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ZERO-ONE MATRICES WITH ZERO TRACE

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Summary. General existence conditions for an $n \times n$ zero-one matrix having given row and column sums and zero trace consist of a set of $2^n - 1$ inequalities. These are shown to simplify to the following set of n inequalities in case the row sums a_i and column sums b_i satisfy $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$:

$$\sum_{i=1}^{k} b_i \leq \sum_{i=1}^{k} a_i^{**} \qquad (k = 1, \cdots, n)\ ,$$

where a_j^{**} is the number of a_i such that $i < j$ and $a_i \geq j - 1$ plus the number of a_i such that $i > j$ and $a_i \geq j$.

Introduction. Ryser [5] and Gale [3] have established simple arithmetic conditions that are necessary and sufficient for the existence of a matrix of zeros and ones having prescribed row and column sums. Here we vary the problem slightly and look for conditions under which there will be an $n \times n$ matrix of zeros and ones having given row and column sums and zero trace. While it is not difficult to derive such conditions from known feasibility theorems for network flows, they appear not to simplify greatly except under special circumstances.

One motivation for considering this problem lies in a graph theoretical interpretation. The existence problem for a directed, loopless graph on n nodes having given local degrees (i.e., for each node two non-negative integers are given, the first specifying the number of issuing arcs, the second the number of entering arcs, and arcs leading from a node to itself are not allowed) is equivalent to that for a zero-one matrix having given row and column sums and zero trace.

1. General existence conditions.

Let

$$a_1, a_2, \cdots, a_n$$
$$b_1, b_2, \cdots, b_n$$

be given non-negative integers. We seek conditions under which the constraints

$$\sum_{j=1}^{n} x_{ij} \leq a_i \qquad (i = 1, \cdots, n) \tag{1a}$$

$$\sum_{i=1}^{n} x_{ij} \geq b_j \qquad (j = 1, \cdots, n) \tag{1b}$$

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(1c) $$x_{ij} = \begin{cases} 0, & i = j \\ 0 \text{ or } 1, & i \neq j \end{cases}$$

have a solution x_{ij}. Thus if $\sum a_i = \sum b_i$, we are looking for an $n \times n$ matrix having row sums a_i, column sums b_i, and zero trace.

Feasibility conditions for the constraints (1) can be deduced, for example, either from known results on the subgraph problem for directed graghs [4, 2], or from the supply-demand theorem [3] for network flows. We shall use the latter.

Applied to our problem, the supply-demand theorem and the integrity theorem for network flows [1, 2] assert that the constraints (1) are feasible if and only if, corresponding to each non-empty subset $I \subseteq \{1, \cdots, n\}$ of columns, the "aggregate demand" $\sum_I b_i$ of I can be fulfilled without violating the "supply" limitations (1a) on individual rows, or the "capacity" constraints (1c) on individual cells of the matrix. Thus, for each I we need to determine the maximal number of ones that can be placed in the columns corresponding to I (ignoring all other columns), subject to the restrictions that at most a_i ones can be used from the ith row, and no ones can be put along the main diagonal. It follows that the maximal number of ones that can be put in the I-columns is given by

$$\sum_I \min (a_i, |I| - 1) + \sum_{\bar{I}} \min (a_i, |I|)$$

where $|I|$ denotes the number of indices in I and $\bar{I}$ is the complement of I in $\{1, \cdots, n\}$. Consequently feasibility conditions for (1) are

(2) $$\sum_I b_i \leq \sum_I \min (a_i, |I| - 1) + \sum_{\bar{I}} \min (a_i, |I|) ,$$

for all non-empty $I \subseteq \{1, \cdots, n\}$, a set of $2^n - 1$ inequalities.

If we let $N = \{1, \cdots, n\}$ and define, for $I \subseteq N$ and each $k = 1, \cdots, n$,

(3) $$a_k^*(I) = | \{i | i \in I \text{ and } a_i \geq k\} | ,$$

then (2) may be written as

(4) $$\sum_I b_i \leq \sum_{i=1}^{|I|-1} a_i^*(N) + a_{|I|}^*(\bar{I}) .$$

(A convenient way to see this is to represent the integers a_i by rows of dots. For example, if in Figure 1 we take $I = \{2, 4, 5\}$, then the right side of (2) is given by the number of dots lying to the left of the indicated line. On the other hand, since $a_k^*(I)$ is the number of dots in the kth column that also lie in the I-rows, the first term on the right of (4) counts all dots in the first $|I| - 1 = 2$ columns, and the second term counts the remaining dots that lie to the left of the line.)

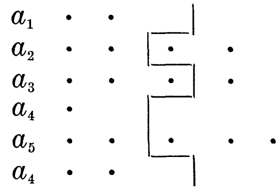

Fig. 1

Henceforth we shall let

$$a_k^* = a_k^*(N) ,$$

retaining the functional notation only when proper subsets of N are being considered. The sequence (a_i^*) is called the conjugate sequence to (a_i); it does not depend on the ordering of the a_i. In the next section we shall make use of another sequence (a_i^{**}) that will depend on the ordering of the a_i.

2. Simplification of existence conditions for a's and b's that are monotone together. The inequalities (4) simplify considerably if we assume that

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 , \qquad (5)$$
$$b_1 \geq b_2 \geq \cdots \geq b_n \geq 0 ,$$

or, what is the same thing, if there is some common renumbering for which (5) holds.

For if (5) holds, and if we rewrite (4) as

$$\sum_I b_i - a_{|I|}^*(\bar{I}) \leq \sum_{i=1}^{|I|-1} a_i^* , \qquad (6)$$

it is apparent that the left side of (6) is maximized, over all I with $|I| = k$, by selecting $I = \{1, \cdots, k\}$, because this I simultaneously maximizes $\sum_I b_i$ and minimizes $a_k^*(\bar{I})$. Since the right side of (6) depends only on $|I|$, it follows that the $2^n - 1$ inequalities (6) are equivalent, under the assumption (5), to the n inequalities

$$\sum_{i=1}^{k} b_i \leq \sum_{i=1}^{k-1} a_i^* + a_k^*(\{k+1, \cdots, n\}), \qquad k = 1, \cdots, n . \qquad (7)$$

The right side of (7) has a convenient and natural interpretation, again in terms of rows of dots, except that this time no dots are placed in the main diagonal of the schema. (See Figure 2.)

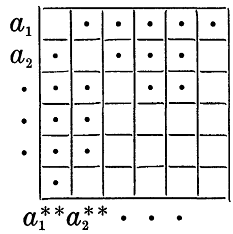

Fig. 2

If we define, for $k = 1, \cdots, n$, the sets of indices

$$(8) \qquad \begin{aligned} I_k &= \{i \mid i < k \text{ and } a_i \geq k-1\} \\ J_k &= \{i \mid i > k \text{ and } a_i \geq k\} , \end{aligned}$$

and let

$$(9) \qquad a_k^{**} = |I_k| + |J_k| ,$$

so that a_k^{**} is the number of dots in the kth column of the diagonally restricted array of dots, we see that the right side of (7) is just $\sum_{i=1}^{k} a_i^{**}$. Consequently feasibility conditions are simply that the partial sums of the sequence (b_i) be dominated by those of the sequence (a_i^{**}), and we have conditions analogous to those found by Ryser and Gale for the case in which ones can be placed along the main diagonal.

We shall refer to the sequence (a_i^{**}) defined by (8) and (9) as the *diagonally restricted conjugate* of (a_i) in the statement of the following theorem.

THEOREM. *Let a_i, b_i $(i = 1, \cdots, n)$ be monotone non-increasing sequences of non-negative integers. Then the following are equivalent:*

(i) *There is an $n \times n$ matrix of zeros and ones with zero trace whose ith row [column] sum is bounded above [below] by $a_i[b_i]$;*

(ii) $\sum_{i=1}^{k} b_i \leq \sum_{i=1}^{k} a_i^{**}$, $(k = 1, \cdots, n)$, *where the sequence (a_i^{**}) is the diagonally restricted conjugate of (a_i).*

Particular cases under which the rows and columns of the matrix can be rearranged so that the theorem applies are if the row sum upper bounds are constant, or if the column sum lower bounds are constant. In each of these cases, the obvious necessary conditions for feasibility turn out to be sufficient.

3. Constant row or column sum bounds. Unlike the sequence (a_i^*), the sequence (a_i^{**}) is not necessarily monotone (see Figure 2), but, as we shall prove in the following lemma, it is almost so, provided the a_i are monotone. Here the phrase "almost monotone" means that the sequence is either monotone or else it has at most one point of increase, and that increase is one. We shall make use of this fact in the proof of Corollary 1 below. First we state and prove the lemma.

LEMMA. *If* $a_1 \geq a_2 \geq \cdots \geq a_n$, *then the diagonally restricted conjugate sequence* (a_i^{**}) *is either monotone non-increasing or else, for some* $k = 1, \cdots, n-1$,

$$a_1^{**} \geq \cdots \geq a_k^{**}; \, a_{k+1}^{**} = a_k^{**} + 1 \geq a_{k+2}^{**} \geq \cdots \geq a_n^{**} .$$

Proof. It follows from (8) that

$$|I_k| \geq |I_{k+1}| - 1$$

$$|J_k| \geq |J_{k+1}|$$

and hence from (9) that

$$a_k^{**} \geq a_{k+1}^{**} - 1$$

equality holding if and only if $|I_k| = |I_{k+1}| - 1, |J_k| = |J_{k+1}|$. Moreover, since $a_1 \geq \cdots \geq a_n$, we have

$$a_k^{**} = a_{k+1}^{**} - 1$$

if and only if $a_k \geq k$ and $a_{k+1} < k$. Thus, if there were two points of increase in the sequence (a_i^{**}), say k and l with $k < l$, then we should have

$$a_l \geq l > k > a_{k+1} ,$$

contradicting $a_{k+1} \geq a_l$. This completes the proof of the lemma.

COROLLARY 1. *There is an* $n \times n$ *zero-one matrix with zero trace whose column sums are bounded below by* b, *and whose* i^{th} *row sum is bounded above by* a_i, *if and only if* $nb \leq \sum_1^n a_i^{**}$, *where* (a_i^{**}) *is the diagonally restricted conjugate of a monotone non-increasing rearrangement of* (a_i).

Proof. By effecting a rearrangement of rows and the same rearrangement of columns, we may assume $a_1 \geq a_2 \geq \cdots \geq a_n$ and hence apply the theorem. Thus, necessity being obvious, we need to show that the inequalities

$$kb \leq \sum_1^k a_i^{**} \qquad (k = 1, \cdots, n)$$

follow from $nb \leq \sum_1^n a_i^{**}$. This can be established by induction on n, as follows. For $n = 1$, there is nothing to prove. Assume the proposition for $n-1$ and consider the case for n. If $b < a_n^{**}$, then the lemma, together with the fact that we are dealing with integers, implies that $b \leq a_i^{**}$ for all i. Consequently

$$kb \leq \sum_{i=1}^{k} a_i^{**} .$$

If, on the other hand, $b \geq a_n^{**}$, then

$$(n-1)b \leq \sum_{i=1}^{n-1} a_i^{**} ,$$

and the induction hypothesis applies.

COROLLARY 2. *There is an $n \times n$ zero-one matrix with zero trace whose row sums are bounded above by a, and whose i^{th} column sum is bounded below by b_i, if and only if $\sum_{i=1}^{n} b_i \leq na$, $b_i \leq n-1$.*

Proof. Again necessity is obvious. To prove sufficiency, we need to show that $\sum_{i=1}^{n} b_i \leq na$ and $b_i \leq n-1$ imply

$$\sum_{i=1}^{k} b_i \leq \sum_{i=1}^{k} a_i^{**} ,$$

where $b_1 \geq \cdots \geq b_n$. If $k \leq a$, then $a_i^{**} = n-1$ for $i \leq k$, and hence

$$\sum_{i=1}^{k} b_i \leq k(n-1) = \sum_{1}^{k} a_i^{**} .$$

If, on the other hand, $k > a$, then

$$a_i^{**} = \begin{cases} n-1 & \text{for } i \leq a \\ a & \text{for } i = a+1 \\ 0 & \text{for } i > a+1 , \end{cases}$$

and hence

$$\sum_{i=1}^{k} b_i \leq \sum_{i=1}^{n} b_i \leq na = \sum_{i=1}^{k} a_i^{**} .$$

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