

Pacific Journal of Mathematics

**PROJECTIONS ONTO THE SUBSPACE OF COMPACT
OPERATORS**

EDWARD OAKLEY THORP

PROJECTIONS ONTO THE SUBSPACE OF COMPACT OPERATORS

E. O. THORP

Introduction. The purpose of this paper is to establish the following theorem.

THEOREM. *Suppose U and V are Banach spaces and that there are bounded projections P_1 from U onto X and P_2 from V onto Y . Then there are no bounded projections from the space of bounded operators on U into V onto the closed subspace of compact operators, in the following cases:*

1. X is isomorphic [1] to ℓ^p , $1 \leq p < \infty$; Y is isomorphic to ℓ^q , $1 \leq p \leq q \leq \infty$ or c_0 or c .
2. X is isomorphic to c_0 ; Y is isomorphic to ℓ^∞ , c_0 or c .
3. X is isomorphic to c ; Y is isomorphic to ℓ^∞ .

NOTATION. If X and Y are Banach spaces, $[X, Y]$ is the set of bounded linear operators from X into Y . ℓ^∞ is the set of bounded sequences with the sup norm.

A space X is said to have a countable basis if there is a countable subset of elements of X , called a basis, such that each $x \in X$ is uniquely expressible as

$$x = \sum_{i=1}^{\infty} \xi_i \varphi_i$$

in the sense that

$$\lim_{n \rightarrow \infty} \left\| x - \sum_{i=1}^n \xi_i \varphi_i \right\| = 0.$$

If X and Y are spaces with countable bases (φ_i) and (ψ_i) respectively and A is a bounded linear transformation from X into Y , then A can be represented by an infinite matrix (a_{ij}) , with

$$A\varphi_j = \sum_{i=1}^{\infty} a_{ij} \psi_i$$

[2]. In what follows, the basis used for ℓ^p will be given by $\varphi_j = (0, 0, \dots, 0, 1, 0, 0, \dots)$ where there is a 1 in the j th place and 0 elsewhere. Similarly for ψ_i . The matrix representations of operators will all be with respect to these bases.

Received April 29, 1959. The author thanks Professor Angus Taylor for proposing this problem and thanks both him and Professor Richard Arens for helpful discussions.

Proof of the theorem. The details of the proof are given below only for $X = \ell^p, 1 \leq p < \infty$, and $Y = \ell^q, 1 \leq p \leq q < \infty$. The proof for the remaining pairs is similar and is indicated in a remark at the end.

DEFINITION. Let E be the function on $[\ell^p, \ell^q], 1 \leq p \leq q < \infty$, which sends an operator whose matrix is (a_{ij}) into the operator whose matrix is $(a_{ij}\delta_{ij})$, i.e. the non-diagonal matrix elements are replaced by zero and the diagonal elements are unaltered.

LEMMA 1. E is a projection with $\|E\| = 1$, range the diagonal operators, and null-space the operators with $a_{ii} = 0$, all i .

Proof. E is additive and homogeneous as easily follows from [2]. $E^2 = E$, and the characterization of the range and null-spaces are apparent.

From the chain

$$\begin{aligned} \infty > \|A\| &= \sup_{\|x\|_p \leq 1} \|Ax\|_q \geq \sup_j \|A\varphi_j\|_q \\ &= \sup_j \left\| \sum_i a_{ij} \psi_j \right\|_q \geq \sup_j \|a_{jj} \psi_j\|_q = \sup_j |a_{jj}| \\ &\geq \sup_{\sum |\xi_i|^p \leq 1} (\sum |a_{ii} \xi_i|^p)^{1/p} \geq \sup_{\|x\|_p \leq 1} (\sum |a_{ii} \xi_i|^q)^{1/q} = \|EA\|, \end{aligned}$$

where the last \geq is by Jensen's inequality, we see that E sends bounded operators into bounded operators and, further, $\|E\| = 1$. Also

$$\|EA\| \leq \sup_j |a_{jj}|.$$

In fact,

$$\|EA\| = \sup_j |a_{jj}|$$

because

$$\|EA\| \geq \sup_j \|EA\varphi_j\| = \sup_j |a_{jj}|.$$

LEMMA 2. The mapping γ from the set of diagonal operators onto ℓ^∞ defined by $\gamma(a_{ii}) = (a_{11}, a_{22}, \dots)$ is an isometry which carries the compact diagonal operators onto c_0 .

Proof. That γ is an isometry from the diagonal operators onto ℓ^∞ follows from the previous observation that $\|EA\| = \sup_j |a_{jj}|$. Hence it suffices to show that the compact diagonal operators are exactly those with the additional condition $\lim_i |a_{ii}| = 0$. This condition is necessary;

otherwise for some $\varepsilon > 0$ there is an infinite index set I such that $|a_{ii}| \geq \varepsilon$ whenever $i \in I$. Then the bounded sequence $(\rho_i)_{i \in I}$ would be carried into the sequence $(a_{ii}\rho_i)_{i \in I}$, which has no convergent subsequence, showing (a_{ii}) is not compact. The condition is sufficient because, if $\|x\|_p \leq 1$ then

$$\left(\sum_{i=1}^{\infty} |a_{ii}\xi_i|^q\right)^{1/q} \leq (\sup_{i \geq n} |a_{ii}|) \|x\|_q \leq \sup_{i \geq n} |a_{ii}|$$

and [2; Th. 2] applies. The last inequality follows from Jensen's inequality and our assumptions $p \leq q, \|x\|_p \leq 1$.

LEMMA 3. *Suppose X is a Banach space with a closed subspace \mathfrak{M} onto which there is a bounded projection E . Let \mathfrak{N} be the null-space of E . Let \mathfrak{A} be any closed linear manifold of X such that if $f \in \mathfrak{A}$ then $f = g + h$, with $g \in \mathfrak{A} \cap \mathfrak{M}$ and $h \in \mathfrak{A} \cap \mathfrak{N}$. Then, given any bounded projection F onto \mathfrak{A} , EF is a bounded projection onto $\mathfrak{A} \cap \mathfrak{M}$ such that $\|EF\| \leq \|E\| \|F\|$.*

The proof is an obvious modification of [3; Lemma 1.2.1].

Let \mathfrak{A} be the set of compact operators, \mathfrak{M} the set of diagonal operators, E the projection of Lemma 1, and \mathfrak{N} its null-space. In order to apply Lemma 3 it remains to show: given any compact operator f , Ef and $f - Ef$ are compact. Ef is compact because, if f is compact,

$$\lim_n \left\| \sum_{i=n}^{\infty} a_{ij}\psi_i \right\| = \lim_n \left(\sum_{i=n}^{\infty} |a_{ij}|^q \right)^{1/q} = 0$$

uniformly in j . This implies $\lim_n |a_{nn}| = 0$, which shows that Ef is compact. Hence $f - Ef$ is compact.

To prove the theorem for $[\sphericalangle^p, \sphericalangle^q], 1 \leq p \leq q < \infty$, assume there is a bounded projection F from $[\sphericalangle^p, \sphericalangle^q]$ onto \mathfrak{A} . By Lemma 3, the restriction of EF to \mathfrak{M} is a bounded projection from \mathfrak{M} onto $\mathfrak{M} \cap \mathfrak{A}$. By Lemma 2 there must be a corresponding bounded projection from \sphericalangle^∞ onto c_0 . This contradicts [4; Cor. 7.5]. For the remaining X and Y pairs of the main theorem, the proof is similar except that the existence of expressions for $\|A\|$ in terms of the matrix coefficients (e.g., see [5]) makes some of the work simpler.

Next we extend the theorem to $[U, V]$. Let \tilde{E} be the function on $[U, V]$ defined by $\tilde{E}f = P_2fP_1$ for all f in $[U, V]$. \tilde{E} is linear and homogeneous and bounded. $\tilde{E}^2f = P_2(P_2fP_1)P_1 = P_2fP_1 = \tilde{E}f$ so \tilde{E} is a projection. The range of \tilde{E} is the set of operators g such that $P_2gP_1 = g$ and is isomorphic with $[X, Y]$. The null-space of \tilde{E} is the set of operators h such that $P_2hP_1 = 0$. If Q_i is the projection $I - P_i$, the

decomposition $f = g + h$ is given by

$$f = (P_2 + Q_2)f(P_1 + Q_1) = \underbrace{P_2fP_1}_g + \underbrace{(P_2fQ_1 + Q_2fP_1 + Q_2fQ_1)}_h.$$

If f is compact, so are g and h . We apply Lemma 3 with $X = [U, V]$, \mathfrak{M} the range of \tilde{E}, \tilde{E} acting as the projection E of that lemma, and \mathfrak{P} the set of compact operators from U to V . The conclusion is that if there were a bounded projection F from X to \mathfrak{P} , the restriction of $\tilde{E}F$ to \mathfrak{M} would be a bounded projection from \mathfrak{M} onto $\mathfrak{P} \cap \mathfrak{M}$, contradicting our result for $[X, Y]$.

REMARK. The problem of finding a bounded projection onto the compact operators is trivial when all the bounded operators are compact. This happens, for example, for $[\not\prec^p, \not\prec^q]$, $\infty > p > q \geq 1$, [2, p. 700], or $p = \infty, q = 1$, and for $[c_0, \not\prec^q]$, $[c, \not\prec^q]$, $\infty > q \geq 1$. Whether there exists a pair of normed spaces with a bounded proper projection from the bounded operators onto the compact operators seems to be unknown.

REFERENCES

1. S. Banach, *Theorie des opérations linéaires* Warsaw, 1932.
2. L. W. Cohen and N. Dunford, *Transformations in sequence spaces*, Duke Math. J., **3** (1937), 689-701.
3. F. J. Murray, *On complementary manifolds and projections in spaces L_p and $\not\prec_p$* , Trans. Amer. Math. Soc., **41** (1937), 138-152.
4. R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc., **48** (1940), 516-541.
5. A. E. Taylor, *Introduction to functional analysis*, John Wiley and Co., (1958).

UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG
Stanford University
Stanford, California

F. H. BROWNELL
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

L. J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY
D. DERRY

E. HEWITT
A. HORN
L. NACHBIN

M. OHTSUKA
H. L. ROYDEN
M. M. SCHIFFER

E. SPANIER
E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 10, No. 2

October, 1960

Maynard G. Arsove, <i>The Paley-Wiener theorem in metric linear spaces</i>	365
Robert (Yisrael) John Aumann, <i>Acceptable points in games of perfect information</i>	381
A. V. Balakrishnan, <i>Fractional powers of closed operators and the semigroups generated by them</i>	419
Dallas O. Banks, <i>Bounds for the eigenvalues of some vibrating systems</i>	439
Billy Joe Boyer, <i>On the summability of derived Fourier series</i>	475
Robert Breusch, <i>An elementary proof of the prime number theorem with remainder term</i>	487
Edward David Callender, Jr., <i>Hölder continuity of n-dimensional quasi-conformal mappings</i>	499
L. Carlitz, <i>Note on Alder's polynomials</i>	517
P. H. Doyle, III, <i>Unions of cell pairs in E^3</i>	521
James Eells, Jr., <i>A class of smooth bundles over a manifold</i>	525
Shaul Foguel, <i>Computations of the multiplicity function</i>	539
James G. Glimm and Richard Vincent Kadison, <i>Unitary operators in C^*-algebras</i>	547
Hugh Gordon, <i>Measure defined by abstract L_p spaces</i>	557
Robert Clarke James, <i>Separable conjugate spaces</i>	563
William Elliott Jenner, <i>On non-associative algebras associated with bilinear forms</i>	573
Harold H. Johnson, <i>Terminating prolongation procedures</i>	577
John W. Milnor and Edwin Spanier, <i>Two remarks on fiber homotopy type</i>	585
Donald Alan Norton, <i>A note on associativity</i>	591
Ronald John Nunke, <i>On the extensions of a torsion module</i>	597
Joseph J. Rotman, <i>Mixed modules over valuations rings</i>	607
A. Sade, <i>Théorie des systèmes démosiens de groupoïdes</i>	625
Wolfgang M. Schmidt, <i>On normal numbers</i>	661
Berthold Schweizer, Abe Sklar and Edward Oakley Thorp, <i>The metrization of statistical metric spaces</i>	673
John P. Shanahan, <i>On uniqueness questions for hyperbolic differential equations</i>	677
A. H. Stone, <i>Sequences of coverings</i>	689
Edward Oakley Thorp, <i>Projections onto the subspace of compact operators</i>	693
L. Bruce Treybig, <i>Concerning certain locally peripherally separable spaces</i>	697
Milo Wesley Weaver, <i>On the commutativity of a correspondence and a permutation</i>	705
David Van Vranken Wend, <i>On the zeros of solutions of some linear complex differential equations</i>	713
Fred Boyer Wright, Jr., <i>Polarity and duality</i>	723