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ON NORMAL NUMBERS

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1. Introduction. A real number ξ , $0 \leq \xi < 1$, is said to be *normal in the scale of r* (or *to base r*), if in $\xi = 0 \cdot a_1 a_2 \dots$ expanded in the scale of $r^{(1)}$ every combination of digits occurs with the proper frequency. If $b_1 b_2 \dots b_k$ is any combination of digits, and Z_N the number of indices i in $1 \leq i \leq N$ having

$$b_1 = a_i, \dots, b_k = a_{i+k-1},$$

then the condition is that

$$(1) \quad \lim_{N \rightarrow \infty} Z_N N^{-1} = r^{-k}.$$

A number ξ is called *simply normal* in the scale of r if (1) holds for $k = 1$. A number is said to be *absolutely normal* if it is normal to every base r . It is well-known (see, for example, [6], Theorem 8.11) that almost every number ξ is absolutely normal.

We write $r \sim s$, if there exist integers n, m with $r^n = s^m$. Otherwise, we put $r \not\sim s$.

In this paper we solve the following problem. *Under what conditions on r, s is every number ξ which is normal to base r also normal to base s ?* The answer is given by

THEOREM 1. *A Assume $r \sim s$. Then any number normal to base r is normal to base s .*

B If $r \not\sim s$, then the set of numbers ξ which are normal to base r but not even simply normal to base s has the power of the continuum.

The A-part of the Theorem is rather trivial, but I shall sketch a proof of it, since I could not find one in the literature.

Next, let I be an interval of length $|I|$ contained in the unit-interval $U = [0, 1]$. We write $M_N(\xi, r, I)$ for the number of indices i in $1 \leq i \leq N$ such that the fractional part $\{r^i \xi\}$ of $r^i \xi$ lies I . A sequence $\xi, r\xi, r^2\xi, \dots$ has *uniform distribution modulo 1* if

$$R_N(\xi, r, I) = M_N(\xi, r, I) - N|I| = o(N)$$

for any I . It was proved by Wall [8] (the most accessible proof in [6], Theorem 8.15) that ξ is normal to base r if and only if $\xi, r\xi, r^2\xi, \dots$ has uniform distribution modulo 1.

Write $T_{s,t}$, where $1 < t < s$, for the following mapping in U : If $\xi = 0 \cdot a_1 a_2 \dots$ in the scale of t , then $T_{s,t} \xi = 0 \cdot a_1 a_2 \dots$ in the scale of s .

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¹ In case of ambiguity we take the representation with an infinity of a_i less than $r - 1$. But this does not affect the property of ξ to be normal or not.

THEOREM 2. *Assume $r \not\sim s$. Then there exists a constant $\alpha_1 = \alpha_1(r, s, t) > 0$ such that for almost every ξ there exists a $N_0(\xi)$ with*

$$(2) \quad R_N(T_{s,t}\xi, r, I) \leq N^{1-\alpha_1}$$

for every $N \geq N_0(\xi)$ and any I .

Thus $T_{s,t}\xi$ is normal to base r for almost all ξ . Since $T_{s,t}\xi$ is not simply normal to base s part B of Theorem 1 follows. It does not follow immediately for $s = 2$, but instead of $T_{2,t}$, which does not exist, we may take $T_{4,t}$.

We can interpret our results as follows. Write $C_{s,t}$ for the image set $T_{s,t}U$ of the unit-interval U under the mapping $T_{s,t}$. $C_{s,t}$ is essentially a Cantor set. In $C_{s,t}$ we define a measure $\mu_{s,t}$ by

$$(3) \quad \int_{C_{s,t}} f(\xi) d\mu_{s,t} = \int_0^1 f(T_{s,t}\xi) d\xi,$$

where $f(\xi)$ is any real-valued function such that the integral on the right hand side of (3) exists. Then it follows from Theorem 2 that with respect to $\mu_{s,t}$ almost every ξ in $C_{s,t}$ is normal in the scale of r .

Throughout this paper, lower case italics stand for integers. $\alpha_1 = \alpha_1(r, s, t)$, $\alpha_2, \alpha_3, \dots$ will be positive constants depending on some or all the variables r, s, t .

1. The case $r \sim s$. First, it follows almost from the definition that any number normal to base s^n is normal to base s .

Next, assume ξ is normal to base r , we shall show it is normal in the scale of r^m . If $\xi = 0 \cdot a_1 a_2 \dots$ in the scale of r , $b_1 \dots b_{mk}$ is any combination of mk digits and $Z_N^{(1)}$ is the number of indices i in $1 \leq i \leq N$ with $i \equiv 1 \pmod m$ satisfying

$$b_1 = a_i, \dots, b_{mk} = a_{i+mk-1},$$

then it was shown in [7] and in [3] that

$$\lim_{N \rightarrow \infty} Z_N^{(1)} N^{-1} = r^{-mk} \mathfrak{m}^{-1}$$

and hence

$$\lim_{N \rightarrow \infty} Z_{mN}^{(1)} N^{-1} = (r^m)^{-k}.$$

Thus ξ is normal to base r^m .

Combining the above remarks we obtain the A-part of Theorem 1.

2. The measure $\mu_{s,t}$. We define *numbers of order h* to be the number $0 \cdot a_1 \dots a_h$ with $0 \leq a_i < t$ in the scale of s . There are t^h numbers of order h , we denote them in ascending order by $\theta_1^{(h)}, \dots, \theta_{t^h}^{(h)}$.

LEMMA 1. Let $f(\xi)$ be a step-function, having a finite number of steps. Then

$$\int_{c_{s,t}} f(\xi) d\mu_{s,t} = \int_0^1 f(T_{s,t}\xi) d\xi = \lim_{h \rightarrow \infty} t^{-h} \sum_{k=1}^{t^h} f(\theta_k^{(h)}) .$$

The integrals and the limit exist and are finite.

Proof. It will be sufficient to prove the lemma for $f(\xi) = \{\xi, \gamma\}$, where $0 \leq \gamma \leq 1$ and

$$\{\xi, \gamma\} = \begin{cases} 1, & \text{if } \{\xi\} < \gamma \\ 0 & \text{otherwise.} \end{cases}$$

$\xi_k^{(h)} = \int_0^1 \{T_{s,t}\xi, \theta_k^{(h)}\} d\xi$ is the least upper bound of numbers ξ having $T_{s,t}\xi \leq \theta_k^{(h)}$. Thus if $\theta_k^{(h)} = 0 \cdot a_1 \cdots a_n$ in the scale of s , then $\xi_k^{(h)} = 0 \cdot a_1 \cdots a_n$ in the scale of t and therefore $\xi_k^{(h)} = (k - 1)t^{-h}$.

Hence if $\theta_k^{(h)} \leq \gamma \leq \theta_{k+1}^{(h)}$, or if $\theta_k^{(h)} \leq \gamma$ with $k = t^h$, then

$$\int_0^1 \{T_{s,t}\xi, \gamma\} d\xi = kt^{-h} - \varepsilon ,$$

where $0 \leq \varepsilon \leq t^{-h}$. We can rewrite this in the form

$$\int_0^1 \{T_{s,t}\xi, \gamma\} d\xi = t^{-h} \sum_{k=1}^{t^h} \{\theta_k^{(h)}, \gamma\} - \varepsilon ,$$

and Lemma 1 follows.

Particularly, for

$$\begin{aligned} \mu(\gamma, x) &= \int_0^1 \{xT_{s,t}\xi, \gamma\} d\xi \\ \mu(\gamma, x, y) &= \int_0^1 \{xT_{s,t}\xi, \gamma\} \{yT_{s,t}\xi, \gamma\} d\xi \end{aligned}$$

we have

$$(4) \quad \mu(\gamma, x) = \lim_{h \rightarrow \infty} t^{-h} \sum_{k=1}^{t^h} \{x\theta_k^{(h)}, \gamma\} ,$$

$$(5) \quad \mu(\gamma, x, y) = \lim_{h \rightarrow \infty} t^{-h} \sum_{k=1}^{t^h} \{x\theta_k^{(h)}, \gamma\} \{y\theta_k^{(h)}, \gamma\} .$$

3. Exponential sums. Write $e(\xi)$ for $e^{2\pi i\xi}$. There exist ([5], pp. 91-92, 99) for any $\gamma, 0 \leq \gamma \leq 1$, and any $\eta > 0$ functions $f_1(\xi), f_2(\xi)$ periodic in ξ with period 1, such that $f_1(\xi) \leq \{\xi, \gamma\} \leq f_2(\xi)$, having Fourier expansions

$$f_1(\xi) = \gamma - \eta + \sum'_u A_u^{(1)} e(u\xi)$$

$$f_2(\xi) = \gamma + \eta + \sum'_u A_u^{(2)} e(u\xi),$$

where the summation is over all $u \neq 0$ and $A_u^{(i)}$ is majorized by

$$(6) \quad |A_u| \leq \frac{1}{u^2 \eta}.$$

Applying this to (5) we obtain

$$\mu(\gamma, x, y) \leq (\gamma + \eta)^2 + \overline{\lim}_{h \rightarrow \infty} t^{-h} \sum'_{\substack{u, v \\ \neq 0, 0}} \left| A_u^{(2)} \| A_v^{(2)} \| \sum_{k=1}^{t^h} e((ux + vy)\theta_k^{(h)}) \right|,$$

where we put $A_0^{(2)} = \gamma + \eta$ and take the sum over all pairs u, v of numbers not both being zero. Since

$$\left| t^{-h} \sum_{k=1}^{t^h} e((ux + vy)\theta_k^{(h)}) \right| \leq 1,$$

and since the double sum over u, v is uniformly convergent in h , we may change the order of limit and summation and obtain

$$\mu(\gamma, x, y) \leq (\gamma + \eta)^2 + \sum'_{u, v} |A_u^{(2)}| |A_v^{(2)}| \overline{\lim}_{h \rightarrow \infty} t^{-h} \left| \sum_{k=1}^{t^h} e((ux + vy)\theta_k^{(h)}) \right|.$$

The numbers $\theta_k^{(h)}$ are the numbers

$$\frac{a_1}{s} + \frac{a_2}{s^2} + \dots + \frac{a_h}{s^h},$$

where $0 \leq a_i < t$. Hence

$$\sum_{k=1}^{t^h} e(w\theta_k^{(h)}) = \prod_{j=1}^h \left(1 + e\left(\frac{w}{s^j}\right) + e\left(\frac{2w}{s^j}\right) + \dots + e\left(\frac{(t-1)w}{s^j}\right) \right).$$

If we keep w fixed, and if j is large, then

$$\left| \left(1 + e\left(\frac{w}{s^j}\right) + \dots + e\left(\frac{(t-1)w}{s^j}\right) \right) t^{-1} - 1 \right| < \frac{t|w|}{s^i}.$$

Therefore

$$(7) \quad \Pi(s, t; w) = \prod_{j=1}^{\infty} \left| \left(1 + e\left(\frac{w}{s^j}\right) + \dots + e\left(\frac{(t-1)w}{s^j}\right) \right) t^{-1} \right|$$

exists and

$$(8) \quad \mu(\gamma, x, y) \leq (\gamma + \eta)^2 + \sum'_{u, v} |A_u^{(2)}| |A_v^{(2)}| \Pi(s, t; ux + vy).$$

The next three sections will be devoted to finding bounds for sums like

$$\sum_{N_1 < n, m \leq N_2} \Pi(s, t; ur^n + vr^m).$$

4. Two lemmas on digits.

LEMMA 2. Write $w = c_0 \cdots c_2 c_1$ in the scale of s . Assume there are at least z pairs of digits $c_{i+1}c_i$ with

$$(9) \quad 1 \leq c_{i+1}c_i \leq s^2 - 2.$$

(Here $c_{i+1}c_i = sc_{i+1} + c_i$). Then

$$H(s, t; w) \leq \alpha_2^z,$$

where $\alpha_2 = \alpha_2(s, t)$, $0 < \alpha_2 < 1$.

Proof. There are at least z numbers i having

$$\frac{1}{s^2} \leq \left\{ \frac{w}{s^i} \right\} \leq 1 - \frac{1}{s^2}.$$

For such an i we have

$$\left| 1 + e\left(\frac{w}{s^i}\right) + \cdots + e\left(\frac{(t-1)w}{s^i}\right) \right| \leq \left| 1 + e\left(\frac{1}{s^2}\right) \right| + t - 2 = t\alpha_2$$

and the Lemma is proved.

There exists an $\alpha_3(s)$, $0 < \alpha_3 < 1/4$, such that

$$\frac{(s^2 - 2)^{\alpha_3} 2^{1/2 - \alpha_3}}{(2\alpha_3)^{\alpha_3} (1 - 2\alpha_3)^{1/2 - \alpha_3}} < 2^{3/4}.$$

LEMMA 3. If k is large, $k > \alpha_4(s)$, then the number of combinations of digits $c_k c_{k-1} \cdots c_1$ in the scale of s with less than $\alpha_3(s)k$ indices i satisfying (9) is not greater than $2^{(3/4)k}$.

Proof. It will be sufficient to show that the number of combinations with less than $\alpha_3(s)k$ indices i satisfying both (9) and $i \equiv 1 \pmod{2}$ is not greater than $2^{(3/4)k}$. We first assume k is even. There exist

$$\binom{k}{l} (s^2 - 2)^l 2^{k/2 - l}$$

combinations $c_k \cdots c_1$ with exactly l indices i having both (9) and $i \equiv 1 \pmod{2}$. Hence the number of combinations with less than $\alpha_3(s)k$ indices i satisfying (9) and $i \equiv 1 \pmod{2}$ does not exceed

$$k \binom{k}{[\alpha_3 k]} (s^2 - 2)^{[\alpha_3 k]} 2^{(k/2) - [\alpha_3 k]}.$$

Using Stirling's formula for the binomial coefficient we obtain for large enough k the upper bound

$$\alpha_3(s)k \frac{(s^2 - 2)^{\alpha_3 k} 2^{((1/2) - \alpha_3)k}}{(2\alpha_3)^{\alpha_3 k} (1 - 2\alpha_3)^{((1/2) - \alpha_3)k}} < 2^{(3/4)k} .$$

Actually, the expression on the left hand side is $< 2^{\alpha_6 k}$, where $\alpha_6 < 3/4$. This permits us to extend the result to odd k .

5. The order of r modulo p^k as a function of k .

LEMMA 4. Assume p is a prime with $p \nmid r$. Then the order $o(r, p^k)$, of r modulo p^k satisfies

$$o(r, p^k) \geq \alpha_7(r, p)p^k .$$

COROLLARY. Let n run through a residue system modulo p^k . Then at most $\alpha_8(r, p)$ of the numbers r^n will fall into the same residue class modulo p^k .

Proof. Write

$$g = g(p) = \begin{cases} p - 1, & \text{if } p \text{ is odd} \\ 2, & \text{if } p = 2. \end{cases}$$

There exists an $\alpha_9 = \alpha_9(r, p)$ such that

$$(10) \quad r^g \equiv 1 + qp^{\alpha_9 - 1} \pmod{p^{\alpha_9}} ,$$

where $q \not\equiv 0 \pmod{p}$. We have necessarily $\alpha_9 > 1$ and even $\alpha_9 > 2$ if $p = 2$. It follows from (10) by standard methods (see, for instance, [4], § 5.5) that

$$r^g p^e \equiv 1 + qp^{\alpha_9 - 1 + e} \pmod{p^{\alpha_9 + e}}$$

for any $e \geq 0$. Thus for $k \geq \alpha_9$ we have

$$r^g p^{k - \alpha_9} \equiv 1 + qp^{k - 1} \pmod{p^k}$$

and

$$o(r, p^k) \geq gp^{k - \alpha_9} = \alpha_7(r, p)p^k .$$

Assume $r \not\sim s$. Write

$$\begin{aligned} r &= p_1^{d_1} p_2^{d_2} \cdots p_h^{d_h} \\ s &= p_1^{e_1} p_2^{e_2} \cdots p_h^{e_h} , \end{aligned}$$

where we may assume that never both $d_i = 0, e_i = 0$. We also may assume that the primes p_1, \dots, p_h are ordered in such a way that

$$\frac{e_1}{d_1} \geq \frac{e_2}{d_2} \geq \dots \geq \frac{e_h}{d_h},$$

where we put $(e_i/d_i) = +\infty$ if $d_i = 0$. Since $r \not\sim s$, we have

$$r_1 = \frac{r^{e_1}}{s^{d_1}} > 1.$$

From now on, $p = p_1(r, s)$ is the prime defined above. We have $p \mid s$ but $p \nmid r_1$. For any $x \neq 0, y > 1$ we define two new numbers x_y and x'_y by $x = x_y x'_y$, where x_y is a power of y and $y \nmid x'_y$.

LEMMA 5. A. Assume $r \not\sim s, v \neq 0$. Let m run through a system $K(s^k)$ of non-negative representatives modulo s^k . Then at most

$$\alpha_{10}(r, s) \left(\frac{s}{2}\right)^k v_p$$

of the numbers

$$v(r^m)'_s$$

are in the same residue class modulo s^k .

B. Assume $r \not\sim s$, furthermore $p \nmid r$. Suppose $u \neq 0, v \neq 0, n$ are fixed. Then, if m runs through $K(s^k)$, at most

$$\alpha_{11}(r, s) \left(\frac{s}{2}\right)^k v_p$$

of the numbers

$$ur^n + vr^m$$

will fall into the same residue class modulo s^k .

Proof. A. Write $m = m_1 e_1 + m_2, 0 \leq m_2 < e_1$. Then $r^m = r^{m_1 e_1 + m_2} = s^{m_1 d_1} r_1^{m_1} r^{m_2}$ and $v(r^m)'_s = v r_1^{m_1} (r^{m_2})'_s$. The equation

$$r_1^{m_1} \equiv a \pmod{p^k}$$

has for fixed a at most $e_1 \alpha_s(r_1, p)$ solutions in $m = m_1 e_1 + m_2$, if m runs through a system $K(p^k)$ of residues modulo p^k . This follows from the corollary of Lemma 4. The equation

$$av(r^{m_2})'_s \equiv b \pmod{p^k}$$

has for fixed b, m_2 at most

$$\text{g.c.d.}(v(r^{m_2})'_s, p^k) \leq v_p r^{m_2}$$

solutions in a . Hence the number of solutions of

$$vr_1^{m_1}(r^{m_2})'_s \equiv b \pmod{p^k}$$

in $m = m_1e_1 + m_2 \in K(p^k)$ does not exceed

$$e_1\alpha_8v_p(1 + r + \dots + r^{e_1-1}) = \alpha_{10}(r, s)v_p.$$

But this implies that the number of solutions of

$$vr_1^{m_1}(r^{m_2})'_s \equiv b \pmod{s^k}$$

in $m = m_1e_1 + m_2 \in K(s^k)$ is not greater than

$$\alpha_{10}(r, s)v_p\left(\frac{s}{p}\right)^k \leq \alpha_{10}(r, s)\left(\frac{s}{2}\right)^k v_p.$$

B. The equation

$$ur^n + vr^m \equiv b \pmod{p^k}$$

has according to the corollary of Lemma 4 at most

$$\alpha_8(r, p)v_p$$

solutions in $m \in K(p^k)$. The result follows as before.

The following conjecture seems related to our results: *Assume $r \not\sim s$. Then for any ε and k almost all the numbers r, r^2, \dots are (ε, k) -normal to the base s in the sense of Besicovitch [1]; that is, the number of $n \leq N$ for which r^n is not (ε, k) -normal is $o(N)$ as $N \rightarrow \infty$ for fixed ε and k .*

6. Bounds for exponential sums.

LEMMA 6. A. *Let r, s, v be as in Lemma 5A. Then*

$$\sum_{m \in K(s^k)} \Pi(s, t; vr^m) \leq \alpha_{12}v_p s^{(1-\alpha_{13})k}$$

B. *Let r, s, u, v, n be as in Lemma 5B. Then*

$$\sum_{m \in K(s^k)} \Pi(s, t; ur^n + vr^m) \leq \alpha_{14}v_p s^{(1-\alpha_{15})k}.$$

Proof. **A.** Write $v(r^m)'_s = c_v \dots c_k \dots c_1$ in the scale of s . Lemma 5A implies that any digit combination $c_k c_{k-1} \dots c_1$ will occur at most $\alpha_{10}(r, s)(s/2)^k v_p$ times. According to Lemma 3, there are for large k not more than $2^{(3/4)k}$ digit-combinations $c_k \dots c_1$ with less than $\alpha_3 k$ indices i satisfying (9). Thus of all the numbers $v(r^m)'_s$, $m \in K(s^k)$, and hence of all the numbers vr^m there will be at most

$$\alpha_{10}(r, s)(s/2)^k v_p 2^{(3/4)k} = \alpha_{10}(r, s)v_p (s/2^{1/4})^k = \alpha_{10}(r, s)v_p s^{(1-\alpha_{16})k}$$

having less than $\alpha_3 k$ digits c_i in their expansion in the scale of s satisfying (9). Thus Lemma 2 yields

$$\Pi(s, t; vr^m) \leq \alpha_2^{k\alpha_3}$$

for all but at most

$$\alpha_{10}(r, s)v_p s^{(1-\alpha_{16})k}$$

numbers $m \in K(s^k)$. This gives

$$\sum_{m \in K(s^k)} \Pi(s, t; vr^m) \leq s^k \alpha_2^{k\alpha_3} + \alpha_{10} v_p s^{(1-\alpha_{16})k} \leq \alpha_{12} v_p s^{(1-\alpha_{13})k}.$$

B is proved similarly, using Lemma 5B.

LEMMA 7. A. Assume $r \not\sim s, v \neq 0$. Then

$$(11) \quad \sum_{N_1 < n \leq N_2} \Pi(s, t; vr^n) \leq \alpha_{17}(N_2 - N_1)^{1-\alpha_{18}} v_p.$$

B. Assume $r \not\sim s, u \neq 0, v \neq 0$. Then

$$(12) \quad \sum_{N_1 < n, m \leq N_2} \Pi(s, t; ur^n + vr^m) \leq \alpha_{19}(N_2 - N_1)^{2-\alpha_{20}} \max(u_p, v_p).$$

Proof. A. There exists a k having $s^{2k} \leq N_2 - N_1 < s^{2(k+1)}$, hence there exists a w satisfying $s^k w \leq N_2 - N_1 < s^k(w + 1)$, where $s^k \leq w < s^{k+2}$. Thus if m runs from N_1 to N_2 , then m runs through w systems $K(s^k)$ of residue classes modulo s^k and at most s^k other numbers. Hence by Lemma 6A

$$\sum_{N_1 < m \leq N_2} \Pi(s, t; vr^m) \leq w \alpha_{12} v_p s^{(1-\alpha_{13})k} + s^k \leq \alpha_{17}(N_2 - N_1)^{1-\alpha_{18}} v_p.$$

B. If $p \nmid r$, then we proceed as in part A. We first take the sum over m and use Lemma 6B.

If $p \mid r$, then our argument is as follows. Consider, for example, the part of the sum with $n \leq m$. Changing the notation in n, m , we see that this part of the sum (12) equals

$$\sum_{n=0}^{N_2-N_1-1} \sum_{m=N_1+1}^{N_2-n} \Pi(s, t; (ur^n + v)r^m).$$

Except for possibly one exceptional n we have $(ur^n)_p \neq v_p$ and therefore $(ur^n + v)_p \leq v_p \leq \max(u_p, v_p)$. If n is not exceptional, then the already proved Lemma 7A can be applied to the inner sum and we obtain the bound

$$\alpha_{17}(N_2 - N_1 - n)^{1-\alpha_{18}} \max(u_p, v_p).$$

Taking the sum over n we obtain (12).

7. A fundamental lemma. Generalizing $M_N(\xi, r, I)$ we write ${}_{N_1}M_{N_2}(\xi, r, I)$ for the number of indices i in $N_1 < i \leq N_2$ such that $\{r^i\xi\}$ lies in I . We put

$${}_{N_1}R_{N_2}(\xi, r, I) = {}_{N_1}M_{N_2}(\xi, r, I) - (N_2 - N_1)|I|.$$

Fundamental lemma. *Assume $r \not\sim s$. Then*

$$\int_0^1 {}_{N_1}R_{N_2}^2(T_{s,t}\xi, r, I) d\xi \leq \alpha_{21}(N_2 - N_1)^{2-\alpha_{22}}.$$

Proof. It is enough to prove this for intervals of the type $I = [0, \gamma)$. Then

$${}_{N_1}M_{N_2}(\xi, r, I) = \sum_{N_1 < n \leq N_2} \{r^n \xi, \gamma\}$$

and

$$(13) \quad \int_0^1 {}_{N_1}M_{N_2}(T_{s,t}\xi, r, I) d\xi = \sum_{N_1 < n \leq N_2} \mu(\gamma, r^n)$$

$$(14) \quad \int_0^1 {}_{N_1}M_{N_2}^2(T_{s,t}\xi, r, I) d\xi = \sum_{N_1 < n, m \leq N_2} \mu(\gamma, r^n, r^m).$$

Now we combine (8) and Lemma 7. We obtain, together with (6),

$$\begin{aligned} \sum_{N_1 < n, m \leq N_2} \mu(\gamma, r^n, r^m) &\leq (\gamma + \eta)^2(N_2 - N_1)^2 \\ &+ 2(\gamma + \eta) \sum_{v \neq 0} \frac{v_p}{\eta v^2} \alpha_{17}(N_2 - N_1)^{2-\alpha_{18}} \\ &+ \sum_{u \neq 0} \sum_{v \neq 0} \frac{\max(u_p, v_p)}{\eta u^2 \eta v^2} \alpha_{19}(N_2 - N_1)^{2-\alpha_{20}}. \end{aligned}$$

Since the sums

$$\sum_{v \neq 0} \frac{v_p}{v^2}, \quad \sum_{u \neq 0} \sum_{v \neq 0} \frac{\max(u_p, v_p)}{u^2 v^2}$$

are convergent, and since η was arbitrary, we have

$$\sum_{N_1 < n, m \leq N_2} \mu(\gamma, r^n, r^m) - (N_2 - N_1)^2 \gamma^2 \leq \alpha_{23}(N_2 - N_1)^{2-\alpha_{24}}.$$

In the same fashion we can prove

$$\begin{aligned} \left| \sum_{N_1 < n, m \leq N_2} \mu(\gamma, r^n, r^m) - (N_2 - N_1)^2 \gamma^2 \right| &\leq \alpha_{23}(N_2 - N_1)^{1-\alpha_{24}} \\ \left| \sum_{N_1 < n \leq N_2} \mu(\gamma, r^n) - (N_2 - N_1) \gamma \right| &\leq \alpha_{25}(N_2 - N_1)^{1-\alpha_{26}}. \end{aligned}$$

These two inequalities, together with (13) and (14), give the **Fundamental Lemma**.

8. **Proof of the theorems.** Once the Fundamental Lemma is shown, we can prove Theorem 2 by the standard method developed in [2].

By J_B , $B > 0$, we denote the set of intervals $[\beta, \gamma)$, $0 \leq \beta < \gamma < 1$ of the type $\beta = a2^{-b}$, $\gamma = (a + 1)2^{-b}$, where $0 \leq b \leq \alpha_{22}B/2$. By P_B we denote the set of all pairs of integers N_1, N_2 having $0 \leq N_1 < N_2 \leq 2^B$ of the type $N_1 = a2^b$, $N_2 = (a + 1)2^b$ for integers a and $b \geq 0$.

LEMMA 8. *Assume $r \neq s$. Then*

$$\sum_{(N_1, N_2) \in P_B} \sum_{I \in J_B} \int_0^1 R_{N_1}^2 R_{N_2}^2(T_{s,t}\xi, r, I) d\xi \leq \alpha_{27} 2^{2B(1-\alpha_{28})}.$$

Proof. Because of the Fundamental Lemma the left hand side is not greater than

$$\alpha_{21} 2^{\alpha_{22}B/2+1} \Sigma,$$

where $2^{\alpha_{22}B/2+1}$ is an upper bound for the number of intervals in J_B and

$$(15) \quad \Sigma = \sum_{(N_1, N_2) \in P_B} (N_2 - N_1)^{2-\alpha_{22}}.$$

In (15) each value of $N_2 - N_1 = 2^b$ occurs 2^{B-b} times, so that

$$\Sigma = \sum_{b=0}^B 2^{B-b+b(2-\alpha_{22})} \leq \alpha_{20} 2^{2B(1-\alpha_{22}/2)}.$$

Hence Lemma 8 is true with $\alpha_{28} = \alpha_{22}/4$.

LEMMA 9. *For large B there exists a set E_B of measure not greater than $2^{-\alpha_{30}B}$ such that*

$$(16) \quad R_N(T_{s,t}\xi, r, I) \leq 2^{B(1-\alpha_{31})}$$

for all $I, N \leq 2^B$ and all ξ in $[0, 1)$ but not in E_B .

Proof. We define E_B to be the set consisting of all ξ in $[0, 1)$ for which it is not true that

$$(17) \quad \sum_{(N_1, N_2) \in P_B} \sum_{I \in J_B} N_1 R_{N_1}^2 R_{N_2}^2(T_{s,t}\xi, r, I) \leq 2^{2B(1-\alpha_{28}/2)}.$$

Lemma 8 assures that the measure of E_B does not exceed

$$\alpha_{27} 2^{-2B\alpha_{28}/2} < 2^{-\alpha_{30}B}$$

for large B . We have to show that (16) is a consequence of (17).

We first assume I to be of the type $I = [0, \gamma)$, $\gamma = a2^{-b}$, where $0 \leq b \leq \alpha_{22}B/2$. Then the interval $[0, \gamma)$, is the sum of at most $b < B$ intervals $I, I \in J_B$, as may be seen by expressing a in the binary scale.

Expressing N in the binary scale we see that the interval $[0, N)$ can be expressed as a union of at most B intervals $[N_1, N_2)$, where the pair $N_1, N_2 \in P_B$. Hence we can write $R_N(T_{s,t}\xi, r, I)$ as a sum of $_{N_1}R_{N_2}(T_{s,t}\xi, r, I)$ over at most B^2 sets N_1, N_2, I , where $N_1, N_2 \in P_B, I \in J_B$:

$$R_N(T_{s,t}\xi, r, I) = \sum_{N_1} R_{N_2}(T_{s,t}\xi, r, I).$$

Hence by (17) and Cauchy's inequality,

$$R_N^2(T_{s,t}\xi, r, I) \leq B^2 2^{2B(1-\alpha_{28}/2)} < 2^{2B(1-\alpha_{32})}$$

for large B .

Next, let $I = [0, \gamma)$ be of the type $a2^{-b} \leq \gamma \leq (a+1)2^{-b}$, where $\alpha_{22}B/4 < b \leq \alpha_{22}B/2$. Then

$$\begin{aligned} |R_N(T_{s,t}\xi, r, [0, \gamma))| &= |M_N(T_{s,t}\xi, r, [0, \gamma)) - \gamma N| \\ &\leq |R_N(T_{s,t}\xi, r, [0, (a+1)2^{-b}))| + |R_N(T_{s,t}\xi, r, [0, a2^{-b}))| + 2^{-b}N \\ &\leq 2 \cdot 2^{B(1-\alpha_{33})} + 2^{(1-\alpha_{22}/4)B} < 2^{B(1-\alpha_{33})}. \end{aligned}$$

The Lemma now follows from

$$|R_N(, , [\beta, \gamma))| \leq |R_N(, , [0, \beta))| + |R_N(, , [0, \gamma))|.$$

Proof of Theorem 2. Since $\sum 2^{-\alpha_{30}B}$ is convergent, there exists for almost all ξ a $B_0 = B_0(\xi)$ such that $\xi \notin E_B$ for $B \geq B_0$. If $N \geq 2^{B_0}$, then we can find a $B \geq B_0$ satisfying $2^{B-1} < N \leq 2^B$ and Lemma 9 yields

$$R_N(T_{s,t}\xi, r, I) < 2^{B(1-\alpha_{31})} < 2N^{1-\alpha_{31}} < N^{1-\alpha_1}$$

for large enough N .

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