ON NON-ASSOCIATIVE ALGEBRAS ASSOCIATED WITH BILINEAR FORMS

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If \( \mathcal{B}_0 \) is a vector space over a field \( k \), then with any non-degenerate bilinear form \( f_0 \) on \( \mathcal{B}_0 \times \mathcal{B}_0 \) is associated the group \( \mathcal{G} \) of linear transformations of \( \mathcal{B}_0 \) which keep \( f_0 \) invariant. In this paper a procedure is given for associating with such a bilinear form an algebra \( \mathcal{A} \), non-associative in general, whose automorphism group is isomorphic to \( \mathcal{G} \) and which is right and left simple provided \( \mathcal{B}_0 \) has dimension at least 2. In case \( k \) is the field of real numbers, then \( \mathcal{G} \) is a Lie group and its Lie algebra is the Lie algebra of derivations of \( \mathcal{A} \). In case the form \( f_0 \) is degenerate, and either symmetric or alternating, then the analogue of the Wedderburn Principal Theorem holds for \( \mathcal{A} \). The results obtained apply, in particular, to the orthogonal and symplectic groups.

Let \( \mathcal{B}_0 \) be a vector space of dimension \( n \) over a field \( k \) with basis \( u_1, \ldots, u_n \). It is assumed that \( \lambda v = v \lambda \) for all \( v \in \mathcal{B}_0 \) and \( \lambda \in k \). Suppose \( f_0 \) is a bilinear form on \( \mathcal{B}_0 \times \mathcal{B}_0 \). Define \( \mathcal{A} \) to be the algebra over \( k \) with basis \( e_0, e_1, \ldots, e_n \) and multiplication table\footnote{Received May 8, 1959.}

**Theorem 1.** Suppose that \( f \) is non-degenerate and that \( n \geq 2 \). Then \( \mathcal{A} \) is right and left simple.

**Proof.** Let \( \mathcal{U} \) be a non-zero left ideal of \( \mathcal{A} \) and let \( u \) be a non-zero element of \( \mathcal{U} \). Suppose first that \( u \in \mathcal{B} \). Then there exists an element \( v \in V \) such that \( f(v, u) \neq 0 \). Then \( v \cdot u = f(v, u)e_0 \). Therefore \( e_0 \in \mathcal{U} \) and so \( \mathcal{U} = \mathcal{A} \). Next suppose \( u = \alpha e_0 + v \) where \( \alpha \neq 0 \) in \( k \) and \( v \in V \). Then one can assume \( \alpha = 1 \). Since \( n \geq 2 \) it follows that \( e_i \cdot u = e_i + \lambda e_0 \) and \( e_i \cdot u = e_i + \lambda e_0 \) where \( \lambda_1, \lambda_2 \in k \). If \( \lambda_1 = 0 \) then \( e_i \in \mathcal{U} \) and the first part of the proof applies; similarly if \( \lambda_2 = 0 \). Consequently one can suppose \( \lambda_1 \lambda_2 \neq 0 \). Then \( \lambda_2 e_1 u - \lambda_1 e_2 u = \lambda_2 e_1 - \lambda_1 e_2 \) is a non-zero element in \( \mathcal{U} \cap \mathcal{B} \). Thus the first part of the proof again applies and so \( \mathcal{U} = \mathcal{A} \). Therefore \( \mathcal{U} \) is left simple; similarly \( \mathcal{A} \) is right simple.

If \( \mathcal{A} \) is any (non-associative) algebra over \( k \) then left (right) multiplication by an element \( a \in \mathcal{A} \) determines a linear transformation \( L_a(R_a) \) of the underlying vector space of \( \mathcal{A} \) by \( a \cdot u = L_a u (u \cdot a = R_a u) \), \( u \in \mathcal{A} \). The set of linear transformations \( L_a \) (\( R_a \)) for \( a \in \mathcal{A} \) generate an associative algebra \( L(\mathcal{A}) \) (\( R(\mathcal{A}) \)) over \( k \). The algebras \( L(\mathcal{A}) \) and \( R(\mathcal{A}) \) together
THEOREM 2. If $f$ is non-degenerate and $n \geq 2$ then $L(\mathfrak{A}) = R(\mathfrak{A}) = T(\mathfrak{A}) = [k]_{n+1}$.

Proof. The proof of Theorem 1 shows that for any $u \neq 0$ in $\mathfrak{A}$ there is an element of $L(\mathfrak{A})$ mapping $u$ into any arbitrarily assigned element of $\mathfrak{A}$. Therefore $L(\mathfrak{A}) = [k]_{n+1}$; similarly for $R(\mathfrak{A})$, and so also for $T(\mathfrak{A})$.

Albert has introduced in [1] the concept of isotopy of non-associative algebras. Suppose $\mathfrak{U}$ is an algebra with left multiplications $L_a$ defined by $a \cdot u = L_a u$. Then an isotope of $\mathfrak{U}$ is an algebra $\mathfrak{U}'$ with the same underlying vector space and multiplication defined by $a \circ u = PL_{uQ}Su$ where $P, Q, S$ are invertible linear transformations of the underlying vector space of $\mathfrak{U}$. An algebra $\mathfrak{U}$ is said to be isotopically left (right) simple if every isotope of $\mathfrak{U}$ is left (right) simple.

THEOREM 3. Suppose $f$ is non-degenerate and that $n \geq 2$. Then $\mathfrak{U}$ is isotopically left and right simple.

Proof. Suppose $\mathfrak{U}$ is a subspace of $\mathfrak{U}$ such that $PL_{uQ}Su \subseteq \mathfrak{U}$ for all $x \in \mathfrak{U}$. Now choose $x \in \mathfrak{U}$ such that $L_x = L_o = I$, the identity transformation. Then $PSu \subseteq \mathfrak{U}$. Therefore $PSu = \mathfrak{U}$ and $Su = P^{-1}Su$ since $P$ and $S$ are invertible. Then for any $u \in \mathfrak{U}$, $LuQSu \subseteq P^{-1}Su = Su$ and so $Su$ is a left ideal of $\mathfrak{U}$. Therefore either $\mathfrak{U} = (0)$ or $\mathfrak{U}$. Consequently $\mathfrak{U}$ is isotopically left simple; similarly it is isotopically right simple.

REMARK. Bruck has shown in [2] that left and right isotopic simplicity follow from left and right simplicity if the algebra has a unit element. The proof has been given here for sake of completeness.

THEOREM 4. Suppose that $f$ is non-degenerate and that $n \geq 2$. Let $\mathfrak{G}$ be the group of linear transformations of $\mathfrak{B}$ which keep $f$ invariant. Then the group of automorphisms of $\mathfrak{U}$ is isomorphic to $\mathfrak{G}$. In case $k$ is the field of real numbers the Lie group $\mathfrak{G}$ has for its Lie algebra the Lie algebra of derivations of $\mathfrak{U}$.

Proof. Let $\varphi$ be an automorphism of $\mathfrak{U}$. It is understood that $\varphi$ is a $k$-automorphism so that $\varphi$ keeps scalar multiples of $e_0$ fixed. Suppose $\varphi e_i = \lambda_i e_0 + v_i$ where $\lambda_i \in k, v_i \in \mathfrak{B}$ and $i = 1, 2, \ldots, n$. Then each product $\varphi e_i \cdot \varphi e_j = \mu_i e_0 + \lambda_i v_j + \lambda_j v_i, \mu_i, j \in k$, must be a scalar multiple of $e_0$. Therefore $\lambda_i v_j + \lambda_j v_i = 0$ and so $\varphi (\lambda_i e_i + \lambda_j e_j - 2\lambda_i \lambda_j e_0) = 0$, which implies that $\lambda_i = \lambda_j = 0$ if $i \neq j$. Therefore $\varphi \mathfrak{B} \subseteq \mathfrak{B}$. Then $\varphi e_i \cdot \varphi e_j = f(\varphi e_i, \varphi e_j)e_0 = \varphi (e_i, e_j) = \varphi f(e_i, e_j)e_0 = f(e_i, e_j)e_0$ for $i, j = 1, \ldots, n$. Therefore $f(\varphi e_i, \varphi e_j) = f(e_i, e_j)$ for $i, j = 1, \ldots, n$. Therefore the restriction of $\varphi$ to $\mathfrak{B}$ is an element of $\mathfrak{G}$. Conversely any element of $\mathfrak{G}$ can be extended uniquely to an automorphism of $\mathfrak{U}$. Thus $\mathfrak{G}$ is isomorphic to the group of automorphisms of $\mathfrak{U}$. Note that if these two groups are
realized as groups of matrices with respect to the given basis, then
the isomorphism is trivially birational and biregular in the sense of al-
gebraic geometry, so that the groups are isomorphic as algebraic groups.
The last statement of the theorem follows from a classical result in

**Theorem 5** (Wedderburn Principal Theorem). Suppose that \( f \) is
degenerate and either symmetric or alternating. Then \( \mathfrak{A} \) has a semisim-
ple subalgebra \( \mathfrak{A}_0 \) and a nilpotent ideal \( \mathfrak{N} \) such that \( \mathfrak{A} = \mathfrak{A}_0 + \mathfrak{N} \) (vec-
tor space direct sum).

**Proof.** If \( f \) is identically zero take \( \mathfrak{N} = \mathfrak{B} \) and \( \mathfrak{A}_0 \) to be the sub-
algebra spanned by \( e_0 \). Otherwise let \( \mathfrak{N}_0 \) be the set of elements \( u \in \mathfrak{B} \)
such that \( f(u, v) = 0 \) for all \( v \in \mathfrak{B} \). Choose a basis \( e_1, \ldots, e_{r+1}, \ldots, e_n \)
for \( \mathfrak{B} \) such that \( e_{r+1} \cdots, e_n \) span \( \mathfrak{N}_0 \). Suppose first that \( r \geq 2 \). Then
\( e_0, e_1, \ldots, e_r \) span a subalgebra \( \mathfrak{A}_0 \) which is isotopically left and right
simple by Theorem 3. Taking \( \mathfrak{N} = \mathfrak{N}_0 \) it follows that \( \mathfrak{A} = \mathfrak{A}_0 + \mathfrak{N} \)
with \( \mathfrak{N} \) a nilpotent ideal of index two. Now suppose \( r = 1 \). Then
\( e_0^2 = \lambda e_0 \) where \( \lambda \neq 0 \) in \( k \). If the subalgebra spanned by \( e_0 \) and \( e_1 \)
is semisimple, then \( \mathfrak{A}_0 \) and \( \mathfrak{N} \) may be taken as before. Otherwise, suppose
that \( e_0 + \beta e_1, \beta \neq 0 \) in \( k \), spans the one-dimensional radical of this sub-
algebra. Then take \( \mathfrak{N} \) to be the ideal of \( \mathfrak{A} \) spanned by \( e_0 + \beta e_1, e_2,
\ldots, e_n \) and \( \mathfrak{A}_0 \) to be the subalgebra spanned by \( e_0 \).

**Remark.** The use of the terms "semisimple" and "nilpotent ideal"
does not seem yet to be standardized in the literature on non-associative
algebras. Although in the present case all of the customary interpre-
tations of these terms are equivalent, nevertheless it desirable to give
explicit definitions. An algebra is said to be semisimple if it is a direct
sum of simple algebras, none of which is the zero algebra of dimension
1. An ideal is said to be nilpotent if there is an integer \( m > 0 \) such
that every product of \( m \) elements of the ideal, irrespective of the
manner of bracketing, is zero.

**References**

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Robert (Yisrael) John Aumann, *Acceptable points in games of perfect information* ................................................................. 381
A. V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them* ........................................ 419
Dallas O. Banks, *Bounds for the eigenvalues of some vibrating systems* .......... 439
Billy Joe Boyer, *On the summability of derived Fourier series* .................. 475
Robert Breusch, *An elementary proof of the prime number theorem with remainder term* .................................................. 487
Edward David Callender, Jr., *Hölder continuity of n-dimensional quasi-conformal mappings* .............................................. 499
L. Carlitz, *Note on Alder’s polynomials* ........................................ 517
P. H. Doyle, III, *Unions of cell pairs in $E^3$* ....................................... 521
James Eells, Jr., *A class of smooth bundles over a manifold* .................. 525
Shaul Foguel, *Computations of the multiplicity function* ....................... 539
James G. Glimm and Richard Vincent Kadison, *Unitary operators in $C^*$-algebras* .................................................. 547
Hugh Gordon, *Measure defined by abstract $L_p$ spaces* ....................... 557
Robert Clarke James, *Separable conjugate spaces* .............................. 563
William Elliott Jenner, *On non-associative algebras associated with bilinear forms* .................................................. 573
Harold H. Johnson, *Terminating prolongation procedures* ....................... 577
John W. Milnor and Edwin Spanier, *Two remarks on fiber homotopy type* .. 585
Donald Alan Norton, *A note on associativity* .................................... 591
Ronald John Nunke, *On the extensions of a torsion module* .................. 597
Joseph J. Rotman, *Mixed modules over valuations rings* ....................... 607
A. Sade, *Théorie des systèmes demosiens de groupoï des* ...................... 625
Wolfgang M. Schmidt, *On normal numbers* ....................................... 661
Berthold Schweizer, Abe Sklar and Edward Oakley Thorp, *The metrization of statistical metric spaces* ................................. 673
John P. Shanahan, *On uniqueness questions for hyperbolic differential equations* .................................................. 677
A. H. Stone, *Sequences of coverings* ............................................ 689
Edward Oakley Thorp, *Projections onto the subspace of compact operators* . 693
L. Bruce Treybig, *Concerning certain locally peripherally separable spaces* . 697
Milo Wesley Weaver, *On the commutativity of a correspondence and a permutation* .................................................. 705
David Van Vranken Wend, *On the zeros of solutions of some linear complex differential equations* .................................................. 713
Fred Boyer Wright, Jr., *Polarity and duality* ................................... 723