

Pacific Journal of Mathematics

**ON UNIQUENESS QUESTIONS FOR HYPERBOLIC
DIFFERENTIAL EQUATIONS**

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1. Statement of results. This note is concerned with the existence, uniqueness, and successive approximations for solutions of the initial value problem

$$z_{xy} = f(x, y, z, p, q), \quad z(x, 0) = \sigma(x), \quad z(0, y) = \tau(y),$$

where $\sigma(0) = \tau(0) = z_0$, on a rectangle $R: 0 \leq x \leq a, 0 \leq y \leq b$. By a solution is meant a continuous function having partial derivatives almost everywhere and satisfying the integral equation

$$(1) \quad z(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s, t, z(s, t), z_x(s, t), z_y(s, t)) ds dt.$$

Actually it will be clear from the conditions imposed on σ, τ and f that any solution of (1) is uniformly Lipschitz continuous. Let D be the five-dimensional set $D = \{(x, y, z, p, q) : (x, y) \in R \text{ and } z, p, q \text{ arbitrary}\}$. Let $f(x, y, z, p, q)$ be defined and continuous on D , such that $|f(x, y, z, p, q)| < N = \text{const.}$ for $(x, y, z, p, q) \in D$. Let $\sigma(x), \tau(y)$ be defined and uniformly Lipschitz continuous on $0 \leq x \leq a, 0 \leq y \leq b$, respectively (so that $|\sigma(x) - \sigma(\bar{x})| \leq K|x - \bar{x}|, |\tau(y) - \tau(\bar{y})| \leq K|y - \bar{y}|$ for some constant K) and let $\sigma(0) = \tau(0) = z_0$. In addition, for $(x, y) \in R$ and arbitrary $z, p, q, \bar{z}, \bar{p}, \bar{q}$ assume that

$$(2) \quad |f(x, y, z, p, q) - f(x, y, \bar{z}, \bar{p}, \bar{q})| \leq \varphi(x, y, |z - \bar{z}|, |p - \bar{p}|, |q - \bar{q}|),$$

where $\varphi(x, y, z, p, q)$ is a continuous, non-negative function defined for $(x, y) \in R$ and non-negative z, p, q , non-decreasing in each of the variables z, p, q , and with the property that for every (α, β) , where $0 < \alpha \leq a, 0 < \beta \leq b$, the only solution of

$$(3) \quad z(x, y) = \int_0^x \int_0^y \varphi(s, t, z(s, t), z_x(s, t), z_y(s, t)) ds dt$$

in the rectangle $R_{\alpha\beta}: 0 \leq x \leq \alpha, 0 \leq y \leq \beta$ is $z \equiv 0$.

THEOREM (*). *Under the above assumptions on σ, τ, f and φ , (1) possesses one and only one solution on R . This solution is the uniform limit of the successive approximations defined by*

Received June 25, 1959. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF 18 (603)-41. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$$(4_0) \quad z_0(x, y) = \sigma(x) + \tau(y) - z_0$$

and, for $n = 1, 2, 3, \dots$, by

$$(4_n) \quad z_n(x, y) = z_0(x, y) + \int_0^x \int_0^y f(x, y, z_{n-1}(s, t), z_{n-1}{}_x(s, t), z_{n-1}{}_y(s, t)) ds dt .$$

The existence assertion of (*) neither implies nor is implied by that in Hartman-Wintner [3] and its generalizations due to Conti, Szymdt, Ciliberto, Kisynski (for references, see [6] and [2]). The uniqueness assertion of (*) can be considered as a crude analogue of Kamke’s uniqueness theorem (cf. [5], p. 139) in the theory of ordinary differential equations. Finally, the assertion concerning the convergence of successive approximations is an analogue of a result on ordinary differential equations (cf. Viswanatham [8] and references there to van Kampen, to Wintner and to Dieudonne, and Coddington and Levinson [1]).

A theorem similar to (*), in which f and φ do not depend on p, q is proved by Guglielmino [2]. The proof of (*) below will be a generalization of that of [2]. A uniqueness theorem for (1) involving a majorant function of the form $\varphi(z, p, q) = \varphi(|z| + |p| + |q|)$ is given in [6]. (After the completion of this manuscript, I learned¹ of a paper “On the existence theorem of Caratheodory for ordinary and hyperbolic differential equations” by W. Walter, written at about the same time, which contains a theorem in the direction of the uniqueness assertion of (*). Walter’s assumptions, however, are somewhat different.)

REMARK. It will be clear from the proofs that (*) *remains valid* if f, z, p, q, σ, τ are n -vectors (say, with the norm $|z| = \sum_{k=1}^n |z^k|$ or $|z| = \max(|z^1|, \dots, |z^n|)$ if $z = (z^1, \dots, z^n)$). Of course φ will still be a function of 5 variables, (not of $(3n + 2)$ variables as f is).

A theorem suggested by Nagumo’s uniqueness theorem (cf. [5], p. 97) for ordinary differential equations is the following:

THEOREM (**). *Let $f(x, y, z, p, q)$ be defined, continuous and bounded on D , and satisfy, for $xy > 0$ and arbitrary $z, p, q, \bar{z}, \bar{p}, \bar{q}$,*

$$(5) \quad |f(x, y, z, p, q) - f(x, y, \bar{z}, \bar{p}, \bar{q})| \leq c_1(x, y)|z - \bar{z}|/xy + c(x, y)|p - \bar{p}|/y + c_3(x, y)|q - \bar{q}|/x ,$$

where $c_i(x, y), i = 1, 2, 3$, are non-negative, continuous functions such that

$$c_1 + c_2 + c_3 \equiv 1 .$$

Let $\sigma(x), \tau(y)$ be as in (*), and, in addition, let

¹ Added in proof, 4 April 1960. Since this paper was accepted for publication, the following related articles have appeared: W. L. Walter, *Ueber die Differentialgleichung $u_{xy} = f(x, y, u, u_x, u_y)$* , I and II, *Math. Zeit.*, **71** (1959), 308-324 and 436-453; my attention has also been called to the paper of J. B. Diaz and W. L. Walter, *On uniqueness theorems for ordinary differential equations and for partial differentiale equatitions of hyperbolic type*, to appear in *Trans. A.M.S.*.

$$(6) \quad \sigma_x(+0) = \lim_{x \rightarrow +0} \sigma_x(x), \quad \tau_y(+0) = \lim_{y \rightarrow +0} \tau_y(y)$$

exist. Then (1) has at most one solution $z = z(x, y)$. Furthermore, if

$$(6^*) \quad c_1(0,0) > 0,$$

then a solution exist and is the uniform limit of the successive approximations (4).

In (6), x [or y] tends to $+0$ through the set of values on which σ_x [or τ_y] exists.

Nagumo's theorem follows from Kamke's (with $\varphi(x, y) = y/x$). However $(**)$ does not follow from $(*)$ because $\varphi(x, y, z, p, q)$ is assumed continuous on $x = 0$ and on $y = 0$.

REMARK 1. $(**)$ is valid if f, z, p, q, σ, τ are n -vectors (say $z = (z^1, \dots, z^n)$) and either $|z| = \sum_{k=1}^n |z^k|$ or $|z| = \max(|z^1|, \dots, |z^n|)$.

REMARK 2. A modification of an example of Perron [7] in the theory of ordinary differential equations will show that $(**)$ is false if $c_1 = \text{const.} > 1, c_2 \equiv c_3 \equiv 0$ (so that f does not depend on p, q). Also, a modification of an example of Haviland [4] shows that successive approximations need not converge if $c_1 = \text{const.} > 1, c_2 = c_3 \equiv 0$.

The proof of $(*)$ will be given in §§ 2-4 below; that of $(**)$ in §§ 5-6; finally, the proof of the last remark will be indicated in § 7.

The results above answer some questions suggested by Professor P. Hartman. I also wish to acknowledge helpful discussions with him.

2. Proof of $(*)$. Preliminaries. In the proof of $(*)$ below, there is no loss of generality in supposing that φ is bounded, say $0 \leq \varphi(x, y, z, p, q) \leq 2N$ on D . For otherwise φ can be replaced by $\bar{\varphi}$, where $\bar{\varphi}(x, y, z, p, q)$ equals $\varphi(x, y, z, p, q)$ or $2N$ according as $\varphi(x, y, z, p, q)$ does not or does exceed $2N$. It is clear that $\bar{\varphi}$ is continuous and non-decreasing in each of the variables z, p, q . Furthermore, the only solution $z(x, y)$ of

$$(3') \quad z(x, y) = \int_0^x \int_0^y \bar{\varphi}(s, t, x(s, t), z_x(s, t), z_y(s, t)) ds dt$$

on any rectangle $R_{\alpha\beta} : 0 \leq x \leq \alpha (\leq a), 0 \leq y \leq \beta (\leq b)$ is $z \equiv 0$.

In order to see this, note that $\varphi(x, y, 0, 0, 0) \equiv 0$ because $z = 0$ is a solution of (3). Hence there exists an $\epsilon > 0$ such that $0 \leq \varphi(x, y, z, p, q) \leq 2N$ if $|z|, |p|, |q| < \epsilon$. Suppose that $z(x, y) \not\equiv 0$ is a solution of (3') on $R_{\alpha\beta}$. Let $d, 0 \leq d \leq (\alpha^2 + \beta^2)^{\frac{1}{2}}$, be the largest value of r for which $z(x, y) \equiv 0$ in the intersection S_r of $x^2 + y^2 \leq r^2$ and $R_{\alpha\beta}$. If U is any neighborhood of S_d (relative to $R_{\alpha\beta}$), there exists a rectangle $R_{\gamma\delta}$ in U on which $z \not\equiv 0$. Since $z \equiv 0$ on S_d , it is clear that if U is "sufficiently small", then, on U (hence on $R_{\gamma\delta}$), $|z| < \epsilon$ and, almost everywhere, $|z_x| + |z_y| < \epsilon$. But then $z \not\equiv 0$ is a solution of (3) on $R_{\gamma\delta}$. Since this is impossible, the only solution of (3') on $R_{\alpha\beta}$ is $z \equiv 0$.

It will be convenient to have the following notation. R_1 denotes a subset (not always the same) of R of the form $E \times [0, b]$, where E is a (Lebesgue) measurable subset of $[0, a]$ with means $E = a$. Similarly, R_2 is a subset (not always the same) of the form $[0, a] \times E$, where E is a measurable subset of $[0, b]$ and means $E = b$. Partial derivatives z_x, z_y of a function z will be denoted by p, q .

3. Lemma for (*). The proof of (*) will depend on the following lemma.

LEMMA 1. *Let $\alpha(x, y), \beta(x, y), \gamma(x, y)$ be non-negative, measurable functions defined on R, R_1, R_2 , respectively, such that α is continuous, β is uniformly Lipschitz continuous with respect to y and γ is uniformly Lipschitz continuous with respect to x . In addition, let*

$$(7) \quad \alpha(x, y) \leq \int_0^x \int_0^y \varphi(s, t, \alpha(s, t), \beta(s, t), \gamma(s, t)) ds dt,$$

$$(8) \quad \beta(x, y) \leq \int_0^y \varphi(s, t, \alpha(x, t), \beta(x, t), \gamma(x, t)) dt,$$

$$(9) \quad \gamma(x, y) \leq \int_0^x \varphi(s, y, \alpha(s, y), \beta(s, y), \gamma(s, y)) ds,$$

where φ satisfies the conditions of (*) and is bounded. Then $\alpha \equiv \beta \equiv \gamma \equiv 0$.

Note that the Lipschitz continuity of β [or α] with respect to y [or x] is assumed to be uniform with respect to x and y .

The proof of the lemma below follows a suggestion made by R. Sacksteder. My original proof, which will be omitted, depended on two results. The first result is an existence theorem for

$$(10) \quad z(x, y) = \psi(x, y) + \int_0^x \int_0^y \varphi(s, t, z(s, t), p(s, t), q(s, t)) ds dt,$$

where ψ is a non-negative, uniformly Lipschitz continuous function which is non-decreasing in x and in y . This existence theorem is proved by using the successive approximations $z_0 = \psi(x, y)$ and

$$(11) \quad z_n(x, y) = z_0(x, y) + \int_0^x \int_0^y \varphi(s, t, z_{n-1}, p_{n-1}, q_{n-1}) ds dt$$

which satisfy

$$(12) \quad z_n \leq z_{n+1}, p_n \leq p_{n+1}, q_n \leq q_{n+1}.$$

The second result is the fact that if ψ is replaced by another function $\bar{\psi}$ with similar properties and, almost everywhere,

$$(13) \quad \psi \leq \bar{\psi}, \psi_x \leq \bar{\psi}_x, \psi_y \leq \bar{\psi}_y,$$

then the corresponding solution \bar{z} satisfies

$$(14) \quad z \leq \bar{z}, p \leq \bar{p}, q \leq \bar{q}.$$

Proof. Define sequences of successive approximations as follows:
Let

$$(15) \quad z_0(x, y) = \alpha(x, y), \quad u_0(x, y) = \beta(x, y), \quad v_0(x, y) = \gamma(x, y)$$

and, for $n \geq 1$,

$$(16) \quad z_n(x, y) = \int_0^x \int_0^y \varphi(s, t, z_{n-1}(s, t), u_{n-1}(s, t), v_{n-1}(s, t)) ds dt,$$

$$(17) \quad u_n(x, y) = \int_0^y \varphi(x, t, z_{n-1}(x, t), u_{n-1}(x, t), v_{n-1}(x, t)) dt,$$

$$(18) \quad v_n(x, y) = \int_0^x \varphi(s, y, z_{n-1}(s, y), u_{n-1}(s, y), v_{n-1}(s, y)) ds.$$

The functions z_n, u_n, v_n are defined on sets R, R_1, R_2 , respectively, which can be taken independent of n . The inequalities (7), (8), (9) give the case $n = 0$ of

$$(19) \quad z_n \leq z_{n+1}, \quad u_n \leq u_{n+1}, \quad v_n \leq v_{n+1}.$$

The cases $n > 0$ of these inequalities follow by induction by virtue of the monotony of φ .

The boundedness of φ implies the uniform boundedness of the functions z_n, u_n, v_n . Hence, as $n \rightarrow \infty$

$$(20) \quad z = \lim z_n, \quad u = \lim u_n, \quad v = \lim v_n,$$

exist on R, R_1, R_2 , respectively. It is clear from (15) and (19), (20) that

$$(21) \quad 0 \leq \alpha \leq z, \quad 0 \leq \beta \leq u, \quad 0 \leq \gamma \leq v.$$

Lebesgue's theorem on term-by-term integration under bounded convergence implies

$$(22) \quad z(x, y) = \int_0^x \int_0^y \varphi(s, t, z(s, t), u(s, t), v(s, t)) ds dt,$$

$$(23) \quad u(x, y) = \int_0^y \varphi(x, t, z(x, t), u(x, t), v(x, t)) dt,$$

$$(24) \quad v(x, z) = \int_0^x \varphi(s, y, z(s, y), u(s, y), v(s, y)) ds.$$

It is clear that $z_y = u, z_x = v$ almost everywhere. Thus the assumption on φ concerning (3) shows that $z \equiv u \equiv v \equiv 0$. Lemma 1 follows from (21).

4. Proof of (*). (i). Let $z(x, y)$ be a solution of (1). There exist functions $u(x, y), v(x, y)$ defined on sets R_1, R_2 , respectively, such that

$$(25) \quad z(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s, t, z(s, t), u(s, t), v(s, t)) ds dt,$$

$$(26) \quad u(x, y) = \sigma_x(x) + \int_0^y f(x, t, z(x, t), u(x, t), v(x, t)) dt,$$

$$(27) \quad v(x, y) = \tau_y(y) + \int_0^x f(s, y, z(s, y), z_x(s, y), z_y(s, y)) ds ,$$

and the relations $u = z_x$ and $v = z_y$ hold almost everywhere. In order to see this, note that almost everywhere on R ,

$$z_x(x, y) = \sigma_x(x) + \int_0^y f(x, t, z(x, t), z_x(x, t), z_y(x, t)) dt ,$$

$$z_y(x, y) = \sigma_y(y) + \int_0^x f(s, y, z(s, y), z_x(s, y), z_y(s, y)) ds ,$$

The expressions on the right side of these equations are defined for (x, y) on sets R_1, R_2 , respectively. Define $u(x, y), v(x, y)$ to be these expressions on R_1, R_2 . In particular $z_x = u$ and $z_y = v$ almost everywhere. Hence (26), (27) hold on (possibly different) sets R_1, R_2 . Clearly (25) is valid for all (x, y) on R .

(ii). *Uniqueness in (*)*. Suppose that (1) possesses two solutions $z = z_1(x, y), z_2(x, y)$ on R . Let $u_1(x, y), v_1(x, y)$ and $u_2(x, y), v_2(x, y)$ be the functions associated with z_1, z_2 by (i). Let $\alpha = |z_1 - z_2|, \beta = |u_1 - u_2|, \gamma = |v_1 - v_2|$. If the relations (25) for $z = z_1, z_2$ are subtracted, it is seen that the inequality (2) for f implies (7). Similarly (26), (27) imply (8), (9) respectively.

The functions α, β, γ satisfy the assumptions of Lemma 1. Hence the uniqueness assertion in (*) follows from Lemma 1.

(iii). *Existence and successive approximations*. Let $z_0(x, y), z_1(x, y), \dots$ be the successive approximations defined by (4). Corresponding to each $z_n(x, y)$, it is possible to introduce functions $u_n(x, y), v_n(x, y)$ defined on sets R_1, R_2 , respectively, and satisfying $u_0 = \sigma_x(x), v_0 = \tau_y(y)$,

$$(28_n) \quad z_n(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s, t, z_{n-1}(s, t), u_{n-1}(s, t), v_{n-1}(s, t)) ds dt ,$$

$$(29_n) \quad u_n(x, y) = \sigma_x(x) + \int_0^y f(x, t, z_{n-1}(x, t), u_{n-1}(x, t), v_{n-1}(x, t)) dt ,$$

$$(30_n) \quad v_n(x, y) = \tau_y(y) + \int_0^x f(s, y, z_{n-1}(s, y), u_{n-1}(s, y), v_{n-1}(s, y)) ds .$$

The sets R_1, R_2 can be assumed to be independent of n .

Let $Z_{mn} = |z_m - z_n|, U_{mn} = |u_m - u_n|, V_{mn} = |v_m - v_n|$ and

$$(31) \quad \alpha_k(x, y) = \text{l.u.b.}_{m, n \geq k} Z_{mn}, \quad \beta_k(x, y) = \text{l.u.b.}_{m, n \geq k} U_{mn}, \quad \gamma_k(x, y) = \text{l.u.b.}_{m, n \geq k} V_{mn} .$$

It is clear that Z_{mn}, U_{mn}, V_{mn} are uniformly Lipschitz continuous with respect to $(x, y), x, y$, respectively, and that a corresponding statement holds for $\alpha_k, \beta_k, \gamma_k$.

By subtracting the relation (28_n) from (28_{n-1}) and using the inequal-

ity (2) for f , it is seen that

$$Z_{mn}(x, z) \leq \int_0^x \int_0^y \varphi(s, t, Z_{m-1, n-1}(s, t), U_{m-1, n-1}(s, t), V_{m-1, n-1}(s, t)) ds dt .$$

Thus, if $m, n \geq k$, the monotony of φ shows that

$$Z_{mn}(x, y) \leq \int_0^x \int_0^y \varphi(s, t, \alpha_{k-1}(s, t), \beta_{k-1}(s, t), \gamma_{k-1}(s, t)) ds dt .$$

Hence

$$\alpha_k(x, y) \leq \int_0^x \int_0^y \varphi(s, t, \alpha_{k-1}(s, t), \beta_{k-1}(s, t), \gamma_{k-1}(s, t)) ds dt .$$

Similarly

$$\beta_k(x, y) \leq \int_0^y \varphi(x, t, \alpha_{k-1}(x, t), \beta_{k-1}(x, t), \gamma_{k-1}(x, t)) dt ,$$

$$\gamma_k(x, y) \leq \int_0^x \varphi(s, y, \alpha_{k-1}(s, y), \beta_{k-1}(s, y), \gamma_{k-1}(s, y)) ds .$$

By (31), the sequences $\{\alpha_k(x, y)\}$, $\{\beta_k(x, y)\}$, $\{\gamma_k(x, y)\}$ are non-increasing (and non-negative). Let $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$ denote the respective limits of these sequence, The Lipschitz continuity of $\alpha_k, \beta_k, \gamma_k$ is preserved under the limiting process. Lebesgue's theorem on term-by-term integration under bounded convergence gives the inequalities (7), (8), (9). Hence Lemma 1 shows that $\alpha \equiv 0, \beta \equiv 0, \gamma \equiv 0$ on R, R_1, R_2 , respectively. This implies the existence of the functions $z = \lim z_n, u = \lim u_n, v = \lim v_n$ on R_1, R_2 , as $n \rightarrow \infty$, satisfying (25), (26), (27). It is clear that the limit function $z(x, y)$ is a solution of (1).

Finally, the equicontinuity of the functions $z_n(x, y)$ (implied by their uniform Lipschitz continuity) shows that $z(x, z)$ is the uniform limit of the $z_n(x, y)$. This proves (*).

5. Lemma for ().** The proof of (**) will depend on the following lemma:

LEMMA 2. *Let $\alpha(x, y), \beta(x, y), \gamma(x, y)$ be non-negative, measurable functions defined on R, R_1, R_2 , respectively, so that α is continuous, β is uniformly Lipschitz continuous with respect to y and γ is uniformly Lipschitz continuous with respect to x . Furthermore, assume that*

$$(32) \quad \alpha(x, y)/xy \rightarrow 0 \text{ as } 0 < xy \rightarrow 0$$

and that, uniformly with respect to x and y , respectively,

$$(33) \quad \beta(x, y)/y \rightarrow 0 \text{ as } y \rightarrow 0 \text{ and } \gamma(x, y)/x \rightarrow 0 \text{ as } x \rightarrow 0 .$$

Finally, suppose that

$$(34) \quad \alpha(x, y) \leq \int_0^x \int_0^y \{c_1(s, t) \alpha(s, t)/st + c_2(s, t) \beta(s, t)/t + c_3(s, t) \gamma(s, t)/s\} ds dt ,$$

$$(35) \quad \beta(x, y) \leqq \int_0^y \{c_1(x, t)\alpha(x, t)/xt + c_2(x, t)\beta(x, t)/y + c_3(x, t)\gamma(x, t)/x\} dt ,$$

$$(36) \quad \gamma(x, y) \leqq \int_0^x \{c_1(s, y)\alpha(s, y)/sy + c_2(s, y)\beta(s, y)/y + c_3(s, y)\gamma(s, y)/s\} ds ,$$

where c_1, c_2, c_3 are as in the first part of (**). Then $\alpha \equiv \beta \equiv \gamma \equiv 0$.

Proof. By (32), if $\alpha(x, y)/xy$ is defined as 0 when $xy = 0$, it becomes a continuous function on R . Hence, it assumes its maximum M_1 at some point $(x^1, y^1) \in R$. Let $M_2 = 1.u.b. \beta(x, y)/y$ and $M_3 = 1.u.b. \gamma(x, y)/x$ for $(x, y) \in R$.

Note that there exist numbers M_{jk} , where $j, k = 1, 2, 3$, satisfying

$$(37) \quad M_{jk} \geqq 0 \text{ and } \sum_{k=1}^3 M_{jk} = 1 \quad \text{for } j = 1, 2, 3 ,$$

and

$$(38_j) \quad M_j \leqq \sum_{k=1}^3 M_{jk} M_k .$$

If $M_1 \neq 0$, then $M_1 = \alpha(x^1, y^1)/x^1y^1$ holds for some point (x^1, y^1) of R with $x^1y^1 > 0$. In this case, (38₁) follows from (34) with $(x, y) = (x^1, y^1)$ if

$$(39) \quad M_{1k} = (x^1y^1)^{-1} \int_0^{x^1} \int_0^{y^1} c_k(s, t) ds dt .$$

If $M_1 = 0$, let $M_{1k} = c_k(0, 0)$.

In order to obtain (38₂), let (x_j, y_j) , where $j = 1, 2, \dots$, be points of R such that $\lim (x_j, y_j) = (x^2, y^2)$ exists, $\lim \beta(x_j, y_j)/y_j = M_2$ and $\lim \beta(x_j, y) = \beta(y)$ exists uniformly for $0 \leqq y \leqq b$. Then (35) leads to (38₂) with

$$(40) \quad M_{2k} = (y^2)^{-1} \int_0^{y^2} c_k(x^2, t) dt \text{ or } M_{2k} = c_k(x^2, 0)$$

according as $y^2 > 0$ or $y^2 = 0$. A relation of the type (38₃) is obtained similarly.

Let $M_J = \max(M_1, M_2, M_3)$. Suppose, if possible, that $M_J > 0$. Assume, for the moment, that $M_J > M_j$ if $j \neq J$. Then, by (37) and (38_J), $M_{JJ} = 1$ and $M_{Jk} = 0$ for $k \neq J$. But the derivation of (38_J) can then be modified to obtain $M_J < M_J$. For example, if $J = 1$, then $c_1(s, t) \equiv 1$ and $c_2(s, t) = c_3(s, t) = 0$ in (34) when $(x, y) = (x^1, y^1)$, while $\alpha(s, t)/st$ is nearly zero for small st , so that one obtains $M_1 < M_1$. Or if $J = 2$, then $y^2 > 0$ and $c_1(x^2, t) = 1, c_2(x^2, t) = c_3(x^2, t) = 0$ for $0 \leqq t \leqq y^2$, while the relations

$$\beta(y) \leqq \int_0^y \beta(t) dt/t, \quad \beta(y^2)/y^2 = M_2$$

give $M_2 < M_2$ since $\beta(t)/t$ is nearly 0 for small t by the uniformity of

the first limit relation in (33).

Similar arguments show that if two or three of the numbers M_1, M_2, M_3 are equal to $M_j > 0$, one is led to a contradiction. Hence $M_j = 0$. This proves the lemma.

6. Proof of ().** (i). *Uniqueness* in (**). Let $z = z_1(x, y), z_2(x, y)$ be two solutions of (1) on R . Let $u_1(x, y), v_1(x, y)$ and $u_2(x, y), v_2(x, y)$ be the functions associated with them as in the proof of (*). Let $\alpha = |z_1 - z_2|, \beta = |u_1 - u_2|, \gamma = |v_1 - v_2|$. It will be verified that, as x (or y) $\rightarrow 0$, then, except for sets of measure zero,

$$(41) \quad \alpha(x, y), \beta(x, y), \gamma(x, y) \rightarrow 0 .$$

Consider the case $x \rightarrow 0$. The assertions (41) concerning α and γ are clear. In order to verify assertion (41) for the function β , it will first be shown that if $z = z(x, y)$ is any solution of (1) (say, $z = z_1$ or $z = z_2$) and if $u(x, y), v(x, y)$ are its associated functions, then

$$(42) \quad \lim u(x, y) = \rho(y), \text{ as } x \rightarrow 0, \text{ exists uniformly in } y .$$

To see this, let x_j , where $j = 1, 2, 3, \dots$ be a sequence of x values such that $\lim x_j = 0$ and $\lim u(x_j, y) = \rho(y)$ exists uniformly as $j \rightarrow \infty$. Putting $x = x_j$ in (26) and letting $j \rightarrow \infty$, it is seen that

$$(43) \quad \rho(y) = \sigma_x(+0) + \int_0^y f(0, t, \tau(t), \rho(t), \tau_y(t)) dt .$$

We note that $\rho(y)$ is continuous. Furthermore, $\rho(y)$ does not depend on the sequence x_1, x_2, \dots . Suppose that another sequence leads to a different limit $\bar{\rho}(y) \neq \rho(y)$. By substituting $\bar{\rho}$ for ρ in (43), and subtracting, we get

$$(44) \quad |\bar{\rho}(y) - \rho(y)| \leq \int_0^y |f(0, t, \tau(t), \bar{\rho}(t), \tau_y(t)) - f(0, t, \tau(t), \rho(t), \tau_y(t))| dt .$$

Since $f, \rho, \bar{\rho}$ are continuous and $\rho(0) = \bar{\rho}(0) = \sigma_x(+0)$, the integrand of (44) can be made small by making y small. Hence

$$(45) \quad |\bar{\rho}(y) - \rho(y)|/y \rightarrow 0, \text{ as } y \rightarrow 0 .$$

By relation (5),

$$|\bar{\rho}(y) - \rho(y)|/y \leq y^{-1} \int_0^y c_2(0, t) |\bar{\rho}(t) - \rho(t)| dt/t ,$$

Using (45) as before, this leads to a contradiction. Hence $\bar{\rho} \equiv \rho$. Therefore every sequence, for which the limit in (42) exists, leads to the same limit. Hence (42) holds.

If $\lim u_1(x, y) = \rho_1(y)$ and $\lim u_2(x, y) = \rho_2(y)$, as $x \rightarrow 0$, we can repeat the above argument and obtain $\rho_1 \equiv \rho_2$. This completes the verification of (41).

We now verify assumptions (32) and (33) of Lemma 2. Consider, for example, the assertion

$$(46) \quad \beta(x, y)/y \rightarrow 0 \text{ as } y \rightarrow 0 .$$

By putting $u = u_1, u_2$ in (26) and subtracting we get

$$(47) \quad \beta(x, y) \leq \int_0^y |f(x, t, z_1(x, t), u_1(x, t), v_1(x, t)) - f(x, t, z_2(x, t), u_2(x, t), v_2(x, t))| dt .$$

Now the integrand of (47) can be made small, by making y small, and using (41). This proves (46). The other limits in (32) and (33) are verified similarly. The other assumptions of Lemma 2 are quite straightforward. Therefore $\alpha \equiv \beta \equiv \gamma \equiv 0$. This proves ‘‘uniqueness’’.

(ii). *Existence and successive approximations in (**)*. Let $z_0(x, y), z_1(x, y), \dots$, be the successive approximations defined by (4). Corresponding to $z_n(x, y)$ it is possible to introduce, as in the proof of (*), functions $u_n(x, y), v_n(x, y)$ defined on sets R_1, R_2 (independent of n) and satisfying $u_0 = \sigma_x(x), v_0 = \tau_y(y), (28_n), (29_n)$ and (30_n) . Let Z_{mn}, U_{mn}, V_{mn} be defined as in the existence proof (*) above. It will be verified that, given ϵ , there exists a $\delta(\epsilon)$ and an $N(\epsilon)$, such that

$$(48) \quad Z_{mn}(x, y), U_{mn}(x, y), V_{mn}(x, y) < \epsilon$$

for $x < \delta(\epsilon)$ and for all $m, n > N(\epsilon)$. A similar statement will be seen to hold when x is replaced by y . The assertion (48) concerning Z_{mn} and V_{mn} is clear. In order to verify (48) for the function U_{mn} it will first be shown that

$$(49) \quad \lim u_n(x, y) = h_n(y), \text{ as } x \rightarrow 0, \text{ exists uniformly in } y \text{ and } n .$$

It is easily verified, by induction, that $h_n(y)$ exists uniformly in y for fixed n , where

$$(50_n) \quad h_n(y) = \sigma_x(+0) + \int_0^y f(0, t, \tau(t), h_{n-1}(y), \tau_y(t)) dt .$$

To see the uniformity in n , define

$$(51_n) \quad \bar{z}_n(x, y) = z_n(x, y) - \sigma(x) - \tau(y) + z_0; \bar{u}_n(y, y) = u_n(y, y) - \sigma_x(y); \\ \bar{v}_n(x, y) = v_n(x, y) - \tau_y(y) ;$$

$$(52) \quad g(x, y, z, p, q) = f(x, y, z + \sigma(x) + \tau(y) - z_0, p + \sigma_x(x), q + \tau_y(y)) .$$

For \bar{u}_n we define \bar{h}_n corresponding to h . Clearly g satisfies a condition analogous to (5), $\bar{u}_0(x, y) = \bar{h}_0(y) \equiv 0$, and

$$(53_n) \quad \bar{u}_n(x, y) = \int_0^y g(x, t, \bar{z}_{n-1}(x, t), \bar{u}_{n-1}(x, t), \bar{v}_{n-1}(x, t)) dt, n \geq 1$$

$$(54_n) \quad \bar{h}_n(y) = \int_0^y g(0, t, 0, \bar{h}_{n-1}(t), 0) dt, n \geq 1 .$$

To prove (49) it suffices to verify that

$$(55) \quad \lim \bar{u}_n(x, y) = \bar{h}_n(y), \text{ as } x \rightarrow 0, \text{ exists uniformly in } y \text{ and } n.$$

By subtracting (54_n) from (53_n), it is seen that

$$(56) \quad |\bar{u}_n(x, y) - \bar{h}_n(y)| \leq \int_0^y \{|g_1 - g_2| + |g_2 - g_3|\} dt$$

where $g_1 = g(x, t, \bar{z}_{n-1}(x, t), \bar{u}_{n-1}(x, t), \bar{v}_{n-1}(x, t))$, $g_2 = g(0, t, 0, \bar{u}_{n-1}(x, t), 0)$ and $g_3 = g(0, t, 0, \bar{h}_{n-1}(t), 0)$. We note that, given $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that $|g_1 - g_2| < \varepsilon$ if $x < \delta$ for all y and n . Hence, noting (5),

$$(57_n) \quad |\bar{u}_n(x, y) - \bar{h}_n(z)| \leq \int_0^y \{\varepsilon + t^{-1} c_2(0, t) |\bar{u}_{n-1}(x, t) - \bar{h}_{n-1}(t)|\} dt .$$

By continuity, because of (6*), $c_2(0, t) < 1$ for small $t > 0$. Hence there exists a number $\theta, 0 < \theta < 1$, such that

$$\int_0^y c_2(0, t) dt \leq \theta y \text{ for } 0 < y \leq b .$$

A simple induction shows that

$$(58) \quad |\bar{u}_n(x, y) - \bar{h}_n(y)| \leq (1 - \theta^n) \varepsilon y / (1 - \theta) \leq b \varepsilon / (1 - \theta) .$$

This proves (55). Hence (49) is established.

Next we note that $h_n(y), n = 0, 1, 2, \dots$, are the successive approximations for the initial value problem

$$(59) \quad dw/dt = F(t, w), w(0) = \sigma_x(+0) ,$$

where $F(t, w) = f(0, t, \tau(t), w, \tau_y(t))$ is bounded, measurable and continuous in w (for almost all fixed t). By (5),

$$(60) \quad |F(t, w) - F(t, \bar{w})| \leq |w - \bar{w}|/t .$$

Note that the existence of $\tau_y(+0)$ implies that $F(t, w) \rightarrow F(0, w) = f(0, 0, \tau(0), w, \tau_y(+0))$ as $t \rightarrow +0$. The proof of the main theorem in [8] shows that these successive approximations converge uniformly, (60) being Nagumo's uniqueness condition (cf. [5], p. 97). Hence

$$(61) \quad \lim h_n(y) = h(y), \text{ exists uniformly in } y \text{ as } n \rightarrow \infty .$$

Now (61) and (49) together give (48) for $U_{mn}(x, y)$. Hence (48) is established.

By an argument similar to that used in verifying (46) it is seen that, given $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that

$$(52) \quad \begin{aligned} (xy)^{-1} Z_{mn}(x, y) &< \varepsilon \text{ for } xy < \delta(\varepsilon) \text{ and for } m, n > N(\varepsilon) \\ x^{-1} U_{mn}(x, y) &< \varepsilon \text{ for } x < \delta(\varepsilon) \text{ and for } m, n > N(\varepsilon) \\ y^{-1} V_{mn}(x, y) &< \varepsilon \text{ for } y < \delta(\varepsilon) \text{ and for } m, n > N(\varepsilon) . \end{aligned}$$

Now defining $\alpha_k, \beta_k, \gamma_k$ as in (31), we note that we can substitute

them for Z_{mn}, U_{mn}, V_{mn} , respectively, in (62) changing $m, n > N(\varepsilon)$ to $k > N(\varepsilon)$. Proceeding as in the analogous section of the proof of theorem (*), we conclude that α, β, γ , satisfy (34), (35) and (36), also (32) and (33). Therefore, by Lemma 2, the successive approximations converge uniformly to a solution of (1).

7. Counter-examples. (a). Let $a = b = 1, 1 + \varepsilon = \delta^2, \varepsilon > 0, \delta > 1$. Let $f(x, y, z, p, q)$ be independent of p, q and defined by

$$f(x, y, z, p, q) = \begin{cases} 0 & \text{if } (x, y) \in R, z \leq 0, \\ (1 + \varepsilon)z/xy & \text{if } (x, y) \in R, 0 < z < (xy)^\delta, \\ (1 + \varepsilon)(xy)^{\delta-1} & \text{if } (x, y) \in R, (xy)^\delta \leq z. \end{cases}$$

Then $f(x, y, z, p, q)$ is continuous and satisfies (5) for $c_1(x, y) = 1 + \varepsilon$, (and $c_2 = c_3 \equiv 0$). Let $\sigma(x) = \tau(y) \equiv 0$. Then (1) has an infinity of solutions, namely, $z = c(xy)^\delta$, where $0 < c < 1$.

(b). Let $a = b = 1, R^0 = \{(x, y) : 0 < x, y \leq 1\}, 1 + \varepsilon = \delta^2, \varepsilon > 0, \delta > 0$ and

$$f(x, y, z, p, q) = \begin{cases} 0 & \text{if } x = 0, y = 0, \\ (xy)^{\delta-1} & \text{if } (x, y) \in R^0, z < 0, \\ (xy)^{\delta-1} - (1 + \varepsilon)z/xy & \text{if } (x, y) \in R^0, 0 \leq z \leq (xy)^\delta, \\ -\varepsilon(xy)^{\delta-1} & \text{if } (x, y) \in R^0, (xy)^\delta < z. \end{cases}$$

Then $f(x, y, z, p, q)$ satisfies the same relation (5) as in example (a). However, in (4), $z_{2n} = 0, z_{2n+1} = (xy)^\delta/\delta^2$, so that successive approximations (4) do not converge.

REFERENCES

1. E. A. Coddington and N. Levinson, *Uniqueness and the convergence of successive approximations*, J. Indian Math. Soc., **16** (1952), 75-81.
2. F. Guglielmino, *Sulla risoluzione del problema de Darboux per l'equazione $s=f(x, y, z)$* , Bollettino della Unione Matematica Italiana, **13** (1958), 308-318.
3. P. Hartman and A. Wintner, *On hyperbolic partial differential equations*, Amer. J. Math. **74** (1952), 832-864.
4. E. K. Haviland, *A note on the convergence of the successive approximations to the solution of an ordinary differential equation*, Amer. J. Math. **54** (1932), 632-634.
5. E. Kamke, *Differentialgleichungen reeler Funktionen*, Leipzig (1930).
6. J. Kisynski, *Sur l'existence et l'unicité des solutions des problèmes classiques relatifs à l'equation $s=F(x, y, z, p, q)$* , Annales Mariae Curie-Sklodowska, **11** (1957), 73-112.
7. O. Perron, *Eine hinreichende Bedingung für die Unität der Lösung von Differentialgleichungen erster Ordnung*, Mathematische Zeitschrift, **23** (1928), 216-219.
8. B. Viswanatham, *The general uniqueness theorem and successive approximations*, J. Indian Math. Soc., **16** (1952), 69-74.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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