PROJECTIONS ONTO THE SUBSPACE OF COMPACT OPERATORS

Edward Oakley Thorp
Introduction. The purpose of this paper is to establish the following theorem.

Theorem. Suppose $U$ and $V$ are Banach spaces and that there are bounded projections $P_1$ from $U$ onto $X$ and $P_2$ from $V$ onto $Y$. Then there are no bounded projections from the space of bounded operators on $U$ into $V$ onto the closed subspace of compact operators, in the following cases:

1. $X$ is isomorphic to $\ell^p$, $1 \leq p < \infty$; $Y$ is isomorphic to $\ell^q$, $1 \leq p \leq q \leq \infty$ or $c_0$ or $c$.
2. $X$ is isomorphic to $c_0$; $Y$ is isomorphic to $\ell^\infty$, $c_0$ or $c$.
3. $X$ is isomorphic to $c$; $Y$ is isomorphic to $\ell^\infty$.

Notation. If $X$ and $Y$ are Banach spaces, $[X, Y]$ is the set of bounded linear operators from $X$ into $Y$. $\ell^\infty$ is the set of bounded sequences with the sup norm.

A space $X$ is said to have a countable basis if there is a countable subset of elements of $X$, called a basis, such that each $x \in X$ is uniquely expressible as

$$x = \sum_{i=1}^{\infty} \xi_i \varphi_i$$

in the sense that

$$\lim_{n \to \infty} \|x - \sum_{i=1}^{n} \xi_i \varphi_i\| = 0.$$ 

If $X$ and $Y$ are spaces with countable bases $(\varphi_i)$ and $(\psi_i)$ respectively and $A$ is a bounded linear transformation from $X$ into $Y$, then $A$ can be represented by an infinite matrix $(a_{ij})$, with

$$A \varphi_j = \sum_{i=1}^{\infty} a_{ij} \psi_i$$

[2]. In what follows, the basis used for $\ell^p$ will be given by $\varphi_j = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ where there is a 1 in the $j$th place and 0 elsewhere. Similarly for $\psi_i$. The matrix representations of operators will all be with respect to these bases.

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Proof of the theorem. The details of the proof are given below only for \( X = \ell^p, 1 \leq p < \infty \), and \( Y = \ell^q, 1 \leq p < q < \infty \). The proof for the remaining pairs is similar and is indicated in a remark at the end.

DEFINITION. Let \( E \) be the function on \([\ell^p, \ell^q], 1 \leq p \leq q < \infty\), which sends an operator whose matrix is \((a_{ij})\) into the operator whose matrix is \((a_{ij}\delta_{ij})\), i.e. the non-diagonal matrix elements are replaced by zero and the diagonal elements are unaltered.

LEMMA 1. \( E \) is a projection with \( \|E\| = 1 \), range the diagonal operators, and null-space the operators with \( a_{ii} = 0 \), all \( i \).

Proof. \( E \) is additive and homogeneous as easily follows from [2]. \( E^2 = E \), and the characterization of the range and null-spaces are apparent.

From the chain
\[
\sup_{\|x\|_p \leq 1} \|Ax\|_q \geq \sup_j \|A\varphi_j\|_q = \sup_j \|\sum a_{jj} \varphi_j\|_q = \sup_j |a_{jj}|
\]
where the last \( \geq \) is by Jensen’s inequality, we see that \( E \) sends bounded operators into bounded operators and, further, \( \|E\| = 1 \). Also
\[
\|EA\| \leq \sup_j |a_{jj}|
\]
In fact,
\[
\|EA\| = \sup_j |a_{jj}|
\]
because
\[
\|EA\| \geq \sup_j \|EA\varphi_j\| = \sup_j |a_{jj}|
\]

LEMMA 2. The mapping \( \gamma \) from the set of diagonal operators onto \( \ell^\infty \) defined by \( \gamma(a_{ii}) = (a_{11}, a_{22}, \cdots) \) is an isometry which carries the compact diagonal operators onto \( c_0 \).

Proof. That \( \gamma \) is an isometry from the diagonal operators onto \( \ell^\infty \) follows from the previous observation that \( \|EA\| = \sup_j |a_{jj}| \). Hence it suffices to show that the compact diagonal operators are exactly those with the additional condition \( \lim_i |a_{ii}| = 0 \). This condition is necessary;
otherwise for some $\varepsilon > 0$ there is an infinite index set $I$ such that $|a_{ii}| \geq \varepsilon$ whenever $i \in I$. Then the bounded sequence $(\varphi_i)_{i \in I}$ would be carried into the sequence $(a_{ii} \varphi_i)_{i \in I}$, which has no convergent subsequence, showing $(a_{ii})$ is not compact. The condition is sufficient because, if $\|x\|_p \leq 1$ then
\[
\left( \sum_{i=1}^{\infty} |a_{ii} x_i|^q \right)^{1/q} \leq \left( \sup_{i \geq n} |a_{ii}| \right) \|x\|_q \leq \sup_{i \geq n} |a_{ii}|
\]
and [2; Th. 2] applies. The last inequality follows from Jensen’s inequality and our assumptions $p \leq q, \|x\|_p \leq 1$.

**Lemma 3.** Suppose $X$ is a Banach space with a closed subspace $\mathcal{M}$ onto which there is a bounded projection $E$. Let $\mathcal{N}$ be the null-space of $E$. Let $\mathcal{P}$ be any closed linear manifold of $X$ such that if $f \in \mathcal{P}$ then $f = g + h$, with $g \in \mathcal{P} \cap \mathcal{N}$ and $h \in \mathcal{P} \cap \mathcal{N}$. Then, given any bounded projection $F$ onto $\mathcal{P}$, $EF$ is a bounded projection onto $\mathcal{P} \cap \mathcal{N}$ such that $\|EF\| \leq \|E\| \|F\|$.

The proof is an obvious modification of [3; Lemma 1.2.1].

Let $\mathcal{P}$ be the set of compact operators, $\mathcal{M}$ the set of diagonal operators, $E$ the projection of Lemma 1, and $\mathcal{N}$ its null-space. In order to apply Lemma 3 it remains to show: given any compact operator $f$, $Ef$ and $f - Ef$ are compact. $Ef$ is compact because, if $f$ is compact,
\[
\lim_{n} \left( \sum_{i=1}^{n} a_{ii} \varphi_i \right) = \lim_{n} \left( \sum_{i=1}^{n} a_{ii} \varphi_i \right)^{1/q} = 0
\]
uniformly in $j$. This implies $\lim_{n} |a_{nn}| = 0$, which shows that $Ef$ is compact. Hence $f - Ef$ is compact.

To prove the theorem for $[p, q], 1 \leq p \leq q < \infty$, assume there is a bounded projection $F$ from $[p, q]$ onto $\mathcal{P}$. By Lemma 3, the restriction of $EF$ to $\mathcal{M}$ is a bounded projection from $\mathcal{M}$ onto $\mathcal{M} \cap \mathcal{P}$. By Lemma 2 there must be a corresponding bounded projection from $\sim$ onto $c_0$. This contradicts [4; Cor. 7.5]. For the remaining $X$ and $Y$ pairs of the main theorem, the proof is similar except that the existence of expressions for $\|A\|$ in terms of the matrix coefficients (e.g., see [5]) makes some of the work simpler.

Next we extend the theorem to $[U, V]$. Let $\tilde{E}$ be the function on $[U, V]$ defined by $\tilde{E}f = P_2 f P_1$ for all $f$ in $[U, V]$. $\tilde{E}$ is linear and homogeneous and bounded. $\tilde{E}^2 f = P_2 (P_2 f P_1) P_1 = P_2 f P_1 = \tilde{E} f$ so $\tilde{E}$ is a projection. The range of $\tilde{E}$ is the set of operators $g$ such that $P_2 g P_1 = g$ and is isomorphic with $[X, Y]$. The null-space of $\tilde{E}$ is the set of operators $h$ such that $P_2 h P_1 = 0$. If $Q_i$ is the projection $I - P_i$, the
decomposition $f = g + h$ is given by

$$f = (P + Q_2)f(P_1 + Q_1) = P_2fP_1 + (P_2fQ_1 + Q_2fP_1 + Q_2fQ_1).$$

If $f$ is compact, so are $g$ and $h$. We apply Lemma 3 with $X = [U, V]$, $\mathfrak{M}$ the range of $\tilde{E}$, $\tilde{E}$ acting as the projection $E$ of that lemma, and $\mathfrak{P}$ the set of compact operators from $U$ to $V$. The conclusion is that if there were a bounded projection $F$ from $X$ to $\mathfrak{P}$, the restriction of $\tilde{E}F$ to $\mathfrak{M}$ would be a bounded projection from $\mathfrak{M}$ onto $\mathfrak{P} \cap \mathfrak{M}$, contradicting our result for $[X, Y]$.

**Remark.** The problem of finding a bounded projection onto the compact operators is trivial when all the bounded operators are compact. This happens, for example, for $[\mathfrak{L}^p, \mathfrak{L}^q]$, $\infty > p > q \geq 1$, [2, p. 700], or $p = \infty$, $q = 1$, and for $[c_0, \mathfrak{L}^q]$, $[c, \mathfrak{L}^q]$, $\infty > q \geq 1$. Whether there exists a pair of normed spaces with a bounded proper projection from the bounded operators onto the compact operators seems to be unknown.

**References**


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