

Pacific Journal of Mathematics

TWO REMARKS ON FIBER HOMOTOPY TYPE

JOHN W. MILNOR AND EDWIN SPANIER

TWO REMARKS ON FIBER HOMOTOPY TYPE

JOHN MILNOR AND EDWIN SPANIER

Section 1 of this note considers the normal sphere bundle of a compact, connected, orientable manifold M^n (without boundary) differentiably imbedded in euclidean space R^{n+k} . (These hypotheses on M^n will be assumed throughout § 1.) It is shown that if k is sufficiently large then the normal sphere bundle has the fiber homotopy type of a product bundle if and only if there exists an S -map from S^n to M^n of degree one (i.e. for some p there exists a continuous map of degree one from S^{n+p} to the p -fold suspension of M^n). The proof is based on the fact that the Thom space of the normal bundle is dual in the sense of Spanier-Whitehead [8] to the disjoint union of M^n and a point.

Section 2 studies the tangent sphere bundle of a homotopy n -sphere. This has the fiber homotopy type of a product bundle if and only if n equals 1, 3 or 7. The proof is based on Adams' work [1].

If X is a space, $S^k X$ will denote the k -fold suspension of X as in [8, 9]. If X has a base point x_0 , then $S_0^k X$ will denote the k -fold reduced suspension and is the identification space $S^k X/S^k x_0$ obtained from $S^k X$ by collapsing $S^k x_0$ to a point (to be used as base point for $S_0^k X$). There is a canonical homeomorphism $S_0^k X \approx S^k \times X/S^k \vee X$.

Two fiber bundles with the same fiber and with projections $p_1: E_1 \rightarrow B$, $p_2: E_2 \rightarrow B$ have the same *fiber homotopy type* [3, 4, 10] if there exist fiber preserving maps $f_i: E_i \rightarrow E_{3-i}$ and fiber preserving¹ homotopies $h_i: E_i \times I \rightarrow E_i$ such that $h_i(x, 0) = f_{3-i} f_i(x)$, $h_i(x, 1) = x$.

Let ξ denote an oriented $(k-1)$ -sphere bundle. The total space of ξ will be denoted by \dot{E} and the total space of the associated k -disk bundle will be denoted by E . The *Thom space* $T(\xi)$ is the identification space E/\dot{E} obtained from E by collapsing \dot{E} to a single point (to be used as base point for $T(\xi)$). The following are easily verified:

(A) If ξ_1, ξ_2 are $(k-1)$ -sphere bundles of the same fiber homotopy type, then $T(\xi_1), T(\xi_2)$ have the same homotopy type.

(B) If ξ is a product bundle, then $T(\xi)$ is homeomorphic to $S_0^k(B \cup p_0)$ (where $B \cup p_0$ is the disjoint union of B and a point, p_0 , which is taken as the base point of $B \cup p_0$).

1. The normal bundle. If X and Y are spaces we let $[X, Y]$ denote the set of homotopy classes of maps of X into Y and we let

Received May 19, 1959. The authors were supported by the Sloan Foundation and by the United States Air Force (Contract No. AF 49(638)-393 monitored by the Air Force Office of Scientific Research), respectively, during the period when this paper was in preparation.

¹ The phrase "fiber-preserving" means that $p_{3-i} f_i(x) = p_i(x)$ and $p_i h_i(x, t) = p_i(x)$.

$\{X, Y\}$ denote the set of S -maps of X into Y as in [8]. Thus, $\{X, Y\}$ is defined to be the direct limit of the sequence

$$[X, Y] \xrightarrow{S} [SX, SY] \xrightarrow{S} \cdots \xrightarrow{S} [S^p X, S^p Y] \xrightarrow{S} \cdots$$

There is a natural map

$$\phi: [X, Y] \longrightarrow \{X, Y\}$$

which assigns to every homotopy class $[f] \in [X, Y]$ the S -map $\{f\}$ represented by any map of $[f]$. The following gives a sufficient condition for ϕ to be onto $\{X, Y\}$.

LEMMA 1. *Let Y be a k -connected CW -complex ($k \geq 1$) and let X be a finite CW -complex with² $H^q(X) = 0$ for $q > 2k + 1$. Then $\phi([X, Y]) = \{X, Y\}$.*

Proof. It suffices to prove that under the hypotheses of the lemma the map $S: [X, Y] \rightarrow [SX, SY]$ is onto $[SX, SY]$ because then, for each $p \geq 0$, the map $S: [S^p X, S^p Y] \rightarrow [S^{p+1} X, S^{p+1} Y]$ is onto $[S^{p+1} X, S^{p+1} Y]$ (because $S^p Y$ is $(p + k)$ -connected and $H^q(S^p X) = 0$ for $q > 2k + p + 1$ and $2(k + p) + 1 \geq 2k + p + 1$).

Choose base points $x_0 \in X$, $y_0 \in Y$ and let $[X, Y]'$ denote the set of homotopy classes of maps $(X, x_0) \rightarrow (Y, y_0)$. Since Y is simply-connected the natural map $[X, Y]' \rightarrow [X, Y]$ is a 1-1 correspondence. Since X, Y are CW -complexes the collapsing maps $SX \rightarrow S_0 X$ and $SY \rightarrow S_0 Y$ are homotopy equivalences (Theorem 12 of [11]) so there are 1-1 correspondences

$$[S_0 X, S_0 Y] \approx [S_0 X, SY] \approx [SX, SY].$$

Since $S_0 Y$ is simply connected, we also have a 1-1 correspondence $[S_0 X, S_0 Y]' \approx [S_0 X, S_0 Y]$. Hence, it suffices to show that $S_0([X, Y]') = [S_0 X, S_0 Y]'$.

Let $\Omega S_0 Y$ denote the space of closed paths in $S_0 Y$ based at y_0 . There is a canonical 1-1 correspondence $[S_0 X, S_0 Y]' \approx [X, \Omega S_0 Y]'$ and a natural imbedding $Y \subset \Omega S_0 Y$ such that the map $S_0: [X, Y]' \rightarrow [S_0 X, S_0 Y]'$ corresponds to the injection (see § 9 of [7])

$$[X, Y]' \longrightarrow [X, \Omega S_0 Y]'$$

Hence, it suffices to show this injection is onto or, equivalently, that the natural injection (without base point condition) $[X, Y] \rightarrow [X, \Omega S_0 Y]$ is onto.

² When no coefficient group appears explicitly in the notation for a homology or cohomology group it is to be understood that the coefficient group is the group of integers. In dimension 0 the groups will be taken reduced.

Since Y is k -connected it follows from the suspension theorem (see § 7 of [9]) that

$$S_0: \pi_i(Y) \longrightarrow \pi_{i+1}(S_0 Y)$$

is 1-1 if $i \leq 2k$ and is onto if $i \leq 2k + 1$. Since S_0 corresponds to the injection map $\pi_i(Y) \rightarrow \pi_i(\Omega S_0 Y)$, this is equivalent to the statement that

$$\pi_i(\Omega S_0 Y, Y) = 0 \text{ for } i \leq 2k + 1 .$$

Since Y is simply-connected the groups $\pi_i(\Omega S_0 Y, Y)$ form a simple system for every i . Now the groups $H^i(X; \pi_i(\Omega S_0 Y, Y))$ vanish for every i because for $i \leq 2k + 1$ the coefficient group vanishes while for $i > 2k + 1$ the groups vanish because of the assumption on the cohomology of X . By Theorem 4.4.2 of [2] it follows that any map $X \rightarrow \Omega S_0 Y$ is homotopic to a map $X \rightarrow Y$, completing the proof.

REMARK. If in Lemma 1 we assume that $H^q(X) = 0$ for $q > 2k$, then a similar argument shows that ϕ is 1-1, however we shall not need this result.

Let $M^n \subset R^{n+k}$ be as in the introduction (i.e. M^n is a differentially imbedded manifold which is compact, connected, orientable, and without boundary). The following result relates the normal bundle of M^n to M^n itself by means of duality.

LEMMA 2. *Let ξ be the normal $(k - 1)$ -sphere bundle of M^n in R^{n+k} . Then the Thom space $T(\xi)$ is weakly $(n + k + 1)$ -dual to the disjoint union $M^n \cup p_0$.*

Proof. Regard S^{n+k} as the one point compactification of R^{n+k} . Let E be a closed tubular neighborhood of M^n and assume E is contained in a large disk D^{n+k} . Then $(D^{n+k}$ -interior E) is a deformation retract of $R^{n+k} - M^n = S^{n+k} - (M^n \cup (\text{point at infinity}))$. Using standard homotopy extension properties and the contractibility of D^{n+k} it follows that if \dot{E} denotes the boundary of E then

$$T(\xi) = E/\dot{E} = D^{n+k}/(D^{n+k} - \text{interior } E)$$

has the homotopy type of the suspension $S(D^{n+k} - \text{interior } E)$. Since $(D^{n+k} - \text{interior } E)$ is an $(n + k)$ -dual of $M^n \cup (\text{point at infinity})$, and the suspension of an $(n + k)$ -dual is an $(n + k + 1)$ -dual, this completes the proof.

REMARK. Lemma 2 shows that the S -type of $T(\xi)$ depends only on that of M^n . If k is sufficiently large this implies that the homotopy type of $T(\xi)$ depends only on that of M^n . This suggests the conjecture

that the fiber homotopy type of the normal bundle of any manifold $M^n \subset R^{n+k}$, k large, is completely determined by the homotopy type of M^n . A similar conjecture can be made for the tangent bundle.

THEOREM 1. *Let $M^n \subset R^{n+k}$ be as before and assume that $H_q(M^n) = 0$ for $q < r$ and that $k \geq \min(n - r + 2, 3)$. The following statements are equivalent:*

(1) *There is an S -map $\alpha \varepsilon \{S^n, M^n\}$ such that*

$$\alpha_*: H_n(S^n) \approx H_n(M^n).$$

(2) *The normal sphere bundle of $M^n \subset R^{n+k}$ has the fiber homotopy type of a product bundle.*

(3) *The disjoint union $M^n \cup p_0$ is weakly $(n + k + 1)$ -dual to $S_0^k(M^n \cup p_0)$.*

Proof. (1) \Rightarrow (2). Let N denote the complement in S^{n+k} of an open tubular neighborhood of M^n . Then N is $(n + k)$ -dual to M^n . The S -map α is $(n + k)$ -dual to an S -map $\beta \varepsilon \{N, S^{k-1}\}$ such that $\beta^*: H^{k-1}(S^{k-1}) \approx H^{k-1}(N)$. Since $H^p(N) \approx H_{n+k-p-1}(M^n)$, we see that $H^p(N) = 0$ if $p > n + k - r - 1$. Since S^{k-1} is $(k - 2)$ -connected, $k - 2 \geq 1$, and $k \geq n - r + 2$ (so $2(k - 2) + 1 \geq n + k - r - 1$), it follows from Lemma 1 that there is a map $f: N \rightarrow S^{k-1}$ representing β . Then $f^*: H^{k-1}(S^{k-1}) \approx H^{k-1}(N)$. Let \dot{E} be the boundary of N (so \dot{E} is the normal $(k - 1)$ -sphere bundle of M^n), and let F be a fiber of \dot{E} . Then the inclusion map $F \subset N$ induces an isomorphism $H^{k-1}(N) \approx H^{k-1}(F)$ (because by Corollaries III. 15 and I.5 of [10] or by Theorems 14 and 21 of [5] we have $H^{k-1}(\dot{E}) \approx H^{k-1}(M^n) + Z$ and the injection $H^{k-1}(N) \rightarrow H^{k-1}(\dot{E})$ maps isomorphically onto the second summand while the injection $H^{k-1}(\dot{E}) \rightarrow H^{k-1}(F)$ maps the second summand isomorphically.) Therefore, the map $f|_{\dot{E}}: \dot{E} \rightarrow S^{k-1}$ has the property that its restriction to a fiber F induces an isomorphism of the cohomology of S^{k-1} onto that of F so is a homotopy equivalence of F with S^{k-1} . This implies (by Corollary 2 on p. 121 of [3]) that \dot{E} has the same fiber homotopy type as a product bundle.

(2) \Rightarrow (3). By Lemma 2, $T(\xi)$ is weakly $(n + k + 1)$ -dual to $M^n \cup p_0$. If ξ is of the same fiber homotopy type as a product bundle, it follows from (A), (B) that $T(\xi)$ is of the same homotopy type as $S_0^k(M^n \cup p_0)$. Combining these two statements gives the result.

(3) \Rightarrow (1) assume $M^n \cup p_0$ is weakly $(n + k + 1)$ -dual to $S_0^k(M^n \cup p_0)$. The map $M^n \cup p_0 \rightarrow S^0$ collapsing each component of $M^n \cup p_0$ to a single point represents an S -map $\beta: S_0^k(M^n \cup p_0) \rightarrow S_0^k(S^0) = S^k$ such that $\beta^*: H^k(S^k) \approx H^k(S_0^k(M^n \cup p_0))$. By duality there is an S -map $\alpha \varepsilon \{S^n, M^n \cup p_0\}$ such that $\alpha_*: H_n(S^n) \approx H_n(M^n \cup p_0) \approx H_n(M^n)$. Since.

$$\{S^n, M^n \cup p_0\} \approx \{S^n, M^n\} + \{S^n, S^0\},$$

the result is proved.

As a corollary we obtain the following result proved by Massey [4].

COROLLARY. *Let M^n be a homology sphere. Then the normal bundle of M^n in R^{n+k} has the same fiber homotopy type as a product bundle.*

Proof. Since $r = n$, the case $k \geq 3$ follows from the theorem. For the cases $k = 1, 2$ it is well known that the normal bundle is, in fact, trivial.

REMARK. Puppe [6] calls a manifold "sphere-like" if the unstable group $\pi_{n+1}(SM^n)$ contains an element of degree one. (The group $\pi_n(M^n)$ can contain an element of degree one if and only if M^n is a homotopy sphere.) Theorem 1 shows that the normal sphere bundle of a sphere-like manifold $M^n \subset R^{n+k}$ has the fiber homotopy type of a product bundle provided k is sufficiently large. An example of a manifold with trivial normal bundle which is not sphere-like is provided by the real projective 3-space.

2. The tangent bundle. Let M^n be as above (i.e. compact, connected, orientable, and without boundary), but let E denote a closed tubular neighborhood of the diagonal in $M^n \times M^n$. If the tangent bundle has the fiber homotopy type of a product bundle, then there exists a map $\dot{E} \rightarrow S^{n-1}$ (where \dot{E} is the boundary of E) having degree one on each fiber. This gives rise to a map $(E, \dot{E}) \rightarrow (D^n, S^{n-1}) \rightarrow (S^n, \text{point})$ of degree one and, hence, to a map

$$M^n \times M^n \longrightarrow M^n \times M^n / (M^n \times M^n\text{-interior } E) = E/\dot{E} \longrightarrow S^n$$

which has degree (1, 1) (the degree is (1, 1) because a generator of $H^n(S^n)$ maps, under the homomorphism induced by the above composite, into a cohomology class of $M^n \times M^n$ dual under Poincaré duality to the diagonal class of $H_n(M^n \times M^n)$).

THEOREM 2. *Suppose that M^n has the homotopy type of an n -sphere. Then the tangent bundle has the fiber homotopy type of a product bundle if and only if n equals 1, 3 or 7 (and in this case the tangent bundle is a product bundle).*

Proof. If a map $S^n \times S^n \rightarrow S^n$ of degree (1, 1) exists, then according to Adams n must be equal to 1, 3 or 7 (see Theorem 1a of [1]).

Conversely, if n equals 1, 3 or 7 then $\pi_{n-1}(SO(n)) = 0$. Using

obstruction theory it follows that any homotopy n -sphere is parallelizable. This completes the proof.

REFERENCES

1. F. Adams, *On the nonexistence of elements of Hopf invariant one*, Bull. Amer. Math. Soc., **64** (1958), 279–282.
2. A. L. Blakers and W. S. Massey, *The homotopy groups of a triad I*, Ann. of Math., **53** (1951), 161–205.
3. A. Dold, *Über faserweise Homotopieäquivalenz von Faserräumen*, Math. Zeits., **62** (1955), 111–136.
4. W. S. Massey, *On the normal bundle of a sphere imbedded in Euclidean space* (to appear).
5. J. Milnor, *Lectures on characteristic classes* (mimeographed), Princeton University, 1958.
6. D. Puppe, *Homotopiemengen und ihre induzierten Abbildungen II*, Math. Zeits., **69** (1958), 395–417.
7. E. Spanier, *Infinite symmetric products, function spaces, and duality*, Ann. of Math., **69** (1959), 142–198.
8. E. Spanier and J. H. C. Whitehead, *Duality in homotopy theory*, Mathematika, **2** (1955), 56–80.
9. E. Spanier and J. H. C. Whitehead, *The theory of carriers and S-theory*, in Algebraic Geometry and Topology (a symposium in honor of S. Lefschetz), Princeton University Press, 1957, pp. 330–360.
10. R. Thom, *Espaces fibrés en sphères et carrés de Steenrod*, Ann. Ecole Norm. Sup., **69** (1952), 109–182.
11. J. H. C. Whitehead, *Combinatorial Homotopy I*, Bull. Amer. Math. Soc., **55** (1949), 213–245.

PRINCETON UNIVERSITY
UNIVERSITY OF CHICAGO AND
THE INSTITUTE FOR ADVANCED STUDY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG

Stanford University
Stanford, California

F. H. BROWNELL

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

L. J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

E. HEWITT

A. HORN

L. NACHBIN

M. OHTSUKA

H. L. ROYDEN

M. M. SCHIFFER

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Maynard G. Arsove, <i>The Paley-Wiener theorem in metric linear spaces</i>	365
Robert (Yisrael) John Aumann, <i>Acceptable points in games of perfect information</i>	381
A. V. Balakrishnan, <i>Fractional powers of closed operators and the semigroups generated by them</i>	419
Dallas O. Banks, <i>Bounds for the eigenvalues of some vibrating systems</i>	439
Billy Joe Boyer, <i>On the summability of derived Fourier series</i>	475
Robert Breusch, <i>An elementary proof of the prime number theorem with remainder term</i>	487
Edward David Callender, Jr., <i>Hölder continuity of n-dimensional quasi-conformal mappings</i>	499
L. Carlitz, <i>Note on Alder's polynomials</i>	517
P. H. Doyle, III, <i>Unions of cell pairs in E^3</i>	521
James Eells, Jr., <i>A class of smooth bundles over a manifold</i>	525
Shaul Foguel, <i>Computations of the multiplicity function</i>	539
James G. Glimm and Richard Vincent Kadison, <i>Unitary operators in C^*-algebras</i>	547
Hugh Gordon, <i>Measure defined by abstract L_p spaces</i>	557
Robert Clarke James, <i>Separable conjugate spaces</i>	563
William Elliott Jenner, <i>On non-associative algebras associated with bilinear forms</i>	573
Harold H. Johnson, <i>Terminating prolongation procedures</i>	577
John W. Milnor and Edwin Spanier, <i>Two remarks on fiber homotopy type</i>	585
Donald Alan Norton, <i>A note on associativity</i>	591
Ronald John Nunke, <i>On the extensions of a torsion module</i>	597
Joseph J. Rotman, <i>Mixed modules over valuations rings</i>	607
A. Sade, <i>Théorie des systèmes demosiens de groupoi des</i>	625
Wolfgang M. Schmidt, <i>On normal numbers</i>	661
Berthold Schweizer, Abe Sklar and Edward Oakley Thorp, <i>The metrization of statistical metric spaces</i>	673
John P. Shanahan, <i>On uniqueness questions for hyperbolic differential equations</i>	677
A. H. Stone, <i>Sequences of coverings</i>	689
Edward Oakley Thorp, <i>Projections onto the subspace of compact operators</i>	693
L. Bruce Treybig, <i>Concerning certain locally peripherally separable spaces</i>	697
Milo Wesley Weaver, <i>On the commutativity of a correspondence and a permutation</i>	705
David Van Vranken Wend, <i>On the zeros of solutions of some linear complex differential equations</i>	713
Fred Boyer Wright, Jr., <i>Polarity and duality</i>	723