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If $\mathfrak{V}_0$ is a vector space over a field k, then with any non-degenerate bilinear form f_0 on $\mathfrak{V}_0 \times \mathfrak{V}_0$ is associated the group $\mathfrak{G}$ of linear transformations of $\mathfrak{V}_0$ which keep f_0 invariant. In this paper a procedure is given for associating with such a bilinear form an algebra $\mathfrak{A}$, non-associative in general, whose automorphism group is isomorphic to $\mathfrak{G}$ and which is right and left simple provided $\mathfrak{V}_0$ has dimension at least 2. In case k is the field of real numbers, then $\mathfrak{G}$ is a Lie group and its Lie algebra is the Lie algebra of derivations of $\mathfrak{A}$. In case the form f_0 is degenerate, and either symmetric or alternating, then the analogue of the Wedderburn Principal Theorem holds for $\mathfrak{A}$. The results obtained apply, in particular, to the orthogonal and symplectic groups.

Let $\mathfrak{V}_0$ be a vector space of dimension n over a field k with basis $u_1, \cdots, u_n$. It is assumed that $\lambda v = v\lambda$ for all $v \in \mathfrak{V}_0$ and $\lambda \in k$. Suppose f_0 is a bilinear form on $\mathfrak{V}_0 \times \mathfrak{V}_0$. Define $\mathfrak{A}$ to be the algebra over k with basis $e_0, e_1, \cdots, e_n$ and multiplication table $e_0^2 = e_0$, $e_i \cdot e_0 = e_0 \cdot e_i = e_i$, $e_i \cdot e_j = f(e_i\, e_j)e_0$ for $i, j = 1, 2, \cdots, n$, where $f(e_i, e_j) = f_0(u_i, u_j)$. Let $\mathfrak{V}$ be the subspace of $\mathfrak{A}$ spanned be $e_1, \cdots, e_n$. Then f is a bilinear form on $\mathfrak{V} \times \mathfrak{V}$.

THEOREM 1. *Suppose that f is non-degenerate and that $n \geqq 2$. Then $\mathfrak{A}$ is right and left simple.*

Proof. Let $\mathfrak{U}$ be a non-zero left ideal of $\mathfrak{A}$ and let u be a non-zero element of $\mathfrak{U}$. Suppose first that $u \in \mathfrak{V}$. Then there exists an element $v \in V$ such that $f(v, u) \neq 0$. Then $v \cdot u = f(v, u)e_0$. Therefore $e_0 \in \mathfrak{U}$ and so $\mathfrak{U} = \mathfrak{A}$. Next suppose $u = \alpha e_0 + v$ where $\alpha \neq 0$ in k and $v \in V$. Then one can assume $\alpha = 1$. Since $n \geqq 2$ it follows that $e_1 \cdot u = e_1 + \lambda_1 e_0$ and $e_2 \cdot u = e_2 + \lambda_2 e_0$ where $\lambda_1, \lambda_2 \in k$. If $\lambda_1 = 0$ then $e_1 \in U$ and the first part of the proof applies; similarly if $\lambda_2 = 0$. Consequently one can suppose $\lambda_1\lambda_2 \neq 0$. Then $\lambda_2 e_1 u - \lambda_1 e_2 u = \lambda_2 e_1 - \lambda_1 e_2$ is a non-zero element in $\mathfrak{U} \cap \mathfrak{V}$. Thus the first part of the proof again applies and so $\mathfrak{U} = \mathfrak{A}$. Therefore $\mathfrak{A}$ is left simple; similarly $\mathfrak{A}$ is right simple.

If $\mathfrak{A}$ is any (non-associative) algebra over k then left (right) multiplication by an element $a \in \mathfrak{A}$ determines a linear transformation $L_a(R_a)$ of the underlying vector space of $\mathfrak{A}$ by $a \cdot u = L_a u (u \cdot a = R_a u)$, $u \in \mathfrak{A}$. The set of linear transformations L_a (R_a) for $a \in \mathfrak{A}$ generate an associative algebra $L(\mathfrak{A})$ $(R(\mathfrak{A}))$ over k. The algebras $L(\mathfrak{A})$ and $R(\mathfrak{A})$ together

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generate the transformation algebra $T(\mathfrak{A})$.

THEOREM 2. *If f is non-degenerate and $n \geqq 2$ then $L(\mathfrak{A}) = R(\mathfrak{A}) = T(\mathfrak{A}) = [k]_{n+1}$.*

Proof. The proof of Theorem 1 shows that for any $u \neq 0$ in $\mathfrak{A}$ there is an element of $L(\mathfrak{A})$ mapping u into any arbitrarily assigned element of $\mathfrak{A}$. Therefore $L(\mathfrak{A}) = [k]_{n+1}$; similarly for $R(\mathfrak{A})$, and so also for $T(\mathfrak{A})$.

Albert has introduced in [1] the concept of isotopy of non-associative algebras. Suppose $\mathfrak{A}$ is an algebra with left multiplications L_a defined by $a \cdot u = L_a u$. Then an isotope of $\mathfrak{A}$ is an algebra $\mathfrak{A}^0$ with the same underlying vector space and multiplication defined by $a \circ u = PL_{aQ}Su$ where P, Q, S are invertible linear transformations of the underlying vector space of $\mathfrak{A}$. An algebra $\mathfrak{A}$ is said to be isotopically left (right) simple if every isotope of $\mathfrak{A}$ is left (right) simple.

THEOREM 3. *Suppose f is non-degenerate and that $n \geqq 2$. Then $\mathfrak{A}$ is isotopically left and right simple.*

Proof. Suppose $\mathfrak{U}$ is a subspace of $\mathfrak{A}$ such that $PL_{xQ}S\mathfrak{U} \subseteqq \mathfrak{U}$ for all $x \in \mathfrak{A}$. Now choose $x \in \mathfrak{A}$ such that $L_{xQ} = L_{e_0} = I$, the identity transformation. Then $PS\mathfrak{U} \subseteqq \mathfrak{U}$. Therefore $PS\mathfrak{U} = \mathfrak{U}$ and $S\mathfrak{U} = P^{-1}\mathfrak{U}$ since P and S are invertible. Then for any $u \in \mathfrak{A}$, $L_{uQ}S\mathfrak{U} \subseteqq P^{-1}\mathfrak{U} = S\mathfrak{U}$ and so $S\mathfrak{U}$ is a left ideal of $\mathfrak{A}$. Therefore either $\mathfrak{U} = (0)$ or $\mathfrak{A}$. Consequently $\mathfrak{A}$ is isotopically left simple; similarly it is isotopically right simple.

REMARK. Bruck has shown in [2] that left and right isotopic simplicity follow from left and right simplicity if the algebra has a unit element. The proof has been given here for sake of completeness.

THEOREM 4. *Suppose that f is non-degenerate and that $n \geqq 2$. Let $\mathfrak{G}$ be the group of linear transformations of $\mathfrak{B}$ which keep f invariant. Then the group of automorphisms of $\mathfrak{A}$ is isomorphic to $\mathfrak{G}$. In case k is the field of real numbers the Lie group $\mathfrak{G}$ has for its Lie algebra the Lie algebra of derivations of $\mathfrak{A}$.*

Proof. Let φ be an automorphism of $\mathfrak{A}$. It is understood that φ is a k-automorphism so that φ keeps scalar multiples of e_0 fixed. Suppose $\varphi e_i = \lambda_i e_0 + v_i$ where $\lambda_i \in k$, $v_i \in \mathfrak{B}$ and $i = 1, 2, \cdots, n$. Then each product $\varphi e_i \cdot \varphi e_j = \mu_{ij} e_0 + \lambda_i v_j, + \lambda_j v_i, \mu_{ij} \in k$, must be a scalar multiple of e_0. Therefore $\lambda_i v_j + \lambda_j v_i = 0$ and so $\varphi(\lambda_j e_i + \lambda_i e_j - 2\lambda_i\lambda_j e_0) = 0$, which implies that $\lambda_i = \lambda_j = 0$ if $i \neq j$. Therefore $\varphi\mathfrak{B} \subseteq \mathfrak{B}$. Then $\varphi e_i \cdot \varphi e_j = f(\varphi e_i, \varphi e_j)e_0 = \varphi(e_i \cdot e_j) = \varphi f(e_i, e_j)e_0 = f(e_i, e_j)e_0$ for $i, j = 1, \cdots, n$. Therefore $f(\varphi e_i, \varphi e_j) = f(e_i, e_j)$ *for* $i, j = 1, \cdots, n$. Therefore the restriction of φ to $\mathfrak{B}$ is an element of $\mathfrak{G}$. Conversely any element of $\mathfrak{G}$ can be extended uniquely to an automorphism of $\mathfrak{A}$. Thus $\mathfrak{G}$ is isomorphic to the group of automorphisms of $\mathfrak{A}$. Note that if these two groups are realized as groups of matrices with respect to the given basis, then the isomorphism is trivially birational and biregular in the sense of algebraic geometry, so that the groups are isomorphic as algebraic groups. The last statement of the theorem follows from a classical result in the theory of Lie groups (cf. [3] p. 137).

THEOREM 5 (Wedderburn Principal Theorem). *Suppose that f is degenerate and either symmetric or alternating. Then $\mathfrak{A}$ has a semisimple subalgebra $\mathfrak{A}_0$ and a nilpotent ideal $\mathfrak{N}$ such that $\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{N}$ (vector space direct sum).*

Proof. If f is identically zero take $\mathfrak{N} = \mathfrak{B}$ and $\mathfrak{A}_0$ to be the subalgebra spanned by e_0. Otherwise let $\mathfrak{N}_0$ be the set of elements $u \in \mathfrak{B}$ such that $f(u, v) = 0$ for all $v \in \mathfrak{B}$. Choose a basis $e_1, \cdots, e_{r+1}, \cdots, e_n$ for $\mathfrak{B}$ such that $e_{r+1} \cdots, e_n$ span $\mathfrak{N}_0$. Suppose first that $r \geqq 2$. Then $e_0, e_1, \cdots, e_r$ span a subalgebra $\mathfrak{A}_0$ which is isotopically left and right simple by Theorem 3. Taking $\mathfrak{N} = \mathfrak{N}_0$ it follows that $\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{N}$ with $\mathfrak{N}$ a nilpotent ideal of index two. Now suppose $r = 1$. Then $e_1^2 = \lambda e_0$ where $\lambda \neq 0$ in k. If the subalgebra spanned by e_0 and e_1 is semisimple, then $\mathfrak{A}_0$ and $\mathfrak{N}$ may be taken as before. Otherwise, suppose that $e_0 + \beta e_1, \beta \neq 0$ in k, spans the one-dimensional radical of this subalgebra. Then take $\mathfrak{N}$ to be the ideal of $\mathfrak{A}$ spannned by $e_0 + \beta e_1$, e_2, $\cdots, e_n$ and $\mathfrak{A}_0$ to be the subalgebra spanned by e_0.

REMARK. The use of the terms "semisimple" and "nilpotent ideal" does not seem yet to be standardized in the literature on non-associative algebras. Although in the present case all of the customary interpretations of these terms are equivalent, nevertheless it desirable to give explicit definitions. An algebra is said to be *semisimple* if it is a direct sum of simple algebras, none of which is the zero algebra of dimension 1. An ideal is said to be *nilpotent* if there is an integer $m > 0$ such that every product of m elements of the ideal, irrespective of the manner of bracketing, is zero.

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