A MULTIMOVE INFINITE GAME WITH LINEAR PAYOFF

Leonard D. Berkovitz and Melvin Dresher
1. Introduction. Games can be classified in terms of the number of moves by each player—unimove or multimove—and in terms of the number of choices—finite or infinite—available at each move. The original work of von Neumann [2] on the existence and structure of solutions of games was, in effect, restricted to unimove finite games. Later, Ville [3] proved the existence of optimal strategies for unimove infinite games with continuous payoff function.

Except for games with perfect information, multimove finite games have been analyzed only very recently; and multimove infinite games with an arbitrary number of moves have hardly been touched upon.

In this paper, we analyze a multimove infinite game with a linear payoff function. The game is symmetric in every respect except that the initial conditions of the two players are different. We prove that one player has an optimal pure strategy and that the other player must randomize on the strategies. The optimal strategies and game value are derived.

Although this game had its origin in a military problem concerning allocation of resources among several tasks, it is presented here solely for its mathematical interest. A complete discussion of the military problem and its solution is given in [1].

2. Description of game. We shall analyze the following multimove zero-sum two-person game. At the \( n \)th move, or stage of the game, Blue has resources given by the state variable \( p_n \) and assigns a value to each of two tactical variables under his control, \( x_n \) and \( u_n \), subject to the constraints

\[
x_n \geq 0, \quad u_n \geq 0, \quad x_n + u_n \leq p_n. \tag{2.1}
\]

At the same time, Red has resources given by the state variable \( q_n \) and controls the values of the tactical variables \( y_n \) and \( w_n \), subject to the constraints

\[
y_n \geq 0, \quad w_n \geq 0, \quad y_n + w_n \leq q_n. \tag{2.2}
\]

Let us number the moves from the end of the game; i.e., the \( n \)th move means \( n \) moves to the end of the game. The state variables at the \((n - 1)\)-st move are defined by

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\[p_{n-1} = \max [0, p_n - \max (0, y_n - u_n)] \]
\[q_{n-1} = \max [0, q_n - \max (0, x_n - w_n)]\]

The payoff to Blue is given by
\[
\sum_{n=1}^{N} [(p_n - x_n - u_n) - (q_n - y_n - w_n)],
\]
where \(N\) is the number of moves in the game.

The play of the game proceeds by first making the \(N\)th move, then the \((N - 1)\)st move, \(\cdots\), the second move, and the first move. An \(n\)th move of the game consists of a choice by Blue of \(x_n\) and \(u_n\) satisfying (2.1) and simultaneously a choice by Red of \(y_n\) and \(w_n\) satisfying (2.2). We assume that each player knows the manner in which the game proceeds from stage to stage; namely, each player has the information expressed by equations (2.3). We also assume that at each stage of the game both players know the state variables and the entire past history of the play; that is, at the \(n\)th move, both players know \(N, p_n, q_n, \) and also know \(x_i, u_i, y_i, w_i\) for \(i = N, N - 1, \cdots, n + 2, n + 1\). It follows that \(p_i, q_i\), for \(i = N, N - 1, \cdots, n + 1, n\), are known at the \(n\)th move.

The strategies of the game in normal form will be defined inductively on the number of moves. First, a strategy for Blue in a one-move game is a point \(X_1 = (x_1, u_1)\), where \(x_1 \geq 0, u_1 \geq 0, \) and \(x_1 + u_1 \leq p_1\). Similarly a strategy for Red in a one move game is a point \(Y_1 = (y_1, w_1)\) where \(y_1 \geq 0, w_1 \geq 0, \) and \(y_1 + w_1 \leq q_1\). Now let \(\sigma_N\) be a strategy for Blue in an \(N\)-move game. Of course, \(\sigma_N\) is a function of \(p_N\) and \(q_N\). Then, in a game of \(N + 1\) moves, at the \((N + 1)\)st move Blue chooses a point \(X_{N+1} = (x_{N+1}, u_{N+1})\) in the triangle \(\Delta_{N+1}\) defined by
\[
x_{N+1} \geq 0, \quad u_{N+1} \geq 0, \quad x_{N+1} + u_{N+1} \leq p_{N+1},
\]
and simultaneously Red chooses a point \(Y_{N+1} = (y_{N+1}, w_{N+1})\) in the triangle \(\Delta_{N+1}\) defined by
\[
y_{N+1} \geq 0, \quad w_{N+1} \geq 0, \quad y_{N+1} + w_{N+1} \leq q_{N+1}.
\]
These choices yield the state variables \(p_{N+1}\) and \(q_{N+1}\), by equations (2.3). A strategy \(\sigma_{N+1}\) for Blue in the \((N + 1)\) move game is then defined as a choice \(X_{N+1}\) in \(\Delta_{N+1}\) and a function \(\Phi_N\) that associates, with each point \((x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1}) = (X_{N+1}, Y_{N+1})\) in the product space \(\Delta_{N+1} \times \Delta_{N+1}\) a strategy \(\sigma_N\) in the \(N\)-move game. Thus \(\sigma_{N+1}\) can be written as
\[
\sigma_{N+1} = (X_{N+1}; \Phi_N) = (x_{N+1}, u_{N+1}; \Phi_N),
\]
where \(\Phi_N\) assigns the strategy \(\sigma_N\) to the point \((x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1})\).

In a like manner, a strategy \(\tau_{N+1}\) for Red in the \((N + 1)\)-move game is defined as a choice \(Y_{N+1}\) and a function \(\bar{\phi}_N\) that associates, with each
A MULTIMOVE INFINITE GAME WITH LINEAR PAYOFF

$(X_{N+1}, Y_{N+1})$, a strategy $\tau_N$ in the $N$-move game. Thus we have

$$\tau_{N+1} = (Y_{N+1}; \psi_N) = (y_{N+1}, w_{N+1}; \psi_N).$$

3. Solution of game. The main result of this paper is the following:

**Theorem 1.** If $N = 1$ or $2$, the value of the game is given by

$$V_N(p_N, q_N) = N(p_N - q_N).$$

Blue has an optimal pure strategy:

$$\bar{x}_m = \bar{u}_m = 0 \text{ for } m \leq N.$$

Red has an optimal pure strategy:

$$\bar{y}_m = \bar{w}_m = 0 \text{ for } m \leq N.$$

If $N \geq 3$, the value of the game is given by the $(N - 2)$-piecewise-linear function.

$$V_N(p_N, q_N) = a_i^N p_N - b_i^N q_N, \quad i = 1, 2, \ldots, N - 2,$$

where the constants $a_i^N$ and $b_i^N$ are positive and monotone decreasing in $i$ for fixed $N$; the value of the superscript $i$ is determined by the ratio $p_N/q_N$. The optimal strategies for the two players are as follows:

(i) At move $m = 1, 2$ (counting from the end) the players choose

$$\bar{x}_m = \bar{u}_m = \bar{y}_m = \bar{w}_m = 0.$$

(ii) At move $m = 3$, if $p_3 \geq q_3$, then Blue chooses $x_3, u_3$ such that

$$q_3 \leq \bar{x}_3 \min \left( \frac{p_3 + q_3}{2}, \frac{3q_3}{2} \right),$$

$$\bar{u}_3 = \bar{x}_3 - q_3.$$

Red chooses either $\bar{y}_3 = q_3$ or $\bar{w}_3 = q_3$, each with probability $1/2$.

(iii) At the $(m + 1)$-st move, where $3 \leq m \leq N - 1$, if $p_{m+1} \geq q_{m+1}$, then the ratio $p_{m+1}/q_{m+1}$ determines an integer $i, 1 \leq i \leq m - 1$, and Blue chooses

$$\bar{x}_{m+1} = \frac{(2m - a_i^m)p_{m+1} - (m - 2b_i^m)q_{m+1}}{m + b_i^m}$$

$$\bar{u}_{m+1} = p_{m+1} - \bar{x}_{m+1}, \quad \text{for } i = 1, 2, \ldots, m - 2,$$

and

$$\bar{x}_{m+1} = \left( 2 - \frac{1}{b_i^{m+2}} \right) q_{m+1}.$$
<table>
<thead>
<tr>
<th>Move m</th>
<th>Ratio of State Variables $p/q$</th>
<th>Associated Superscript $\xi$</th>
<th>Value of Game $V(p, q)$</th>
<th>Blue's Optimal Choice</th>
<th>Red's Optimal Choice</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\pi^*_m$</td>
<td>$\pi^*_m$</td>
</tr>
<tr>
<td>1</td>
<td>1.00 to $\infty$</td>
<td>1</td>
<td>$p - q$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1.00 to $\infty$</td>
<td>1</td>
<td>$2(p - q)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1.00 to 2.00</td>
<td>1</td>
<td>$3(p - q)$</td>
<td>$q$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2.00 to $\infty$</td>
<td>2</td>
<td>$3(p - q)$</td>
<td>$.5q$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1.00 to 2.33</td>
<td>1</td>
<td>$4.5(p - q)$</td>
<td>$.5p + .5q$</td>
<td>$.5p - .5q$</td>
</tr>
<tr>
<td></td>
<td>2.33 to $\infty$</td>
<td>2</td>
<td>$4.00p - 3.33q$</td>
<td>$1.67q$</td>
<td>$.67q$</td>
</tr>
<tr>
<td>5</td>
<td>1.00 to 1.70</td>
<td>1</td>
<td>$6.35p - 6.35q$</td>
<td>$.41p + .59q</td>
<td>$.59p - .59q</td>
</tr>
<tr>
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<td>1.70 to 2.45</td>
<td>2</td>
<td>$5.82p - 5.45q$</td>
<td>$.55p + .36q</td>
<td>$.45p - .36q</td>
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<td></td>
<td>2.45 to $\infty$</td>
<td>3</td>
<td>$5.00p - 3.45q$</td>
<td>$1.70q$</td>
<td>$.75q$</td>
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<tr>
<td>6</td>
<td>1.00 to 1.44</td>
<td>1</td>
<td>$8.39p - 8.39q$</td>
<td>$.32p + .68q</td>
<td>$.68p - .68q</td>
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<tr>
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<td>1.44 to 1.78</td>
<td>2</td>
<td>$8.00p - 7.83q$</td>
<td>$.40p + .56q</td>
<td>$.60p - .56q</td>
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<tr>
<td></td>
<td>1.78 to 2.51</td>
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<td>$7.04p - 6.12q$</td>
<td>$.59p + .22q</td>
<td>$.41p - .22q</td>
</tr>
<tr>
<td></td>
<td>2.51 to $\infty$</td>
<td>4</td>
<td>$6.00p - 3.51q$</td>
<td>$1.71q$</td>
<td>$.80q$</td>
</tr>
<tr>
<td>7</td>
<td>1.00 to 1.29</td>
<td>1</td>
<td>$10.50p - 10.50q$</td>
<td>$.25p + .75q</td>
<td>$.75p - .75q</td>
</tr>
<tr>
<td></td>
<td>1.29 to 1.53</td>
<td>2</td>
<td>$10.26p - 10.19q$</td>
<td>$.29p + .70q</td>
<td>$.71p - .70q</td>
</tr>
<tr>
<td></td>
<td>1.53 to 1.84</td>
<td>3</td>
<td>$9.54p - 9.06q$</td>
<td>$.41p + .51q</td>
<td>$.69p - .51q</td>
</tr>
<tr>
<td></td>
<td>1.84 to 2.55</td>
<td>4</td>
<td>$8.21p - 6.64q$</td>
<td>$.63p + .11q</td>
<td>$.37p - .11q</td>
</tr>
<tr>
<td></td>
<td>2.55 to $\infty$</td>
<td>5</td>
<td>$7.00p - 3.55q$</td>
<td>$1.72q$</td>
<td>$.84q$</td>
</tr>
<tr>
<td>8</td>
<td>1.00 to 1.25</td>
<td>1</td>
<td>$12.06p - 12.60q$</td>
<td>$.20p + .80q</td>
<td>$.80p - .80q</td>
</tr>
<tr>
<td></td>
<td>1.25 to 1.40</td>
<td>2</td>
<td>$12.48p - 12.45q$</td>
<td>$.22p + .78q</td>
<td>$.78p - .78q</td>
</tr>
<tr>
<td></td>
<td>1.40 to 1.59</td>
<td>3</td>
<td>$12.06p - 11.86q$</td>
<td>$.28p + .69q</td>
<td>$.72p - .69q</td>
</tr>
<tr>
<td></td>
<td>1.59 to 1.88</td>
<td>4</td>
<td>$11.03p - 10.22q$</td>
<td>$.42p + .46q</td>
<td>$.58p - .46q</td>
</tr>
<tr>
<td></td>
<td>1.88 to 2.58</td>
<td>5</td>
<td>$9.36p - 7.07q$</td>
<td>$.66p + .01q</td>
<td>$.34p - .01q</td>
</tr>
<tr>
<td></td>
<td>2.58 to $\infty$</td>
<td>6</td>
<td>$8.00p - 3.58q$</td>
<td>$1.72q$</td>
<td>$.86q$</td>
</tr>
</tbody>
</table>
\[ \bar{u}_{m+1} = \left(1 - \frac{1}{m}\right)q_{m+1}, \]  
for \( i = m - 1 \),

where the constants \( a_i^m \) and \( b_i^m \) are those associated with a game of length \( m \) and initial condition \( p_m, q_m \). Red chooses either \( y_{m+1} \) or \( w_{m+1} = q_{m+1} \) with probabilities \( a_i^m = \frac{b_i^m}{m+b_i^m} \) and \( \beta_i^m = \frac{m}{m+b_i^m} \), respectively, for \( i = 1, 2, \ldots, m - 2 \); however, if \( i = m - 1 \), Red chooses \( y_{m+1} = q_{m+1} \) with probability \( \alpha_i^m = \frac{1}{m} \), or \( w_{m+1} = q_{m+1} \) with probability \( \beta_i^m = \frac{1}{b_i^m} \), or \( y_{m+1} = w_{m+1} = 0 \) with probability \( \gamma_i^m = 1 - \frac{1}{m} - \frac{1}{b_i^m} \).

The proof of Theorem 1 will be carried out by induction on \( N \), the number of moves of the game. In the course of this argument, recursive definitions will be given for the constants \( a_i^N \) and \( b_i^N \). As an illustration of the theorem, Table 1 shows the solutions for games with eight or less moves.

### 4. A three-part sufficiency condition with mixed strategies

From the statement of the theorem, it is seen that mixed strategies will have to be introduced, at least for Red. However, it is sufficient to introduce a restricted class of mixed strategies in order to prove the theorem.

For a game of one move, a mixed strategy for Red is a probability distribution \( G_1 \) over \( D \). Now suppose \( G_N \) is a mixed strategy for Red in a game of \( N \) moves and state variables \( p_N \) and \( q_N \). Then a probability distribution \( g_{N+1} \) over \( D_{N+1} \) and a function \( \psi_{N+1} \) that associates \( (x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1}) \) with \( G_N \) is a mixed strategy \( G_{N+1} \) in the \((N + 1)\)-move game. Thus we may write the mixed strategy as

\[ G_{N+1} = (g_{N+1}, \psi_N). \]

Mixed strategies \( F_{N+1} \) for Blue are defined similarly by a distribution function \( f_{N+1} \) and a function \( \phi_N \), and can be written as

\[ F_{N+1} = (f_{N+1}, \phi_N). \]

Let \( \hat{F}_{N+1} \) denote a mixed strategy for Blue in the \((N + 1)\)-move game in which he selects \( X_{N+1} = (x_{N+1}, u_{N+1}) \) with probability 1 at the \((N + 1)\)-st move. Let \( \hat{G}_{N+1} \) denote a mixed strategy for Red in which he selects \( Y_{N+1} = (y_{N+1}, w_{N+1}) \) with probability 1 at the \((N + 1)\)-st move.

Suppose that Theorem 1 is valid for games of length \( N = n \). Let \( F^*_n \) and \( G^*_n \) be optimal strategies for Blue and Red, respectively. Let \( \phi^*_n, \psi^*_n \) denote the functions that associate \( (x_{n+1}, u_{n+1}, y_{n+1}, w_{n+1}) \) with \( F^*_n, G^*_n \), respectively. Suppose, further, that \( p_{n+1} \geq q_{n+1} \) (from symmetry, it suffices to consider this case only).

The theorem asserts that at the \((n + 1)\)-st move Blue’s optimal choice is a point \( (x_{n+1}, u_{n+1}) \) that is determined by the ratio \( p_{n+1}/q_{n+1} \). Denote this point by
and let $F^*_n = (X^*_n, q^*_n)$.

The theorem further asserts that Red chooses $(y_{n+1}, w_{n+1})$ to be

$$(\xi_1, \phi_{n+1} = (x_{n+1}, \rho_{n+1}) \Leftrightarrow (0, 0), \xi_2 = (0, 1), \xi_3 = (1, 0), \xi_4 = (1, 1)),$$

with probabilities $\alpha_{n+1}, \beta_{n+1}$, and $\gamma_{n+1} = (1 - \alpha_{n+1} - \beta_{n+1})$, respectively, the values of $\alpha_{n+1}$ and $\beta_{n+1}$ being determined by the ratio $p_{n+1}/q_{n+1}$. Denote this distribution in $\Delta_{n+1}$ by $g_{n+1}(p_{n+1}, q_{n+1})$, and set $G^*_{n+1} = (g^*_{n+1}(p_{n+1}, q_{n+1}))$.

Define

$$L_{n+1}(X_{n+1}, Y_{n+1}) = p_{n+1} - x_{n+1} - u_{n+1} - q_{n+1} + y_{n+1} + w_{n+1}$$

and

$$M_{n+1}(X_{n+1}, Y_{n+1}) = L_{n+1}(X_{n+1}, Y_{n+1}) + V_n(p_n, q_n),$$

where $p_n, q_n$ are obtained from $p_{n+1}, q_{n+1}$ by means of (2.3) and the choices $x_{n+1}, u_{n+1}, y_{n+1}, w_{n+1}$. Let $E_n(F_n, G_n)$ denote the expected payoff of the game of length $N$ if Blue chooses a strategy $F_n$ and Red chooses a strategy $G_n$. Then

$$E_{n+1}(\hat{F}^*_{n+1}, \hat{G}^*_{n+1}) = L_{n+1}(X^*_n, Y^*_n) + E_n(F^*_n, G^*_n)$$

$$\geq M_{n+1}(X^*_n, Y^*_n), \text{ for all } Y^*_{n+1},$$

where $\phi_n(X_{n+1}, Y_{n+1}) = G_n$. Furthermore, we have

$$E_{n+1}(\hat{F}^*_{n+1}, \hat{G}^*_{n+1}) = \alpha_{n+1}[L_{n+1}(X_{n+1}, Y_{n+1})] + E_n(F^*_n, G^*_n)$$

$$+ \beta_{n+1}[L_{n+1}(X_{n+1}, Y_{n+1})] + E_n(F^*_n, G^*_n)$$

$$+ (1 - \alpha_{n+1} - \beta_{n+1})[L_{n+1}(X_{n+1}, Y_{n+1})] + E_n(F^*_n, G^*_n)$$

$$\leq \alpha_{n+1} M_{n+1}(X_{n+1}, Y_{n+1}) + \beta_{n+1} M_{n+1}(X_{n+1}, Y_{n+1})$$

$$+ (1 - \alpha_{n+1} - \beta_{n+1}) M_{n+1}(X_{n+1}, Y_{n+1}),$$

for all $X_{n+1}$, where $\phi_n(X_{n+1}, Y_{n+1}) = F^*_n$.

The validity of the following lemma is now apparent.

**Lemma 1.** Given that Theorem 1 is true for $N = n$, to prove the theorem for $N = n + 1$ with initial conditions $p_{n+1} \geq q_{n+1}$, it suffices to exhibit the $X^*_{n+1}, \alpha_{n+1}$, and $\beta_{n+1}$ for which

$$E_{n+1}(\hat{F}^*_{n+1}, \hat{G}^*_{n+1}) = V_{n+1}(p_{n+1}, q_{n+1}),$$

$$M_{n+1}(X^*_n, Y^*_n) \geq V_{n+1}(p_{n+1}, q_{n+1}),$$

for all $Y^*_{n+1}$, and

$$\alpha_{n+1} M_{n+1}(X_{n+1}, Y_{n+1}) + \beta_{n+1} M_{n+1}(X_{n+1}, Y_{n+1})$$

$$+ (1 - \alpha_{n+1} - \beta_{n+1}) M_{n+1}(X_{n+1}, Y_{n+1}) \leq V_{n+1}(p_{n+1}, q_{n+1})$$
for all $X_{n+1}$.

5. Some special cases. It will also be useful to tabulate the information given by equations (2.3). We may assume that $p_n \geq q_n$, whence $y_n - u_n > p_n$ is impossible and the equations (2.3) can be tabulated as follows, where the subscript $n$ is suppressed:

Table 2

<table>
<thead>
<tr>
<th>Region in $(X, Y)$ Space</th>
<th>Region Number</th>
<th>$p_{-1}$</th>
<th>$q_{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y - u &lt; 0, \quad x - w &lt; 0$</td>
<td>I</td>
<td>$p$</td>
<td>$q$</td>
</tr>
<tr>
<td>$y - u &lt; 0, \quad 0 \leq x - w \leq q$</td>
<td>II</td>
<td>$p$</td>
<td>$q - x + w$</td>
</tr>
<tr>
<td>$y - u &lt; 0, \quad x - w &gt; q$</td>
<td>III</td>
<td>$p$</td>
<td>0</td>
</tr>
<tr>
<td>$0 \leq y - u \leq p, \quad x - w \leq 0$</td>
<td>IV</td>
<td>$p - y + u$</td>
<td>$q$</td>
</tr>
<tr>
<td>$0 \leq y - u \leq p, \quad 0 \leq x - w \leq q$</td>
<td>V</td>
<td>$p - y + u$</td>
<td>$q - x + w$</td>
</tr>
<tr>
<td>$0 \leq y - u \leq p, \quad x - w &gt; q$</td>
<td>VI</td>
<td>$p - y + u$</td>
<td>0</td>
</tr>
</tbody>
</table>

Games of length $N = 1, 2, 3$ will now be discussed. From the statement of the theorem, it is clear that separate arguments are needed for $N = 1, 2$, and for $N \geq 3$.

For $N = 1$, an examination of the payoff (2.4) shows that optimal play for Blue is to choose $x_1 = u_1 = 0$, and that optimal play for Red is to choose $y_1 = w_1 = 0$.

As a consequence of Lemma 1, for $N = 2$ it suffices to consider

$$M(X_3, Y_3) = p_2 - x_3 - u_3 - q_2 + y_3 + w_3 + (p_1 - q_1) ,$$

with $p_3 \geq q_3$. Using Table 2 and dropping the subscript 2, we may write this

$$M(X, Y) = \begin{cases} 2(p - q) - (x + u) + (y + w) & \text{in region I,} \\ 2(p - q) - u + w & \text{in region II,} \\ 2p - q - (x + u) + (y + w) & \text{in region III,} \\ 2(p - q) - x + w & \text{in region IV,} \\ 2(p - q) & \text{in region V,} \\ 2p - q - x + w, & \text{in region VI,} \end{cases}$$

where the region in the $(X, Y)$ space for which each expression on the right is valid is that given in Table 2. It now follows that the optimal choices at the second move are

$$(\bar{x}, \bar{w}) = (0, 0) \text{ for Blue; } (\bar{y}, \bar{w}) = (0, 0) \text{ for Red;}$$

and that $V' = 2(p_2 - q_2)$. Thus the theorem is proved for $N = 2$.

For $N = 3$, it suffices to consider

$$M_3(X_3, Y_3) = p_3 - x_3 - u_3 - q_3 + y_3 + w_3 + 2(p_2 - q_2) ,$$
where \( p_3 \geq q_3 \). It follows from Table 2 that, dropping the subscript 3, we may write \( M_3(X_3, Y_3) \) as
\[
(5.1) \quad A p - B q + \varepsilon_1 x + \varepsilon_2 u + \varepsilon_3 y + \varepsilon_4 v,
\]
where the values of \( A, B, \varepsilon_i = \pm 1, i = 1, \cdots, 4 \) depend on the region of the \((X, Y)\) space. To prove the theorem for \( N = 3 \), we use (5.1) and Lemma 1 to verify that the choices given for the third move are actually optimal.

Games of length \( N = 3 \), and \( N = 4 \), as can be seen from Table 1, are somewhat transitional in character, and do not exhibit all the characteristics of games of arbitrary length. For \( N = 5 \), however, all of the characteristics of the game's structure become apparent. For the reader who wishes to gain further insight into the structure of the game, as well as to obtain a motivation for the general induction step, the computation of the functions \( m_i(x, u) = \min_i M_i(X_i, Y_i) \), \( i = 3, 4, 5 \), is recommended. By way of illustration we give \( m_5(x, u) \), valid for \( 1 \leq p/q \leq 7/3 \). We have that \( m_5(x, u) \) is given as follows, (where the subscript 5 is omitted):

(i) \( 5p - q - x - u \), if \( x \geq 17q/10, u \geq 3q/4 \);
(ii) \( 5p - 20q/3 + 7x/3 - u \), if \( 2q - 3p/7 \leq x \leq 17q/10, 5x - 6u \leq 4q \);
(iii) \( 11p/2 - 9q + 7x/2 - u \), if \( q \leq x \leq 2q - 3p/7 \), \( 9x - 8u \leq 10q - p \);
(iv) \( 11(p - q)/2 - u \), if \( 0 \leq x \leq q, 9u + 7x \geq 7q \), \( 14x + 24u \geq 3p + 11q \);
(v) \( 11p/2 - 9q + 7(x + u)/2 \), if \( x \geq 0, u \geq 0 \), \( 9u + 7x \leq 7q, 7x + 3u \leq 10q - 3p \);
(vi) \( 5p + 22q/3 + 7x/3 + 3u \), if \( x \leq q, u \geq 0 \), \( 7x + 3u \geq 10q - 3p, 14x + 24u \leq 3p + 11q \);
(vii) \( 5p - 4q - x + 3u \) if \( x \geq q, u \geq 0, 5x - 6u \geq 4q \), \( 9x - 8u \geq 10q - p \).

6. Definitions and properties of constants. The first step of the proof of Theorem 1 is to define the sequences \( \{a^i_n\}, \{b^i_n\}, \{\lambda^i_n\} \). To this end, consider the following sequences defined in the manner and order indicated:
\[
(6.1) \quad a^1_3 = 3, a^2_{n+1} = a^{n-2}_n + 1; n \geq 3 ;
\]
\[
(6.2) \quad b^i_3 = 3, b^{n-1}_{n+1} = 4 - \frac{1}{a^{n-2}_n} - \frac{1}{b^{n-2}_n}, n \geq 3 ;
\]
\[
(6.3) \quad b^n_1 = a^n_1 = 0, n \geq 3 ;
\]
A MULTIMOVE INFINITE GAME WITH LINEAR PAYOFF

\begin{align*}
(6.4) & \quad a_{n+1}^i = \frac{a_n^{n-2}(2b_n^i + a_n^i)}{b_n^i + a_n^{n-2}} \quad i \geq 1; n = i + 2, i + 3, \cdots; \\
(6.5) & \quad b_{n+1}^i = \frac{3a_n^{n-2}b_n^i}{b_n^i + a_n^{n-2}} \quad n \geq 1; \quad n = 3, 4, 5, \cdots; \\
(6.6) & \quad \lambda_{n+1}^{n-1} = +\infty, n = 3, 4, 5, \cdots; \\
(6.7) & \quad \lambda_3 = 1, \\
(6.8) & \quad \lambda_i = \frac{b_{n+1}^{i-1} - b_n^i}{a_{n+1}^{i-1} - a_n^i}, n \geq 4; \quad i = 1, 2, 3, \cdots, n - 3.
\end{align*}

The following properties of the foregoing sequences will be useful in the proof of the theorem; indications of the derivations of the properties are given after the listing:

\begin{align*}
(6.9) & \quad a_n^{n-2} = n, \quad n \geq 3; \\
(6.10) & \quad a_n^i = b_n^i, \quad n \geq 3; \\
(6.11) & \quad 4 > b_{n+1}^{n-1} > b_n^{n-2} \geq 3, \quad n \geq 3; \\
(6.12) & \quad a_n^i \geq b_n^i > 0, \quad i = 1, 2, \cdots, n - 2; \quad n = 3, 4, 5, \cdots; \\
(6.13) & \quad n \leq a_n^i < 2n, \quad i = 1, \cdots, n - 2; \quad n = 3, 4, 5, \cdots; \\
(6.14) & \quad \begin{cases}
\lambda_n^i < \lambda_n^{i-1}, \\
b_n^i < b_n^{i-1}
\end{cases} \quad n \geq 3; \quad i = 2, 3, \cdots, n - 2; \\
(6.15) & \quad \lambda_{n+1}^i = \frac{3n\lambda_n^i}{(2n - a_n^i)(\lambda_n^i + n + b_n^i)}, \quad n \geq 3; \quad i = 1, 2, \cdots, n - 2; \\
(6.16) & \quad 3 > \lambda_{n+1}^{n-1} > \lambda_n^{n-2} \geq 2, \quad n \geq 3; \\
(6.17) & \quad \lambda_{n+1}^{i+1} > \lambda_n^i, \quad n \geq 3; \quad i = 1, 2, \cdots, n - 2; \\
(6.18) & \quad \lambda_{n+1}^i \leq \lambda_n^i, \quad n \geq 3; \quad i = 1, \cdots, n - 2.
\end{align*}

Statements (6.9)-(6.11) follow from the definitions and from trivial inductive arguments.

Inequalities (6.12) and (6.13) are proved by induction on \( n, n \geq i + 2, \) for each fixed \( i. \)

The monotonicity properties in (6.14) are established as follows. The monotonicity of \( \{b_{n+1}^i\}, i = 1, \cdots, n - 2, \) follows, by induction on \( n, \) from the monotonicity of \( \{b_n^i\}, i = 1, \cdots, n - 2. \) To show that \( b_{n+1}^{n-2} > b_n^{n-1}, \) it suffices to show that \( b_{n+1}^{n-1} \geq 4. \) This inequality, however, is obvious.
from (6.5) and (6.11). When we compute $a_{n+1}^i - a_{n+1}^i$, $i = 2, 3, \ldots, n - 2$, we obtain

$$\frac{n}{(n + b^{i-1})(n + b^i)}[b^{i-1}(2n - a^i) - b^i(2n - a^{i-1}) + n(a^{i-1} - a^i)],$$

where the subscript $n$ is omitted. From the inductive hypothesis that

$$\{a_i^n\} \text{ is monotone decreasing in } i,$$

and from (6.13), the bracketed expression in turn is seen to be larger than the positive quantity

$$b^{i-1}(2n - a^{i-1}) - b^i(2n - a^{i-1}) = (b^{i-1} - b^i)(2n - a^{i-1}).$$

Furthermore, for $n \geq 3$, by (6.1), (6.4), and (6.11) we have

$$a_{n+1}^{n-2} - a_{n+1}^{n-1} = \frac{b_n^{n-2}(n - 1) - 1}{b_n^{n-2} + n} \geq \frac{3(n - 1) - n}{b_n^{n-2} + n} > 0.$$

Thus the fact that $\{a_{n+1}^i\}$ is monotone decreasing in $i$ is established.

To prove (6.15), we use (6.4) and (6.5) in the definition (6.8) of $\lambda_{n+1}^i$. Inequality (6.16) is obvious, and (6.17) is established inductively as follows. Suppose $\lambda_{n+1}^{i+1} > \lambda_n^i$ for $i = 1, \ldots, n - 1$. Consider

$$\frac{\lambda_{n+1}^{i+1}}{\lambda_{n+1}^i} = \frac{\lambda^{i+1}[2n - a^i] + (n + b^i)}{\lambda^i[(2n - a^{i+1})\lambda^{i+1} + (n + b^{i+1})]}, \quad i = 1, 2, \ldots, n - 3,$$

where the subscript $n$ is omitted. To prove that this ratio exceeds 1, it therefore suffices to show that

$$n(\lambda^{i+1} - \lambda^i) + \lambda^{i+1}b^i - \lambda^ib^{i+1} > \lambda^i\lambda^{i+1}(a^i - a^{i+1}).$$

Upon replacing $\lambda^{i+1}$ on the right by its definition (6.8), we see that this last inequality is equivalent to $(n + b^i)(\lambda^{i+1} - \lambda^i) > 0$, the validity of which follows from the inductive hypothesis. The chain

$$\lambda_{n+1}^{i+2} = \frac{3n\lambda_{n+1}^{i+2}}{(\lambda^{i+1} + 1)n + b^{i+1}} = \frac{3n\lambda_{n+1}^{i+2}}{b^{i+1}(n + 1)} \leq \frac{3\lambda_{n+1}^{i+2}}{b^{i+1}} \leq \lambda_{n+1}^{i+1}$$

completes the proof of (6.17).

It is seen from (6.15) that to verify (6.18) it suffices to show that

$$(6.19) \quad \lambda_{n}^i \geq \frac{2n - b_{n}^{i}}{2n - a_{n}^{i}}, \quad n \geq 3; \quad i = 1, 2, \ldots, n - 2.$$

This inequality is shown to hold by induction on $i$, as follows. For $i = 1$, equality is obvious. Suppose that the inequality holds for $i = k$. It is then seen to hold for $i = k + 1, 1 \leq i \leq n - 3$, by writing (omitting the subscript $n$)

$$\lambda^{k+1}(2n - a^{k+1}) - (2n - b^{k+1}) = \lambda^{k+1}(2n - a^k) - (2n - b^k)$$

$$+ \lambda^{k+1}(a^k - a^{k+1}) - (b^k - b^{k+1})$$

$$> \lambda^k(2n - a^k) - (2n - b^k) \geq 0.$$
7. Miscellaneous preparations. For $N \leq 3$, Theorem 1 has been proved in §5. The theorem will now be proved inductively for arbitrary $N > 3$. Suppose then that it has been established for $N = n \geq 3$. It is required to show that it holds for $N = n + 1$.

In order to simplify notation, for the remainder of the proof we shall omit the subscript $n$. Thus the symbol $a_n^i$, say, will be written merely as $a^i$, the symbol $\lambda_{n+1}^i$ as $\lambda_{i+1}^i$, $b_{n+1}^i$ as $b_{i+1}^i$, etc.

From the symmetry of Theorem 1, it is clear that it suffices to consider the case $p_i \geq q_i$. Define $X_i^* = X_i^*(p_i/q_i)$ as follows:

$$X_i^* = \tilde{X}_i^* = (\tilde{x}_i^*, \tilde{u}_i^*) \text{ if } \lambda_i^i \leq \frac{p_i}{q_i} \leq \lambda_{i+1}^i, \quad i = 1, \ldots, n - 1,$$

where

$$\tilde{x}_i^i = \frac{(2n - a^i)p_i - (n - 2b^i)q_i}{b^i + n}, \quad \tilde{u}_i^i = p_i - \tilde{x}_i^i, \quad i = 1, \ldots, n - 2,$$

(7.1)

where

$$\tilde{x}_i^{n-1} = \left(2 - \frac{1}{b^{n-2}}\right)q_i,$$

$$\tilde{u}_i^{n-1} = \left(1 - \frac{1}{n}\right)q_i.$$

Define $\alpha_i = \alpha_i(p_i/q_i)$ and $\beta_i = \beta_i(p_i/q_i)$ thus:

(a) if $\lambda_i^i \leq \frac{p_i}{q_i} \leq \lambda_{i+1}^i$, $1 \leq i \leq n - 2$,

(7.2)

then

$$\alpha_i^i = \frac{b^i}{n + b^i}, \quad \beta_i^i = \frac{n}{n + b^i} ;$$

(b) if $\frac{p_i}{q_i} \geq \lambda_{i+1}^{n-1}$,

then

$$\alpha_i^{n-1} = \frac{1}{n}, \quad \beta_i^{n-1} = \frac{1}{b^{n-2}} .$$

Clearly, $\alpha_i^i > 0, \beta_i^i > 0$ for all $i$ satisfying $i \leq 1 \leq n - 1$; $\alpha_i^i + \beta_i^i = 1$ for all $i$ satisfying $1 \leq i \leq n - 2$; and $\alpha_i^i + \beta_i^i < 1$ for $i = n - 1$. Thus $\alpha_i$ and $\beta_i$ are probabilities. Lemma 2 will show that $X_i^*$ is an admissible choice for Blue and will furnish some useful bounds for $\tilde{x}_i^i$ and $\tilde{u}_i^i$.

**Lemma 2.** The point $X_i^*$ is an admissible choice of strategic variable for Blue. Furthermore, for all $i$ satisfying $1 \leq i \leq n - 1$, we have

$$q_i \leq \tilde{x}_i^i \leq 2q_i , \quad 0 \leq \tilde{u}_i^i < q_i.$$  

(7.3)
Since $X^*_i$ is defined piecewise, the first step in showing that $X^*_i$ is admissible is to show that the pieces come together—i.e., that $X^*_i$ is well defined. Substitution of $p_i = \lambda_i^{n-1} q_i$ into the definition (7.1) of $\bar{x}^{n-2}_i$, and application of (6.7), show that if $p_i = \lambda_i^{n-1} q_i$, then $\bar{x}^{n-2}_i = \bar{x}^{n-1}_i$.

Similarly, it is seen that $\bar{u}^{n-2}_i = \bar{u}^{n-1}_i$ for $p_i = \lambda_i^{n-1} q_i$. Substitution of $p_i = \lambda_i^i q_i$, $i = 1, \ldots, n-2$, into the definition (7.1) of $\bar{x}^i$, and use of (6.15), show that, for $p_i = \lambda_i^i q_i$,

$$\bar{x}^i = \left(2 - \frac{\lambda_i^i}{\lambda_i^{i+1}}\right) q_i .$$

Substitution of $p_i = \lambda_i^{i+1} q_i$, and use of (6.15), show that, for $p_i = \lambda_i^{i+1} q_i$, $i = 1, \ldots, n-3$,

$$\bar{x}^i = \left(2 - \frac{\lambda_i^{i+1}}{\lambda_i^{i+2}}\right) q_i .$$

Thus $X^*_i$ is well defined, and for $\lambda_i^i \leq p_i/q_i \leq \lambda_i^{i+1}$, $i = 1, 2, \ldots, n-3$,

$$\left(2 - \frac{\lambda_i^i}{\lambda_i^i}\right) q_i \leq \bar{x}^i \leq \left(2 - \frac{\lambda_i^{i+1}}{\lambda_i^{i+1}}\right) q_i ,$$

with equality on the left occurring for $p_i = \lambda_i^i q_i$, and on the right for $p_i = \lambda_i^{i+1} q_i$. Similarly, we obtain

$$\left(2 - \frac{\lambda_i^{n-2}}{\lambda_i^{n-2}}\right) q_i \leq \bar{x}^{n-2}_i \leq \bar{x}^{n-1}_i .$$

Clearly, (7.3) implies $\bar{x}^i \geq 0, \bar{u}^i \geq 0$ for all $i$ satisfying $1 \leq i \leq n - 1$. By definition, $\bar{x}^i + \bar{u}^i = p_i$ for $1 \leq i \leq n - 2$; and for $i = n - 1$, we obtain $\bar{x}^{n-1}_i + \bar{u}^{n-1}_i = \lambda_i^{n-1} q_i \leq p_i$. Thus the proof of the lemma depends on the proof of (7.3). For $i = n - 1$, (7.3) is obvious. The inequality (6.18) implies $\lambda_i^i \leq \lambda_i^i$, whence it follows that $(2 - \lambda_i^i/\lambda_i^i) q_i \geq q_i$ for $i = 1, 2, \ldots, n - 2$. Clearly, the inequality $(2 - \lambda_i^i/\lambda_i^i) q_i < 2q_i$ holds for $i = 1, \ldots, n - 2$. Hence, we obtain $q_i \leq \bar{x}^i < 2q_i$, for $i = 1, \ldots, n - 2$.

To verify $0 \leq \bar{u}^i \leq q_i$ for $1 \leq i \leq n - 2$, we substitute from the definition of $\bar{x}^i$ into the definition $\bar{u}^i = p_i - \bar{x}^i$ and obtain

$$\bar{u}^i = \frac{(b^i + a^i - n) p_i + (n - 2b^i) q_i}{(b^i + n)} , \quad 1 \leq i \leq n - 2 .$$

Hence, showing that $\bar{u}^i < q_i$ is equivalent to showing that $p_i/q_i < 3b^i/(b^i + a^i - n)$. Since $p_i/q_i \leq \lambda_i^{i+1}$, it clearly suffices to show that $\lambda_i^{i+1} < 3b^i/(b^i + a^i - n)$. If $i = n - 2$ this inequality is obvious. If $i \leq n - 3$, then by using (6.15) and (6.8) this inequality becomes

$$b^i - \lambda_i^{i+1}(a^i - n) > 0 , \quad i = 1, 2, \ldots, n - 3 .$$

Since $a^{n-2} = n$, it follows from (6.8) that for $i = n - 3$ the expression on
the left of this inequality is positive. Further, the left hand member is a monotone decreasing function of \( i \), as can be seen by forming the difference of the left-hand side for superscripts \( i - 1 \) and \( i \) and getting \((\lambda^{i+1} - \lambda^i)(a^i - n) > 0\). This inequality follows from (6.13) and (6.17). Thus it follows that \( \bar{u}_i < q_i \). The inequality \( \bar{u}_i \geq 0 \), follows from

\[
(b^i + a^i - n)p_i + (n - 2b^i)q_i \geq (a^i - b^i)q_i \geq 0; \quad 1 \leq i \leq n - 2,
\]

and the lemma is proved.

It follows from Lemma 1 and the definitions of \( X_i^*, \alpha_i, \beta_i \), and from the inductive hypothesis to the effect that Theorem 1 is valid for \( N = n \), that the validity of Theorem 1 will be established if (4.1), (4.2), and (4.3) of Lemma 1 are shown to hold for this \( X_i^*, \alpha_i, \beta_i \). The next sections of the proof will be devoted to the verification of these three statements.

In the course of this verification, it will be necessary to compute

\[
M_i(X_i, Y_i) = L_i(X_i, Y_i) + V(p, q) = L_i(X_i, Y_i) + a^i p - b^i q, \quad j = 1, \ldots, n - 2,
\]

explicitly in terms of \( p_i, q_i, X_i, Y_i \), for certain choices of \( X_i, Y_i \). For any given fixed initial condition \((p_i, q_i)\), an integer \( 1 \leq i \leq n - 1 \) is determined by the inequality \( \lambda_i \leq p_i/q_i \leq \lambda_{i+1} \). Each choice \((X_i, Y_i)\) by the players falls into one of the six regions enumerated in Table 2\(^1\) and determines \( p \) and \( q \) and hence an integer \( 1 \leq j \leq n - 2 \) via the inequality \( \lambda^j \leq p/q \leq \lambda^{j+1} \). It is this integer \( j \) that appears in (7.6). Clearly, \( j \) is a function of \( p_i, q_i, X_i, \) and \( Y_i \). In computing \( M_i(X_i, Y_i) \) explicitly in terms of the initial conditions and choices \( X_i, Y_i \), it will thus be necessary to take into account the region of the \((X_i, Y_i)\) space and the superscript \( j \). The statement \"\((X_i, Y_i)\) leads to case \( \text{III}^{\text{on}} \)\" will mean that, for the initial condition being considered, the pair \((X_i, Y_i)\) falls into region III of the \((X_i, Y_i)\) space and the ratio \( p/q \) is such that \( j = j_o \). At first glance, it appears that there are \( 6(n - 2) \) cases. Actually, not all of these cases are possible; and since some specialization of \( X_i, Y_i \) will occur, not all of the possible cases will be encountered.

8. Verification, first part of sufficiency condition. We divide the discussion into two cases.

Case 1. \( p_i/q_i \geq \lambda_i^{n-1} \).

For this initial condition, it is readily seen from Table 2 that \((X_i^*, Y_i^{(1)}) \) leads to case \( \text{VI}_{n-2} \), \((X_i^*, Y_i^{(2)}) \) leads to case \( \text{II}_{n-2} \), and \((X_i^*, Y_i^{(3)}) \) leads to case \( \text{III}_{n-2} \). It then follows by straightforward computation

\(^1\) The table is given for passage from \( n \) to \( n - 1 \), whereas the present situation is for passage from \( n + 1 \) to \( n \). The adjustment of subscripts is left to the reader.
and the definition of \( \S 6 \) that

\[
M_i(X^*_i, Y^{(1)}_i) = M_i(X^*_i, Y^{(2)}_i) = M_i(X^*_i, Y^{(3)}_i) = V_i(p_i, q_i).
\]

The equality \( E_i(\hat{F}^*_i, G^*_i) = V_i(p_i, q_i) \) now follows from

\[
E_i(\hat{F}^*_i, G^*_i) = \alpha_i[L_i(X^*_i, Y^{(1)}_i) + E(F^*_i, G^*_i)] + \beta_i[L_i(X^*_i, Y^{(2)}_i) + E(F^*_i, G^*_i)] + (1 - \alpha_i - \beta_i)[L_i(X^*_i, Y^{(3)}_i) + E(F^*_i, G^*_i)]
\]

\[
= \alpha_iM_i(X^*_i, Y^{(1)}_i) + \beta_iM_i(X^*_i, Y^{(2)}_i) + (1 - \alpha_i - \beta_i)M_i(X^*_i, Y^{(3)}_i).
\]

**Case 2.** \( \lambda_i \leq p_i/q_i \leq \lambda_i^{(i+1)} \), \( i = 1, 2, \ldots, n-2 \).

For this initial condition, it is immediately seen from Table 2 that \((X^*_i, Y^{(1)}_i)\) leads to case VI\(^{n-2}\), while \((X^*_i, Y^{(2)}_i)\) leads to case II\(^j\) for appropriate \( j \). To determine the value of \( j \), we first observe that \( p_i/q_i = p_i/(2q_i - \bar{x}_i^2) \). It follows from (7.4) and (7.5) that \( j = i \). Straightforward computation and use of the definitions in \( \S 6 \) now show that

\[
M_i(X^*_i, Y^{(2)}_i) = M_i(X^*_i, Y^{(3)}_i) = V_i(p_i, q_i),
\]

and hence (4.1) follows as before.

**9. Verification, second part of sufficiency condition.** Again, as in \( \S 8 \), we divide the discussion into two cases.

**Case 1.** \( p_i/q_i \geq \lambda_i^{n-1} \).

Since \( \bar{x}_i^{n-2} \geq q_i \), \((X^*_i, Y_i)\) can never fall into region I or IV of the \((X_i, Y_i)\) space. Since \((X^*_i, Y_i)\) lying in region V implies that

\[
y_1 + w_1 \geq \bar{x}_i^{n-2} + \bar{u}_i^{n-2} - q_i = \left( 2 - \frac{1}{b^{n-2}} + \frac{1}{n} \right) q_i > q_i,
\]

this event is also impossible.

For \( Y_i \) such that \((X^*_i, Y_i)\) falls in region II, we have

\[
\frac{p_i}{q_i} = \frac{p_i}{q_i - \bar{x}_i^{n-2} + w_i} \geq \frac{p_i}{2q_i - \bar{x}_i^{n-2}} = \frac{b^{n-2}p_i}{q_i} \geq \lambda_i^{n-2},
\]

the last inequality following from (6.11) and (6.16). Hence, the superscript associated with region II is \( n-2 \), and

\[
M_i(X^*_i, Y_i) = (n + 1)p_i - (b^{n-2} - 1)\bar{x}_i^{n-2} - \bar{u}_i^{n-2} - (1 + b^{n-2})q_i + y_i - (b^{n-2} - 1)w_i \geq M_i(X^*_i, Y^{(3)}_i) = V_i(p_i, q_i),
\]

the last inequality following from (8.1).

The only superscript that can be associated with region III or VI
is \( n - 2 \). Having noted this, we easily see that for \((X^*_i, Y_i)\) in region III, \( \min_{Y_i} M_i(X^*_i, Y_i) \) occurs for \( Y_i = Y_i^{(3)} \), and so (4.2) follows from (8.1). Similarly, for \( Y_i \) such that \((X^*_i, Y_i)\) is in VI,

\[
\min_{Y_i} M_i(X_i, Y_i) = M_i(X^*_i, Y_i^{(3)}) = V_i(p_i, q_i) .
\]

Case 2.

\[
\lambda_i^j \leq p_i/q_i \leq \lambda_i^{j+1}, \quad i = 1, \ldots, n - 2 .
\]

Since \( \bar{v}^i \geq q_i \), \((X^*_i, Y_i)\) cannot fall in region I or IV. If \((X^*_i, Y_i)\) is in region II, then we have

\[
M_i(X^*_i, Y_i) = (a^i + 1)p_i - (b^i + 1)q_i + (b^i - 1)\bar{v}_i - \bar{w}_i + y_i - (b^i - 1)w_i ,
\]

where \( j \) is determined by the ratio \( p_i/q_i = (q_i - \bar{v}_i + w_i) \). Clearly, \( j \) is a nonincreasing function of \( w_i \) alone, \( j = j(w_i) \). From (7.4), it follows that \( j(q_i) = i \). For each \( j \), the minimum of \( M_i(X^*_i, Y_i) \) is achieved at a point \( Y_i = (y_i, w_i) \), where \( y_i = 0 \) and where \( w_i \) is the largest value of \( w \) such that \((X^*_i, Y_i)\), \( Y_i = (0, w) \), leads to case II'. Hence, by the continuity of \( M_i(X^*_i, Y_i) \), it follows that the minimum of \( M_i(X^*_i, Y_i) \), over all \( Y_i \) such that \((X^*_i, Y_i)\) is in region II, occurs at \( Y_i^{(3)} \). Thus, using (8.2), we get

\[
M_i(X^*_i, Y_i) \geq M_i(X^*_i, Y_i^{(3)}) = V_i(p_i, q_i) .
\]

The only superscript possible for \((X_i, Y_i)\) in region III is \( n - 2 \); thus for \( Y_i \) such that \((X^*_i, Y_i)\) is in III, we have

\[
M_i(X^*_i, Y_i) = np_i - q_i + (y_i + w_i) .
\]

The minimum of this expression over region III is assumed at \( Y_i = (0, 0) \) and is \( np_i - q_i \). Since we are considering the case \( p_i/q_i \leq \lambda_i^{j+1} \), the inequality

\[
(b^i - 1) - \lambda_i^{j+1}(a^i - n) \geq 0
\]

implies the inequality \( np_i - q_i \geq a^i p_i - b^i q_i \), and so it suffices to establish (9.1) in order to verify (4.2). With the aid of (6.8) and (6.13), it is easy to see that the left-hand side of (9.1) is a decreasing function of the superscript. Furthermore, for \( i = n - 2 \) it follows from (6.4), (6.5), and (6.7) that the left-hand side of (9.1) is zero, and so (9.1) is verified.

In the event that \( Y_i \) is such that \((X^*_i, Y_i)\) is in region V, we have

\[
M_i(X^*_i, Y_i) = (a^i + 1)p_i - (b^i + 1)q_i + (b^i - 1)\bar{v}_i - \bar{w}_i + (a^i - 1)(\bar{w}_i - y_i) ,
\]

where \( j \) is determined by
\[ \lambda^j \leq \frac{p_1 - q_1 + \bar{w}_1^j}{q_1 - \bar{x}_1^j + w_1} \leq \lambda^{j+1}, \quad j = 1, 2, \ldots, n - 2. \]

Since \( X_i^* \) is fixed, all questions concerned with determining which points in the \((X_i^*, Y_i)\) plane lead to the different cases \( V^j \) are thus seen to devolve upon questions concerning point sets in the \( Y_i \) plane. Clearly, the lines \( L^j \),

\[ y_i = -\lambda^j w_1 + \lambda^j(\bar{x}_1^j - q_1) + p_1 + \bar{w}_1^j, \]

in the \((y_i, w_i)\) plane form a finite pencil through the point \( y_i = p_1 + \bar{w}_1^j, \ w_i = \bar{x}_1^j - q_1 \). From the monotonicity properties of the sequence \( \{\lambda^j\} \), it follows that for any fixed \( y_i = c \) with \( c \leq p_1 + \bar{w}_1^j \) (and so particularly for \( y_i \leq q_1 \)), as one moves along \( y_i = c \) in the direction of increasing \( w_i \), the lines \( L^j \) are encountered in order of decreasing \( j \), with the line \( L^{n-2} \) being intercepted at a value of \( w_1 > \bar{x}_1^j - q_1 \). Thus, the sets in the
Y_1 plane giving rise to the various cases V^j are, in general, as indicated by the hatched regions in Fig. 5.

From the fact that a^j \geq b^j, it follows that the minimum of M_i(X^*_i, Y_i), over each set V^j of Fig. 1, is assumed at the upper left-hand vertex of V^j. Hence, by the continuity of M_i(X^*_i, Y_i) in Y_i, it follows that the minimum of M_i(X^*_i, Y_i), over all Y_i such that (X^*_i, Y_i) is in V, is achieved at \( w_i = \bar{x}_i^1 - q_i, y_i = 2q_i - \bar{x}_i^1 \). Substituting these values into (9.2) and using \( \bar{x}_i^1 + \bar{u}_i^1 = p_i \), we see that the value of the minimum is \( 2n(p_i - q_i) \). It remains to show that \( 2n(p_i - q_i) \geq a^j_i p_i - b^j_i q_i \). Since \( p_i/q_i \geq \lambda_i^j \), this inequality is implied by the inequality \( \lambda_i^j (2n-b^j_i)/(2n-a^j_i) \) which is established by induction in exactly the way that (6.19) was established.

Finally, the case in which (X^*_i, Y_i) lies in region VI must be considered. Examination of Table 2 shows that the only superscript possible is \( j = n - 2 \), and so

\[
M(X^*_i, Y_i) = (n + 1)p_i - q_i - \bar{x}_i^1 + (n - 1)\bar{u}_i^1 - (n - 1)y_i + w_i.
\]

The minimum of this expression is assumed at \( Y_i = Y_i(i) \). Since \( M_i(X^*_i, Y_i(i)) = V_i \), the proof of (4.2) is now concluded.

10. Verification, third part of sufficiency condition. The proof of (4.3) will clearly involve the computation of

\[
(10.1) \quad \mathcal{M}(X_i) = \alpha_i M_i(X_i, Y_i^{(i)}) + \beta_i M_i(X_i, Y_i^{(j)}) + (1 - \alpha_i - \beta_i) M_i(X_i, Y_i^{(j)}).
\]

Thus, for each \( X_i \), it is necessary to know the case to which we are led by each of the points
A: \((X_i, Y_i^{(1)})\), B: \((X_i, Y_i^{(2)})\), C: \((X_i, Y_i^{(3)})\).

Part of this information is tabulated in Fig. 2. In this tabulation, a symbol such as, say, \(AV^j\) in a given region means that, for all \(X_i\) in that region, \((X_i, Y_i^{(1)})\) leads to case \(V^j\). In some instances, the value of \(j\) is indicated; in others, the determination of \(j\) will be made in the discussions of Cases 1 and 2 below.

**Case 1.** \(\lambda_i^{n-1} \leq \frac{p_i}{q_i} \).

First, the values assumed by the superscripts \(j\) will be determined. In the case BI', Table 2 shows that \(\frac{p_i}{q_i} = \frac{p_i}{q_i};\) and since, by (6.16), \(\lambda_i^{n-1} \geq \lambda_i^{n-2}\), it follows that \(j = n - 2\). Also, in the case BI', we have \(j = n - 2\), because

\[
\frac{p}{q} = \frac{p_1}{2q_1 - x_1} \geq \frac{p_1}{q_1}, \quad q_1 \leq x_1 \leq 2q_1.
\]

Similarly, in AI', and CI', we have \(j = n - 2\), because the relation \(p/q = p_i/(q_i - x_i) \geq p_i/q_i\) holds there. In the case AV', the value of \(j\) is determined by the ratio

\[
\lambda^j \leq \frac{p}{q} = \frac{p_i - q_i + u_i}{q_i - x_i} \leq \lambda^j+1, \quad j = 1, \ldots, n - 2.
\]

The lines

\[l': \lambda'x_1 + u_1 = \lambda'q_1 - (p_i - q_i)\]

form a finite pencil through the point \(x_1 = q_i, u_1 = -(p_i - q_i)\). It follows from the monotonicity of the \(\lambda^j\)'s that if a line \(u_1 = c\) with \(c > -(p_i - q_i)\) is traversed from \(x_1 = q_i\) in the direction of decreasing \(x_1\), then the lines of the pencil are encountered in order of decreasing \(j\), with \(l^{n-2}\) being the first line encountered. Thus, the lines \(l'\) divide the square \(0 \leq x_1 \leq q_1, 0 \leq u_1 \leq q_1\) into subregions over each of which a different superscript \(j\)

<table>
<thead>
<tr>
<th>Region of ((x_1, u_1)) plane</th>
<th>Region Number</th>
<th>(R)</th>
<th>(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1) (u_1)</td>
<td>(2q_1 \leq x_1) (u_1 \geq q_1)</td>
<td>(1)</td>
<td>-1</td>
</tr>
<tr>
<td>(q_1 \leq x_1 \leq 2q_1) (u_1 \geq q_1)</td>
<td>(2)</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>(0 \leq x_1 \leq q_1) (u_1 \geq q_1)</td>
<td>(3)</td>
<td>(b^{n-2} - 2)</td>
<td>-1</td>
</tr>
<tr>
<td>(2q_1 \leq x_1) (u_1 \leq q_1)</td>
<td>(4)</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(q_1 \leq x_1 \leq 2q_1) (u_1 \leq q_1)</td>
<td>(5)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(0 \leq x_1 \leq q_1) (u_1 \leq q_1)</td>
<td>(6)</td>
<td>(\frac{(b^{n-2} - 1)b^j}{b^{n-2}} - 1)</td>
<td>(\frac{a^j}{n} - 1)</td>
</tr>
</tbody>
</table>
is applicable. The number of subregions depends on the ratio $p_i/q_i$. For sufficiently large values of this ratio, the entire square will have the superscript value $n - 2$ associated with it. The important fact to be noted is that the region with superscript $n - 2$ always exists and contains the line segment $x_i = q_i$, $0 \leq u_i \leq q_i$.

Clearly, $\mathcal{M}(X_i)$ is continuous and is of the form

$$\mathcal{M}(X_i) = F(p_i, q_i) + Rx_i + Su_i,$$

where $F(p, q_i)$ is a step function on the $(x_i, u_i)$ plane whose values are expressions involving the constants $a^i, b^i$ and the initial conditions $p_i, q_i$. Its exact form is of no concern here. The coefficients $R$ and $S$ are also step functions on the $(x_i, u_i)$ plane whose values involve the constants $a^i, b^i$. The information concerning $R$ and $S$ shown in Table 3 is easily obtained from Fig. 2, the preceding discussion, Table 2, and the definitions (7.2) of $\alpha_i$ and $\beta_i$.

The superscript $j$ in the entries for region (6) varies as the superscript in $AV^i$, and assumes the same values as the superscript in $AV^i$.

For each of the regions (1) through (6) of this table, the set of points at which the maximum of $\mathcal{M}(X_i)$ is achieved on that region is easily determined from the tabulated values of $R$ and $S$ in the region. It then follows from the continuity of $\mathcal{M}(X_i)$ that the maximum of $\mathcal{M}(X_i)$ is achieved at all points of the square $q_i \leq x_i \leq 2q_i$, $0 \leq u_i \leq q_i$.

In particular, it is achieved at $(x^{i-2}, u^{i-2})$, since by (7.3) this point is in the square. It now follows from (8.1) that

$$\mathcal{M}(X_i) \leq \mathcal{M}(X^i) = V_i(p_i, q_i).$$

**Case 2.** $\lambda_i \leq p_i/q_i \leq \lambda_i^{i+1}$, $i = 1, 2, \ldots, n - 2$

Under these initial conditions, Fig. 2 is modified from the outset as follows. Point $C$ is eliminated since we have $\alpha_i^i + \beta_i^i = 1$ for $1 \leq i \leq n - 2$; and the region $x_i \geq 2q_i, u_i \geq q_i$ need not be considered since $p_i/q_i < 3$.

In determining the superscript $j$ and the modifications of Fig. 2, it will be convenient to distinguish two cases, namely $p_i/q_i \geq 2$ and $p_i/q_i \leq 2$.

Suppose that $p_i/q_i \geq 2$. The superscript $j$ in $BII^i$ is determined by the ratio

$$\frac{p_i}{q_i} = \frac{p_i}{2q_i - x_i},$$

where $q_i \leq x_i \leq \min [2q_i, p_i - u_i]$. Thus $j$ is a nondecreasing step function of $x_i$ alone whose value at $x_i = 2q_i$ is $n - q$, and whose jumps occur at

$$x_i = 2q_i - \frac{p_i}{\lambda^i},$$
where \( j \) is such that \( 2q_1 - p_1/\lambda^j \geq q_1 \). At the jump points, \( j \) is continuous from the right. Let \( j_0 \) denote the lowest value of the superscript \( j \). This is clearly assumed at \( x_i = q_i \), and the defining relation for \( j_0 \) can be taken as

\[
(10.4) \quad \lambda_{j_0}^0 \leq \frac{p_1}{q_1} \lambda_{j_0+1}.
\]

Since, by assumption and (6.18), \( p_1/q_1 \leq \lambda_{i+1}^{i+1} < \lambda_{i+1}^{i+1} \), it follows that \( j_0 \leq i \). It is also necessary to have some information concerning the superscript at \( u_i = q_i, x_i = p_i - q_i \). Substitution of this value of \( x_i \) into (10.2) gives the quantity \( p_i/(3q_1 - p_i) \). It can be shown that \( p_i/(3q_1 - p_i) \leq \lambda_{i+1}^{i+1} \), and hence it follows that \( j \leq i \) at the point \( x_i = p_i - q_i, u_i = q_i \).

![Fig. 3](image)

The superscript \( j \) in \( BI^j \) is determined by the ratio \( p_i/q_i \); and, in view of (10.4), this makes \( j = j_0 \). In \( AII^j \) the superscript is determined by \( p_i/q_i = p_i/(q_i - x_i) \). Thus \( j \) is an increasing step function of \( x_i \) alone, having value \( j_0 \) at \( x_i = 0 \), and \( n-2 \) at \( x_i = q_i \). The remarks made under Case 1 concerning \( AV^{j0} \) are applicable here, too. It is not difficult
to see that the lines $V$ split up the square $0 \leq x_1 \leq q_1$, $0 \leq u_1 \leq q_1$ as indicated in Fig. 3, which summarizes the foregoing discussion.

Suppose now that $p_j/q_1 \leq 2$. Most of the remarks concerning the superscript $j$ in $BII'$ in the case $p_j/q_1 \geq 2$ are also valid here. Now, however, $u_1$ cannot exceed $q_1$ when $x_1 \geq q_1$, and so there is no need to discuss the point $u_1 = q_1$, $x_1 = p_1 - q_1$. If $j_1$ denotes the maximum value of the superscript $j$, it no longer need be true that $j_1 = n - 2$. However, the relation $j_1 \geq i$ does hold. For, the maximum value of $p_j/q_1$ is $p_1/(2q_1 - p_1)$, and so the assertion $j_1 \geq i$ is equivalent to $p_1/(2q_1 - p_1) \geq \lambda^i$. This relation, however, is easily established.

As before, the superscript in $BII'$ takes on the value $j_0$. In $AII'$, it is readily seen that the superscript $j$ is equal to $j_0$ at $x_1 = 0$ and increases to the maximum value of $j_1$. In $AVI'$, the remarks made in the discussion of $p_j/q_1 \geq 2$ still hold, except that for $x_1 \geq p_1 - q_1$ the

regions are truncated by the line $x_1 + u_1 = p_1$. Furthermore, the smallest superscript involved in a truncated region is clearly $j_1$, and so is greater than $i$. This information is summarized in Fig. 4.

Regardless of whether $p_1 \geq 2q_1$, or $p_1 \leq 2q_1$, we may write

$$\mathcal{M}(X_1) = H(p_1, q_1) + Tx_1 + Uu_1,$$
where $H$, $T$, and $U$ are step functions having values that depend on the choice of $X_i$, but do not involve the variables $x_i$ or $u_i$. The functional values of $H$, $T$, and $U$ do involve the constants $a^i, b^i, a^j, b^j$, and those of $H$ involve $p_i$ and $q_i$ in addition. The superscripts $j$, of course, are determined by $X_i$. The values of $T$ and $U$ are shown in Table 4, and the regions of constancy are indicated. Clearly, the regions of constancy of $F$ coincide with those of $T$ and $U$.  

<table>
<thead>
<tr>
<th>Region of $(x, u)$ Plane</th>
<th>Region Number</th>
<th>$T$</th>
<th>$U$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2q_i \leq x_i$ $u_i \leq q_i$</td>
<td>(1)</td>
<td>$-1$</td>
<td>$\frac{nb^i}{b^i + n} - 1$</td>
<td>Only applies if $p_i \geq 2q_i$</td>
</tr>
<tr>
<td>$q_i \leq x_i \leq 2q_i$ $u_i \leq q_i$</td>
<td>(2)</td>
<td>$\frac{nb^i}{b^i + n} - 1$</td>
<td>$-1$</td>
<td>Only applies if $p_i \geq 2q_i$; exponent $j$ varies as superscript in $BI^j$</td>
</tr>
<tr>
<td>$q_i \leq x_i \leq 2q_i$ $u_i \leq q_i$</td>
<td>(3)</td>
<td>$\frac{nb^i}{b^i + n} - 1$</td>
<td>$\frac{nq^i}{b^i + n} - 1$</td>
<td>Exponent $j$ varies as superscript in $BI^j$</td>
</tr>
<tr>
<td>$0 \leq x_i \leq q_i$ $u_i \leq q_i$</td>
<td>(4)</td>
<td>$\frac{nb^i}{b^i + n} - 1$</td>
<td>$-1$</td>
<td>Exponent $j$ varies as superscript in $AI^j$</td>
</tr>
<tr>
<td>$0 \leq x_i \leq q_i$ $u_i \leq q_i$</td>
<td>(5)</td>
<td>$\frac{nb^i}{b^i + n} - 1$</td>
<td>$\frac{nq^i}{b^i + n} - 1$</td>
<td>Exponent $j$ varies as superscript in $AV^j$</td>
</tr>
</tbody>
</table>

Suppose now that $p_i \geq 2q_i$. Since $nb^i > n + b^i$, the maximum of $H(X)$ over region (1) of Table 4 occurs at $x_i = 2q_i$, $u_i = p_i - 2q_i$. Then the following four facts, 

(a) $nb^i > b^i + n$,

(b) the $b^i$'s are decreasing in $j$,

(c) the point $x_i = p_i$, $u_i = q_i$ lies in a set for which the superscript $j$ in $BI^j$ does not exceed $i$, and

(d) the continuity of $H(X)$, have the following implications: (a) the maximum of $H(X)$ over region (2) and that part of region (3) lying below $x_i = p_i - q_i$ is attained at $x_i = p_i - q_i$, $u_i = q_i$; (b) the maximum of $H(X)$ over that part of region (2) lying above $x_i = p_i - q_i$ is attained at all points of the line $x_i + u_i = p_i$ that lie in the strip for which the superscript in $BI^j$ assumes the value $i$. Denote this set of points by $\mathcal{Z}$. Again appealing to the continuity of $H(X)$, we see that the maximum of $H(X)$ over all admissible $X_i$ for which $x_i \geq q_i$ is achieved on $\mathcal{Z}$.

It is now asserted that $X^*_i$ lies in $\mathcal{Z}$. In view of (10.3), this is equivalent to showing that we have

$$2q_i - \frac{p_i}{\lambda^i} \leq x_i^* \leq 2q_i - \frac{p_{i+1}}{\lambda^{i+1}}, \quad i = 1, \ldots, n - 2.$$  

The right-hand inequality follows immediately from (7.4) and the initial
conditions. The left-hand inequality follows from the definition (7.1) of \( \bar{x}_i \), (6.15), and the initial conditions.

Thus, it has been proved that

\[
\mathcal{M}(X^*_i) \leq \mathcal{M}(X_i)
\]

for all \( X_i \) such that \( x_i \geq q_i \). Since \( \mathcal{M}(X^*_i) = V_i(p_i, q_i) \), to complete the proof of (4.3) in the case \( \lambda_i \leq p_i/q_i \leq \lambda_i^{-1}, p_i/q_i \geq 2 \), it is sufficient to show that (10.6) holds for all \( X_i \) such that \( x_i \leq q_i \). From the form of \( S \) in region (5) of Table 4, it is clear that \( \mathcal{M}(X_i) \) attains its maximum along the line \( u_i = q_i \) whenever \( x_i \leq q_i \). From the form of \( R \) in this region, it is clear that if \( b^i b^{n-2} \geq b^i + n \), then the maximum of \( \mathcal{M}(X_i) \) is attained at \( (x_i, u_i) = (q_i, q_i) \). Hence, (10.6) follows for all \( X_i \) in this event. On the other hand, if \( b^i b^{n-2} < b^i + n \), then the maximum of \( \mathcal{M}(X_i) \) will be attained at one of the points \( x_i = q_i - p_i/\lambda_i \geq 0 \), or \( x_i = 0 \), of the line \( u_i = q_i \). In this event, it can be shown by lengthy computation that (10.6) holds for such \( X_i \). Thus (4.3) is established for \( \lambda_i \leq p_i/q_i \leq \lambda_i^{-1}, p_i \geq 2q_i \). By similar methods, which will not be carried out here, (4.3) can be established for \( p_i \leq 2q_i \). Thus the validity of (4.3), and hence that of the theorem, is established.

**References**

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<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glen Earl Baxter, <em>An analytic problem whose solution follows from a simple algebraic identity</em></td>
<td>731</td>
</tr>
<tr>
<td>Leonard D. Berkovitz and Melvin Dresher, <em>A multimove infinite game with linear payoff</em></td>
<td>743</td>
</tr>
<tr>
<td>Earl Robert Berkson, <em>Sequel to a paper of A. E. Taylor</em></td>
<td>767</td>
</tr>
<tr>
<td>Gerald Berman and Robert Jerome Silverman, <em>Embedding of algebraic systems</em></td>
<td>777</td>
</tr>
<tr>
<td>Peter Crawley, <em>Lattices whose congruences form a boolean algebra</em></td>
<td>787</td>
</tr>
<tr>
<td>Daniel T. Finkbeiner, II, <em>Irreducible congruence relations on lattices</em></td>
<td>813</td>
</tr>
<tr>
<td>William James Firey, <em>Isoperimetric ratios of Reuleaux polygons</em></td>
<td>823</td>
</tr>
<tr>
<td>Delbert Ray Fulkerson, <em>Zero-one matrices with zero trace</em></td>
<td>831</td>
</tr>
<tr>
<td>Leon W. Green, <em>A sphere characterization related to Blaschke’s conjecture</em></td>
<td>837</td>
</tr>
<tr>
<td>Israel (Yitzchak) Nathan Herstein and Erwin Kleinfeld, <em>Lie mappings in characteristic 2</em></td>
<td>843</td>
</tr>
<tr>
<td>Charles Ray Hobby, <em>A characteristic subgroup of a p-group</em></td>
<td>853</td>
</tr>
<tr>
<td>R. K. Juberg, <em>On the Dirichlet problem for certain higher order parabolic equations</em></td>
<td>859</td>
</tr>
<tr>
<td>Melvin Katz, <em>Infinitely repeatable games</em></td>
<td>879</td>
</tr>
<tr>
<td>Emma Lehmer, <em>On Jacobi functions</em></td>
<td>887</td>
</tr>
<tr>
<td>D. H. Lehmer, <em>Power character matrices</em></td>
<td>895</td>
</tr>
<tr>
<td>Henry B. Mann, <em>A refinement of the fundamental theorem on the density of the sum of two sets of integers</em></td>
<td>909</td>
</tr>
<tr>
<td>Marvin David Marcus and Roy Westwick, <em>Linear maps on skew symmetric matrices: the invariance of elementary symmetric functions</em></td>
<td>917</td>
</tr>
<tr>
<td>Richard Dean Mayer and Richard Scott Pierce, <em>Boolean algebras with ordered bases</em></td>
<td>925</td>
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<tr>
<td>Trevor James McMinn, <em>On the line segments of a convex surface in ( E_3 )</em></td>
<td>943</td>
</tr>
<tr>
<td>Frank Albert Raymond, <em>The end point compactification of manifolds</em></td>
<td>947</td>
</tr>
<tr>
<td>Edgar Reich and S. E. Warschawski, <em>On canonical conformal maps of regions of arbitrary connectivity</em></td>
<td>965</td>
</tr>
<tr>
<td>Marvin Rosenblum, <em>The absolute continuity of Toeplitz’s matrices</em></td>
<td>987</td>
</tr>
<tr>
<td>Lee Albert Rubel, <em>Maximal means and Tauberian theorems</em></td>
<td>997</td>
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<tr>
<td>Helmut Heinrich Schaefer, <em>Some spectral properties of positive linear operators</em></td>
<td>1009</td>
</tr>
<tr>
<td>Robert Steinberg, <em>The simplicity of certain groups</em></td>
<td>1039</td>
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<tr>
<td>Hisahiro Tamano, <em>On paracompactness</em></td>
<td>1043</td>
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<tr>
<td>Angus E. Taylor, <em>Mittag-Leffler expansions and spectral theory</em></td>
<td>1049</td>
</tr>
<tr>
<td>Marion Franklin Tinsley, <em>Permanents of cyclic matrices</em></td>
<td>1067</td>
</tr>
<tr>
<td>Charles J. Titus, <em>A theory of normal curves and some applications</em></td>
<td>1083</td>
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<tr>
<td>Charles R. B. Wright, <em>On groups of exponent four with generators of order two</em></td>
<td>1097</td>
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