TERMINATING PROLONGATION PROCEDURES

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ON TERMINATING PROLONGATION PROCEDURES

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In the classical treatments [3] of systems of differential equations there are two outstanding techniques—the Cauchy-Kowalewski theorem and completely integrable systems (the latter is really a special case of the former [1, p. 77]). In terms of systems of differential forms the Cauchy-Kowalewski theorem becomes the Cartan-Kahler theorem, and systems with independent variables which satisfy its conditions are called involutive.

Many systems are not involutive, and the central problem of prolongation theory is to construct a procedure by which one can reduce every system to an equivalent involutive system. For total prolongations Kuranishi’s theorem [4, p. 44] gives a precise answer to the question of when total prolongations will lead to involutive systems. If \( S \) is the initial system in euclidean space \( \mathbb{E}^n \), \( P^g(S) \) the \( g^{th} \) total prolongation in the space \( \mathbb{R}^{g_1} \), then for all points \( x \in \mathbb{E}^n \), except possibly on a proper subvariety, there is a number \( g_0 \) such that if \( g \geq g_0 \) and \( y \in R_g \) is a point over \( x \), then \( P^g(S) \) is involutive at \( y \) if and only if \( y \) is an ordinary integral point [4, p. 7] and the 1-forms of \( P^g(S) \) do not imply any dependencies among the independent variables at integral points in a neighborhood of \( y \). Then \( y \) is called a normal point.

The first part of this paper deals with an application of this theorem to certain types of differential systems. We show that under certain conditions the total prolongation process must result in normal points if there are to be any solutions. An application of this leads to a theorem often used in differential geometry [2, p. 14].

The second section is concerned with what can be done if normal points are not obtained for \( P^g(S) \) as is the case with an example of Kuranishi. Here we must distinguish two cases. If \( P^g(S) \) does not contain ordinary integral points, so that its 0-forms are not a regular system of equations [4, p. 7] the Cartan-Kahler theory does not apply. Let us call such systems singular. We shall not consider this aspect of the problem in this paper.

If, however, the problem lies in a dependency among the independent variables implied by 1-forms of \( P^g(S) \), at generic integral points, one would naturally think of restricting the system to those points where dependencies do not occur, since solutions must lie only in these points. Thus one obtains a sort of partial prolongation which could in turn be prolonged. Such a procedure was certainly what Cartan and Kuranishi had in mind. However, it is not clear that the process will ever result

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in an involutive system. One might conceivably go on obtaining non-normal systems indefinitely.

Kuranishi has recently proved a generalization of his prolongation theorem which is used to show that the above procedure does in fact ultimately stop, barring the occurrence of singular systems somewhere along the line.

The first section of this paper is part of the author's doctoral thesis at the University of California at Berkeley, written under the direction of Professor Harley Flanders to whom the author would like to record here his appreciation.

All functions, forms, and manifolds are assumed to be real analytic.

1. Kuranishi's fundamental theorem [4, p. 44] concerns a certain general type of differential system (called normal) which is generated by 1-forms \( \theta^\alpha, \alpha = 1, \ldots, \alpha_i \). If \( \omega^1, \ldots, \omega^p \) is a basis of a system of independent variables and \( \pi^1, \ldots, \pi^m \) any other 1-forms to fill out a basis, then the \( \theta^\alpha \) are normal if

\[
d \theta^\alpha \equiv \sum_{i=1}^p \sum_{\lambda=1}^m A_{\varphi, i \lambda} \omega^i \wedge \pi^\lambda + \sum_{i=1}^p \sum_{j=1}^p \frac{1}{2} B_{\varphi, i j} \omega^i \wedge \omega^j
\]

modulo \( \theta^a \). Suppose that these are defined on \( E^n \) where \( n = \alpha_i + p + m \) of variables \( x^1, \ldots, x^n \). Then \( R_g \) is the euclidean space of variables

\[
x^i, u^{i_1}_1, u^{i_1 i_2}_2, \ldots, u^{i_1 i_2 \ldots i_g}_g,
\]

where \( j = 1, \ldots, n; i, \ldots, i_g = 1, \ldots, p; \lambda = 1, \ldots, m, \) and the \( u^{i_1 i_2 \ldots i_g}_g \) are symmetric in the lower indices.

Then \( P^a(S) \) can be taken to be the system on \( R_g \) generated by the 1-forms

\[
\begin{align*}
\theta^a, \\
\pi^\lambda \quad d \pi^\lambda - \sum_{j=1}^n u^{\lambda}_j \omega^j, \\
\pi^\lambda \quad d u^{\lambda}_1 - \sum_{j=1}^n u^{\lambda}_{i_1} \omega^j, \\
\vdots \\
\pi^\lambda \quad d u^{\lambda}_{i_1 i_2 \ldots i_{g-1}} - \sum_{j=1}^n u^{\lambda}_{i_1 i_2 \ldots i_{g-1} j} \omega^j,
\end{align*}
\]

and certain functions

\[
\theta_{\psi; i_1 i_2 \ldots i_t}, \quad t \leq g - 1.
\]

It turns out that for \( t \leq g - 2 \),

\[
d \theta_{\psi; i_1 i_2 \ldots i_t} \equiv 0 \quad \text{modulo } \pi_g,
\]

while
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\[ d\theta_{\phi;ij;1,...,k,...,g-1} \equiv \sum_{k=1}^{m} (A_{\phi;ij}(du^k_{1,k},...,du^k_{g-1}) - A_{\phi;ij}(du^k_{1,k},...,du^k_{g-1})) \]
\[ + \sum_{k=1}^{p} B_{\phi;ij,k,...,g-1} \omega^k, \]
modulo \( \pi_g \).

These \( B \)'s are defined inductively by
\[
B_{\phi;ij;1,...,k,...,g-1} = \sum_{k=1}^{m} [(D_k A_{\phi;ij})du^k_{1,k},...,du^k_{g-1}]
-D_k(B_{\phi;ij;1,...,k,...,g-1}) + D_k(B_{\phi;ij;1,...,k,...,g-1}),
\]
where \( D_k F \) is defined as follows.

If \( F \) is any function on \( R_{t-1} \) it can be considered to be a function on \( R_s \) for all \( s \geq t - 1 \). If we form \( dF \), then modulo \( \pi_s \), when \( s \geq t \), \( dF \) involves only \( \omega^i, \..., \omega^p \):
\[
dF \equiv \sum_{k=1}^{p} F_k \omega^k \quad \text{modulo } \pi_s,
\]
and the \( F_k \) are independent of \( s \) so long as \( s \geq t \). Then one defines \( D_k F \) to be \( F_k \). \( D_k F \) is a function on \( R_t \).

**Theorem 1.** Let \( S \) be a normal system where
1. The \( A_{\phi;ij} \) are constants,
2. \( dB_{\phi;ij} \equiv 0 \) modulo \( (\omega^i, \omega^p) \).

Then if \( P^s(S) \) is non-singular for all \( g \), there is a \( g_0 \) such that \( P^s(S) \) is involutive for all \( g \geq g_0 \) at ordinary integral points, or else there exist no solutions.

**Proof.** If an ordinary integral point \( y \in R_g \) is not normal, then there must be a dependency among \( \omega^i, \..., \omega^p \) implied by the 1-forms of \( P^s(S) \) at integral points \( y_i \), arbitrarily near \( y \). This can happen only if there is a relation of the type
\[
\Sigma \Gamma^\phi;ij;1,...,g-1(y_i)(d\theta_{\phi;ij;1,...,k,...,g-1})_{y_i} \equiv 0 \quad \text{modulo } (\omega^i),
\]
where the left side does not vanish identically. This can only happen if
\[
0 = \Sigma \Gamma^\phi;ij;1,...,g-1(y_i)(A_{\phi;ij}(y_i)(du^i_{1,i},...,du^i_{g-1})_{y_i}) - A_{\phi;ij}(y_i)(du^i_{1,i},...,du^i_{g-1})_{y_i},
\]
while for some \( k \),
\[
\Sigma \Gamma^\phi;ij;1,...,k-g-1(y_i)B_{\phi;ij;1,...,k-g-1}(y_i) \neq 0.
\]
Since the \( A \) depend only on \( x^1, \..., x^n \), we can choose the \( \Gamma \) to be functions of \( x^1, \..., x^n \).

Now, the functions in \( P^{s+1}(S) \) have the form
\[ \theta_{\psi; i j; k_1 \ldots k_g} = \sum_{\lambda=1}^{m} (A_{\psi; i \lambda} u^\lambda_{k_1 \ldots k_g} - A_{\psi; j \lambda} u^\lambda_{k_1 \ldots k_g}) + B_{\psi; i j; k_1 \ldots k_g}. \]

Hence we have in \( P^{g+1}(S) \) the function which is not in \( P^g(S) \),
\[ \sum I_{\psi; i j; k_1 \ldots k_g} = \sum I_{\psi; i j; k_1 \ldots k_g} B_{\psi; i j; k_1 \ldots k_g}. \]

Consider now these \( B \). Since the \( A \) are constants,
\[ B_{\psi; i j; k_1 \ldots k_1} = D_{k_1} B_{\psi; i j; k_1 \ldots k_1 - 1} \]

where
\[ dB_{\psi; i j; k_1 \ldots k_1 - 1} = \sum_{i=1}^{p} D_{k_i} B_{\psi; i j; k_1 \ldots k_1 - 1} \omega^k \]
modulo \( \pi_t \).

By assumption (2), \( dB_{\psi; i j} \) have the form
\[ dB_{\psi; i j} = \sum_{k=1}^{p} C_{\psi; i j; k} \omega^k + \sum_{\beta=1}^{q} E_{\psi; i j; \beta} \theta^\beta, \]
\[ = \sum_{k=1}^{p} C_{\psi; i j; k} \omega^k \]
modulo \( \pi_t \), hence
\[ B_{\psi; i j; k} = D_{k} B_{\psi; i j} = C_{\psi; i j; k} \]
are functions of \( x^1, \ldots, x^n \) alone. Obviously \( dB_{\psi; i j; k} = 0 \) modulo \( (\theta^s, \omega^t) \) also, so the argument can be repeated to show that the functions
\[ B_{\psi; i j; k} \]
depend only on \( x^1, \ldots, x^n \). But that means that
(I)
\[ \sum I_{\psi; i j; k_1 \ldots k_g} B_{\psi; i j; k_1 \ldots k_g} \]
is a function in \( P^{g+1}(S) \), not in \( P^g(S) \), and dependent only on \( x^1, \ldots, x^n \).

Now, to any integral manifold \( I \) of \( S \) there corresponds a unique integral manifold \( I^{g+1} \) of \( P^{g+1}(S) \), such that if \( \rho^{g+1} \) is the natural fibre bundle mapping on \( R_{g+1} \) to \( E^n \), then \( \rho^{g+1} (I^{g+1}) = I \) [4, p. 15]. \( I^{g+1} \) must annihilate the function (I). Since it is a function of \( x^1, \ldots, x^n \) alone, \( I \) must itself annihilate it.

We conclude; if there exist ordinary integral points in \( R_g \) where \( P^g(S) \) is not normal, then the manifold of integral points of \( S \) where solutions can occur must satisfy an additional condition to any imposed by \( P^t(S), t < g \). Clearly, this can happen at most \( n - p \) times if there are to be solutions.

Since the \( A_{\psi; i \lambda} \) are constants, every point of \( E^n \) is regular of order \( 0 \) [4, p. 36], so by Kuranishi's fundamental theorem there exists an in-
integer $g_1$ such that if $y$ is an ordinary integral point in $R_g$ for $g \geq g_1$, then $P^g(S)$ is involutive at $y$ if and only if $y$ is normal. Taking $g_0 = g_1 + (n - p)$ one obtains the theorem.

Next an application of this theorem will be made to a certain type of system of differential equations.

Let $E^n$ be the euclidean space of variables $x^1, \ldots, x^n, z^1, \ldots, z^m$. Consider the problem of finding $m$ functions $f^g(x^1, \ldots, x^n) = z^g$ which will satisfy a given set of first order partial differential equations

$$\frac{\partial z^\alpha}{\partial x^i} = \psi^\alpha_i(x, z), \quad \alpha = 1, \ldots, m; \quad i = 1, \ldots, p.$$  

In terms of differential forms this is the problem of finding integral manifolds of the system $S$ generated by the 1-forms

$$\theta^a = dz^a - \sum_{i=1}^p \psi^a_i(x, z)dx^i$$

with independent 1-forms $dx^1, \ldots, dx^n$. Here there are no $\pi^\lambda$. Then

$$d\theta^a = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \left( \sum_{\beta=1}^m \frac{\partial \psi^a_j}{\partial z^\beta} \psi^\beta_i - \sum_{\beta=1}^m \frac{\partial \psi^\beta_i}{\partial z^\beta} \psi^a_j + \frac{\partial \psi_i^a}{\partial x^j} - \frac{\partial \psi_i^j}{\partial x^i} \right) dx^i \wedge dx^j \mod (\theta^a).$$

If then

$$B_{\alpha;ij} = \sum_{\beta=1}^m \frac{\partial \psi^a_j}{\partial z^\beta} \psi^\beta_i - \sum_{\beta=1}^m \frac{\partial \psi_i^a}{\partial z^\beta} \psi^\beta_j + \frac{\partial \psi_i^a}{\partial x^j} - \frac{\partial \psi_j^a}{\partial x^i}$$

one can deduce the following theorem from the nature of the forms

$$d\theta_{\psi^{i,j};k_1,\ldots,k_\ell} = B_{\psi^{i,j};k_1,\ldots,k_\ell} \omega^k.$$  

**Theorem 2.** In order that the system of differential equations

$$\frac{\partial z^\alpha}{\partial x^i} = \psi^\alpha_i(x, z)$$

have a solution, given that the equations

$$B_{\psi^{i,j};k_1,\ldots,k_\ell} = 0, \quad t \leq g$$

are non-singular for all $g$, it is necessary and sufficient that for all $\psi, i, j, k_1, \ldots, k_\ell$,

$$B_{\psi^{i,j},k_1,\ldots,k_\ell} \equiv 0 \mod (B_{\psi^{i,j};k_1,\ldots,k_\ell}, t \leq g - 1)$$

for some $g \leq m - 1$. [See 2, p. 14].
2. In Theorem 1, the prolongation process had to yield an involutive system because whenever a non-normal prolongation occurred, this implied additional restrictions on the original system. In general this need not happen. Kuranishi gives an example of a system in which $P^g(S)$ is not normal for any $g \geq 1$ [4, p. 45].

Normality at integral points $y$ of $P^g(S)$ involves two conditions; the set of 0-forms of $P^g(S)$ which define $y$ must define a regular system of equations at $y$, and the 1-forms of $P^g(S)$ must imply no relations among the independent variables at integral points near $y$. This paper will ignore the first problem. It would seem to call for a more delicate approach to the Cartan-Kahler theorem. Let $y$ be a non-normal integral point of $P^g(S)$ such that for all integral points $y$ near $y$ there is a dependency of the type

$$\sum_{i=1}^{p} A_i(y_i)(\omega^i)_{y_i}$$

in $P^g(S)$. Then obviously solutions can occur only at points $y_i$ where $A_i(y_i) = A_2(y_i) = \cdots = A_p(y_i) = 0$. Hence a natural step to solving the system would be to add $A_1, \cdots, A_p$ as 0-forms to the system $P^g(S)$. One would obtain a system having the same solutions as $P^g(S)$.

Observe also that if $P^g(S)$ contains a 0-form which is a function on $R_{g-1}$, obviously any solution of $P^{g-1}(S)$ must annihilate that function; hence, adding it to $P^{g-1}(S)$ would generate a system having the same solutions as $P^{g-1}(S)$.

We introduce the following definition: let the system $T$ in independent variables $x^1, \cdots, x^p$, and dependent variables $y^1, \cdots, y^r, z^1, \cdots, z^m$ be called complete if the 1-forms of $T$ contain no forms of the type $\sum A_i \omega^i$, where $\omega^1, \cdots, \omega^p$ is a basis of independent variables, $A_i$ not in $T$.

**Lemma.** Let $S$ be any system with independent variables $x^1, \cdots, x^p$, and dependent variables $z^1, \cdots, z^m$. Then there exists a sequence $\{S^g\}$ of differential systems $S^g$, closed, on $R_g$ such that

1. $S^g$ has the same solutions as $P^g(S)$,
2. $S^g$ is complete,
3. $P(S^{g-1}) \subseteq S^g$, and
4. the 0-forms of $S^g$ contain no functions on $R_{g-1}$ except those in $S^{g-1}$,
5. $S^g$ is generated by 0-forms, $\pi_g$, and their derivatives.

**Proof.** Let $X$ be the set of all sequences $\{T^g | g = 1, 2, \cdots\}$, where $T^g$ is a closed differential system on $R_g$ generated by 0-forms, $\pi_g$ and their derivatives and having the same solutions as $P^g(S)$ and $P(T^{g-1}) \subseteq T^g$. The elements of $X$ can be partially ordered by inclusion: $\{U^g\} \supseteq \{T^g\}$ if $U^g \supseteq T^g$ for all $g = 1, 2, \cdots$ If $A = \{\{T^g_a\} | a \in A\}$ is a nest in $X$,
then \( \{T^\circ\} \), where \( T^\circ \) is the closed differential system generated by 
\( U \{T^\circ_a \mid a \in A\} \), is in \( X \) and is \( \geq \) every element of \( A \). Hence, \( X \) contains 
a maximal element, \( \{S^\circ\} \). By definition, \( \{S^\circ\} \) satisfies (1) and (3). If 
\( S^h \) were not complete, one could add to \( S^h \) the coefficients of forms of the 
type \( \Sigma A_i \omega^i \) to obtain a still larger system \( S^h \), and \( \{S^\circ\} \), where 
\( S^\circ = S^g \) for \( g < h \), and \( S^\circ = P^{g-h}(S^h) \) for \( g \geq h \), would be properly 
greater than \( \{S^\circ\} \). Similarly, if condition (4) did not hold for some \( S^h \), we 
could enlarge \( S^{h+1} \). Hence the lemma.

The construction of such a sequence, given \( S \), could proceed as follows. Form \( P(S) \) and complete it in the obvious way to form a system 
\( T^1 \). If the resulting system involves any functions on \( R_n \) i.e., depending only 
on the coordinates of \( R_n \), add these to the system \( S \) and begin again. Otherwise, form \( P(T^1) \) and complete to form \( T^2 \). If \( T^2 \) contains functions on \( R_n \), add these to \( T_1 \) and begin again at that step. Observe 
that the addition of new functions to any one system on, say, \( R_n \), is 
limited by the dimension of \( R_n \), since each such addition reduces the 
dimension of the variety of integral points, which must have at least 
dimension \( p \) if there are to be any solutions at all.

Granted that such a sequence \( \{S^g\} \) as given in the lemma exists, it is still not clear whether any \( S^g \) is involutive. Of course, the 0-forms 
might not define a regular system of equations for the integral points. But barring this one can prove that for \( g \) sufficiently large, \( S^g \) is involutive. This follows from a recent extension of Kuranishi’s prolongation theorem [5, Theorem III. 1], where the required conditions are 
precisely those of the lemma.

**Theorem 3.** Given a differential system \( S \) with independent variable \( dx^1, \cdots, dx^p \), there exists a sequence \( \{S^g\} \) of closed differential 
systems, where \( S^g \) is on \( R_n \), \( g = 1, 2, \cdots \), which have the same solutions 
as \( P^0(S) \). Moreover, if for all \( g \geq g_0, S^g \) is non-singular, then there 
exists a \( g_i \) such that for \( g \geq g_i, P(S^{g-1}) = S^g \) and \( S^g \) is involutive.

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