

# Pacific Journal of Mathematics

## **PROJECTIONS ONTO THE SUBSPACE OF COMPACT OPERATORS**

EDWARD OAKLEY THORP

# PROJECTIONS ONTO THE SUBSPACE OF COMPACT OPERATORS

E. O. THORP

**Introduction.** The purpose of this paper is to establish the following theorem.

**THEOREM.** *Suppose  $U$  and  $V$  are Banach spaces and that there are bounded projections  $P_1$  from  $U$  onto  $X$  and  $P_2$  from  $V$  onto  $Y$ . Then there are no bounded projections from the space of bounded operators on  $U$  into  $V$  onto the closed subspace of compact operators, in the following cases:*

1.  $X$  is isomorphic [1] to  $\ell^p$ ,  $1 \leq p < \infty$ ;  $Y$  is isomorphic to  $\ell^q$ ,  $1 \leq p \leq q \leq \infty$  or  $c_0$  or  $c$ .
2.  $X$  is isomorphic to  $c_0$ ;  $Y$  is isomorphic to  $\ell^\infty$ ,  $c_0$  or  $c$ .
3.  $X$  is isomorphic to  $c$ ;  $Y$  is isomorphic to  $\ell^\infty$ .

**NOTATION.** If  $X$  and  $Y$  are Banach spaces,  $[X, Y]$  is the set of bounded linear operators from  $X$  into  $Y$ .  $\ell^\infty$  is the set of bounded sequences with the sup norm.

A space  $X$  is said to have a countable basis if there is a countable subset of elements of  $X$ , called a basis, such that each  $x \in X$  is uniquely expressible as

$$x = \sum_{i=1}^{\infty} \xi_i \varphi_i$$

in the sense that

$$\lim_{n \rightarrow \infty} \left\| x - \sum_{i=1}^n \xi_i \varphi_i \right\| = 0.$$

If  $X$  and  $Y$  are spaces with countable bases  $(\varphi_i)$  and  $(\psi_i)$  respectively and  $A$  is a bounded linear transformation from  $X$  into  $Y$ , then  $A$  can be represented by an infinite matrix  $(a_{ij})$ , with

$$A\varphi_j = \sum_{i=1}^{\infty} a_{ij} \psi_i$$

[2]. In what follows, the basis used for  $\ell^p$  will be given by  $\varphi_j = (0, 0, \dots, 0, 1, 0, 0, \dots)$  where there is a 1 in the  $j$ th place and 0 elsewhere. Similarly for  $\psi_i$ . The matrix representations of operators will all be with respect to these bases.

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*Proof of the theorem.* The details of the proof are given below only for  $X = \ell^p, 1 \leq p < \infty$ , and  $Y = \ell^q, 1 \leq p \leq q < \infty$ . The proof for the remaining pairs is similar and is indicated in a remark at the end.

**DEFINITION.** Let  $E$  be the function on  $[\ell^p, \ell^q], 1 \leq p \leq q < \infty$ , which sends an operator whose matrix is  $(a_{ij})$  into the operator whose matrix is  $(a_{ij}\delta_{ij})$ , i.e. the non-diagonal matrix elements are replaced by zero and the diagonal elements are unaltered.

**LEMMA 1.**  $E$  is a projection with  $\|E\| = 1$ , range the diagonal operators, and null-space the operators with  $a_{ii} = 0$ , all  $i$ .

*Proof.*  $E$  is additive and homogeneous as easily follows from [2].  $E^2 = E$ , and the characterization of the range and null-spaces are apparent.

From the chain

$$\begin{aligned} \infty > \|A\| &= \sup_{\|x\|_p \leq 1} \|Ax\|_q \geq \sup_j \|A\varphi_j\|_q \\ &= \sup_j \left\| \sum_i a_{ij} \psi_j \right\|_q \geq \sup_j \|a_{jj} \psi_j\|_q = \sup_j |a_{jj}| \\ &\geq \sup_{\sum |\xi_i|^p \leq 1} \left( \sum |a_{ii} \xi_i|^p \right)^{1/p} \geq \sup_{\|x\|_p \leq 1} \left( \sum |a_{ii} \xi_i|^q \right)^{1/q} = \|EA\|, \end{aligned}$$

where the last  $\geq$  is by Jensen's inequality, we see that  $E$  sends bounded operators into bounded operators and, further,  $\|E\| = 1$ . Also

$$\|EA\| \leq \sup_j |a_{jj}|.$$

In fact,

$$\|EA\| = \sup_j |a_{jj}|$$

because

$$\|EA\| \geq \sup_j \|EA\varphi_j\| = \sup_j |a_{jj}|.$$

**LEMMA 2.** The mapping  $\gamma$  from the set of diagonal operators onto  $\ell^\infty$  defined by  $\gamma(a_{ii}) = (a_{11}, a_{22}, \dots)$  is an isometry which carries the compact diagonal operators onto  $c_0$ .

*Proof.* That  $\gamma$  is an isometry from the diagonal operators onto  $\ell^\infty$  follows from the previous observation that  $\|EA\| = \sup_j |a_{jj}|$ . Hence it suffices to show that the compact diagonal operators are exactly those with the additional condition  $\lim_i |a_{ii}| = 0$ . This condition is necessary;

otherwise for some  $\varepsilon > 0$  there is an infinite index set  $I$  such that  $|a_{ii}| \geq \varepsilon$  whenever  $i \in I$ . Then the bounded sequence  $(\varphi_i)_{i \in I}$  would be carried into the sequence  $(a_{ii}\varphi_i)_{i \in I}$ , which has no convergent subsequence, showing  $(a_{ii})$  is not compact. The condition is sufficient because, if  $\|x\|_p \leq 1$  then

$$\left(\sum_{i=1}^{\infty} |a_{ii}\xi_i|^q\right)^{1/q} \leq \left(\sup_{i \geq n} |a_{ii}|\right) \|x\|_q \leq \sup_{i \geq n} |a_{ii}|$$

and [2; Th. 2] applies. The last inequality follows from Jensen's inequality and our assumptions  $p \leq q, \|x\|_p \leq 1$ .

**LEMMA 3.** *Suppose  $X$  is a Banach space with a closed subspace  $\mathfrak{M}$  onto which there is a bounded projection  $E$ . Let  $\mathfrak{N}$  be the null-space of  $E$ . Let  $\mathfrak{A}$  be any closed linear manifold of  $X$  such that if  $f \in \mathfrak{A}$  then  $f = g + h$ , with  $g \in \mathfrak{A} \cap \mathfrak{M}$  and  $h \in \mathfrak{A} \cap \mathfrak{N}$ . Then, given any bounded projection  $F$  onto  $\mathfrak{A}$ ,  $EF$  is a bounded projection onto  $\mathfrak{A} \cap \mathfrak{M}$  such that  $\|EF\| \leq \|E\| \|F\|$ .*

The proof is an obvious modification of [3; Lemma 1.2.1].

Let  $\mathfrak{A}$  be the set of compact operators,  $\mathfrak{M}$  the set of diagonal operators,  $E$  the projection of Lemma 1, and  $\mathfrak{N}$  its null-space. In order to apply Lemma 3 it remains to show: given any compact operator  $f$ ,  $Ef$  and  $f - Ef$  are compact.  $Ef$  is compact because, if  $f$  is compact,

$$\lim_n \left\| \sum_{i=n}^{\infty} a_{ij}\varphi_i \right\| = \lim_n \left( \sum_{i=n}^{\infty} |a_{ij}|^q \right)^{1/q} = 0$$

uniformly in  $j$ . This implies  $\lim_n |a_{nn}| = 0$ , which shows that  $Ef$  is compact. Hence  $f - Ef$  is compact.

To prove the theorem for  $[\not\sim^p, \not\sim^q], 1 \leq p \leq q < \infty$ , assume there is a bounded projection  $F$  from  $[\not\sim^p, \not\sim^q]$  onto  $\mathfrak{A}$ . By Lemma 3, the restriction of  $EF$  to  $\mathfrak{M}$  is a bounded projection from  $\mathfrak{M}$  onto  $\mathfrak{M} \cap \mathfrak{A}$ . By Lemma 2 there must be a corresponding bounded projection from  $\not\sim^\infty$  onto  $c_0$ . This contradicts [4; Cor. 7.5]. For the remaining  $X$  and  $Y$  pairs of the main theorem, the proof is similar except that the existence of expressions for  $\|A\|$  in terms of the matrix coefficients (e.g., see [5]) makes some of the work simpler.

Next we extend the theorem to  $[U, V]$ . Let  $\tilde{E}$  be the function on  $[U, V]$  defined by  $\tilde{E}f = P_2fP_1$  for all  $f$  in  $[U, V]$ .  $\tilde{E}$  is linear and homogeneous and bounded.  $\tilde{E}^2f = P_2(P_2fP_1)P_1 = P_2fP_1 = \tilde{E}f$  so  $\tilde{E}$  is a projection. The range of  $\tilde{E}$  is the set of operators  $g$  such that  $P_2gP_1 = g$  and is isomorphic with  $[X, Y]$ . The null-space of  $\tilde{E}$  is the set of operators  $h$  such that  $P_2hP_1 = 0$ . If  $Q_i$  is the projection  $I - P_i$ , the

decomposition  $f = g + h$  is given by

$$f = (P_2 + Q_2)f(P_1 + Q_1) = \underbrace{P_2fP_1}_g + \underbrace{(P_2fQ_1 + Q_2fP_1 + Q_2fQ_1)}_h.$$

If  $f$  is compact, so are  $g$  and  $h$ . We apply Lemma 3 with  $X = [U, V]$ ,  $\mathfrak{M}$  the range of  $\tilde{E}, \tilde{E}$  acting as the projection  $E$  of that lemma, and  $\mathfrak{P}$  the set of compact operators from  $U$  to  $V$ . The conclusion is that if there were a bounded projection  $F$  from  $X$  to  $\mathfrak{P}$ , the restriction of  $\tilde{E}F$  to  $\mathfrak{M}$  would be a bounded projection from  $\mathfrak{M}$  onto  $\mathfrak{P} \cap \mathfrak{M}$ , contradicting our result for  $[X, Y]$ .

REMARK. The problem of finding a bounded projection onto the compact operators is trivial when all the bounded operators are compact. This happens, for example, for  $[\not\prec^p, \not\prec^q]$ ,  $\infty > p > q \geq 1$ , [2, p. 700], or  $p = \infty, q = 1$ , and for  $[c_0, \not\prec^q]$ ,  $[c, \not\prec^q]$ ,  $\infty > q \geq 1$ . Whether there exists a pair of normed spaces with a bounded proper projection from the bounded operators onto the compact operators seems to be unknown.

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