** For any positive integer \( n \), there is a Banach space \( B_n \) such that the \( n \)th conjugate space of \( B_n \) is the first nonseparable conjugate space of \( B_n \).

**Proof.** Let \( B_1 = l^{(1)} \) and \( B_2 = (c_0) \). Then \( B_1 \) has a basis of type \( \alpha \) and \( B_2 \) has a basis of type \( \beta \). In the following, the notation \( R + S \) is used only if \( \| r + s \| \geq \| s \| \) whenever \( r \in R \) and \( s \in S \). It follows from Theorem 1 that there is a separable Banach space \( B_3 \) with a basis of type \( \beta \) for which

\[ B_3^{**} = B_3 + l^{(1)} = B_3 + B_2^* \]

Then \( B_3^{***} \) is nonseparable and \( B_2^* \) has a basis of type \( \alpha \) [3, Theorem 3]. Now suppose that, for \( k \leq n \), \( B_k \) has been found for which

\[ B_k^{**} = B_k + B_{k-1}^* \]

if \( k \geq 3 \), \( B_k \) has a basis of type \( \beta \) if \( k \geq 2 \), and the \( k \)th conjugate space of \( B_k \) is the first nonseparable conjugate space of \( B_k \). Then \( B_k^* \) has a basis of type \( \alpha \) and it follows from Theorem 1 that there exists a separable space \( B_{n+1} \) which has a basis of type \( \beta \) and for which

\[ B_{n+1}^{**} = B_{n+1} + B_n^*. \]

Then \( B_{n+1}^{***} = B_{n+1}^* + B_n + B_{n-1}^* \). The \((n-2)\)nd conjugate space of \( B_{n-1}^* \) is the first nonseparable conjugate space of \( B_{n-1}^* \), while the \((n-2)\)nd conjugate space of \( B_n \) is separable. Hence the \((n+1)\)st conjugate space of \( B_{n+1} \) is the first nonseparable conjugate space of \( B_{n+1} \).

**References**


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