

# Pacific Journal of Mathematics

**ZERO-ONE MATRICES WITH ZERO TRACE**

DELBERT RAY FULKERSON

# ZERO-ONE MATRICES WITH ZERO TRACE

D. R. FULKERSON

**Summary.** General existence conditions for an  $n \times n$  zero-one matrix having given row and column sums and zero trace consist of a set of  $2^n - 1$  inequalities. These are shown to simplify to the following set of  $n$  inequalities in case the row sums  $a_i$  and column sums  $b_i$  satisfy  $a_1 \geq \dots \geq a_n, b_1 \geq \dots \geq b_n$ :

$$\sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i^{**} \quad (k = 1, \dots, n),$$

where  $a_j^{**}$  is the number of  $a_i$  such that  $i < j$  and  $a_i \geq j - 1$  plus the number of  $a_i$  such that  $i > j$  and  $a_i \geq j$ .

**Introduction.** Ryser [5] and Gale [3] have established simple arithmetic conditions that are necessary and sufficient for the existence of a matrix of zeros and ones having prescribed row and column sums. Here we vary the problem slightly and look for conditions under which there will be an  $n \times n$  matrix of zeros and ones having given row and column sums and zero trace. While it is not difficult to derive such conditions from known feasibility theorems for network flows, they appear not to simplify greatly except under special circumstances.

One motivation for considering this problem lies in a graph theoretical interpretation. The existence problem for a directed, loopless graph on  $n$  nodes having given local degrees (i.e., for each node two non-negative integers are given, the first specifying the number of issuing arcs, the second the number of entering arcs, and arcs leading from a node to itself are not allowed) is equivalent to that for a zero-one matrix having given row and column sums and zero trace.

1. **General existence conditions.** Let

$$\begin{aligned} a_1, a_2, \dots, a_n \\ b_1, b_2, \dots, b_n \end{aligned}$$

be given non-negative integers. We seek conditions under which the constraints

$$(1a) \quad \sum_{j=1}^n x_{ij} \leq a_i \quad (i = 1, \dots, n)$$

$$(1b) \quad \sum_{i=1}^n x_{ij} \leq b_j \quad (j = 1, \dots, n)$$

$$(1c) \quad x_{ij} = \begin{cases} 0, & i = j \\ 0 \text{ or } 1, & i \neq j \end{cases}$$

have a solution  $x_{ij}$ . Thus if  $\sum a_i = \sum b_i$ , we are looking for an  $n \times n$  matrix having row sums  $a_i$ , column sums  $b_i$ , and zero trace.

Feasibility conditions for the constraints (1) can be deduced, for example, either from known results on the subgraph problem for directed graphs [4, 2], or from the supply-demand theorem [3] for network flows. We shall use the latter.

Applied to our problem, the supply-demand theorem and the integrity theorem for network flows [1, 2] assert that the constraints (1) are feasible if and only if, corresponding to each non-empty subset  $I \subseteq \{1, \dots, n\}$  of columns, the "aggregate demand"  $\sum_I b_i$  of  $I$  can be fulfilled without violating the "supply" limitations (1a) on individual rows, or the "capacity" constraints (1c) on individual cells of the matrix. Thus, for each  $I$  we need to determine the maximal number of ones that can be placed in the columns corresponding to  $I$  (ignoring all other columns), subject to the restrictions that at most  $a_i$  ones can be used from the  $i$ th row, and no ones can be put along the main diagonal. It follows that the maximal number of ones that can be put in the  $I$ -columns is given by

$$\sum_I \min(a_i, |I| - 1) + \sum_{\bar{I}} \min(a_i, |I|)$$

where  $|I|$  denotes the number of indices in  $I$  and  $\bar{I}$  is the complement of  $I$  in  $\{1, \dots, n\}$ . Consequently feasibility conditions for (1) are

$$(2) \quad \sum_I b_i \leq \sum_I \min(a_i, |I| - 1) + \sum_{\bar{I}} \min(a_i, |I|),$$

for all non-empty  $I \subseteq \{1, \dots, n\}$ , a set of  $2^n - 1$  inequalities.

If we let  $N = \{1, \dots, n\}$  and define, for  $I \subseteq N$  and each  $k = 1, \dots, n$ ,

$$(3) \quad a_k^*(I) = |\{i \mid i \in I \text{ and } a_i \geq k\}|,$$

then (2) may be written as

$$(4) \quad \sum_I b_i \leq \sum_{i=1}^{|I|-1} a_i^*(N) + a_{|I|}^*(\bar{I}).$$

(A convenient way to see this is to represent the integers  $a_i$  by rows of dots. For example, if in Figure 1 we take  $I = \{2, 4, 5\}$ , then the right side of (2) is given by the number of dots lying to the left of the indicated line. On the other hand, since  $a_k^*(I)$  is the number of dots in the  $k$ th column that also lie in the  $I$ -rows, the first term on the right of (4) counts all dots in the first  $|I| - 1 = 2$  columns, and the second term counts the remaining dots that lie to the left of the line.)

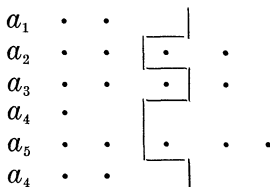


Fig. 1

Henceforth we shall let

$$a_k^* = a_k^*(N) ,$$

retaining the functional notation only when proper subsets of  $N$  are being considered. The sequence  $(a_i^*)$  is called the conjugate sequence to  $(a_i)$ ; it does not depend on the ordering of the  $a_i$ . In the next section we shall make use of another sequence  $(a_i^{**})$  that will depend on the ordering of the  $a_i$ .

**2. Simplification of existence conditions for  $a$ 's and  $b$ 's that are monotone together.** The inequalities (4) simplify considerably if we assume that

$$(5) \quad \begin{aligned} a_1 \geq a_2 \geq \dots \geq a_n \geq 0 , \\ b_1 \geq b_2 \geq \dots \geq b_n \geq 0 , \end{aligned}$$

or, what is the same thing, if there is some common renumbering for which (5) holds.

For if (5) holds, and if we rewrite (4) as

$$(6) \quad \sum_I b_i - a_{|I|}^*(\bar{I}) \leq \sum_{i=1}^{|I|-1} a_i^* ,$$

it is apparent that the left side of (6) is maximized, over all  $I$  with  $|I| = k$ , by selecting  $I = \{1, \dots, k\}$ , because this  $I$  simultaneously maximizes  $\sum_I b_i$  and minimizes  $a_k^*(\bar{I})$ . Since the right side of (6) depends only on  $|I|$ , it follows that the  $2^n - 1$  inequalities (6) are equivalent, under the assumption (5), to the  $n$  inequalities

$$(7) \quad \sum_{i=1}^k b_i \leq \sum_{i=1}^{k-1} a_i^* + a_k^*(\{k + 1, \dots, n\}), \quad k = 1, \dots, n .$$

The right side of (7) has a convenient and natural interpretation, again in terms of rows of dots, except that this time no dots are placed in the main diagonal of the schema. (See Figure 2.)

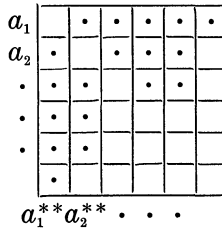


Fig. 2

If we define, for  $k = 1, \dots, n$ , the sets of indices

$$(8) \quad \begin{aligned} I_k &= \{i \mid i < k \text{ and } a_i \geq k - 1\} \\ J_k &= \{i \mid i > k \text{ and } a_i \geq k\} , \end{aligned}$$

and let

$$(9) \quad a_k^{**} = |I_k| + |J_k| ,$$

so that  $a_k^{**}$  is the number of dots in the  $k$ th column of the diagonally restricted array of dots, we see that the right side of (7) is just  $\sum_{i=1}^k a_i^{**}$ . Consequently feasibility conditions are simply that the partial sums of the sequence  $(b_i)$  be dominated by those of the sequence  $(a_i^{**})$ , and we have conditions analogous to those found by Ryser and Gale for the case in which ones can be placed along the main diagonal.

We shall refer to the sequence  $(a_i^{**})$  defined by (8) and (9) as the *diagonally restricted conjugate* of  $(a_i)$  in the statement of the following theorem.

**THEOREM.** *Let  $a_i, b_i$  ( $i = 1, \dots, n$ ) be monotone non-increasing sequences of non-negative integers. Then the following are equivalent:*

- (i) *There is an  $n \times n$  matrix of zeros and ones with zero trace whose  $i$ th row [column] sum is bounded above [below] by  $a_i$ [ $b_i$ ];*
- (ii)  *$\sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i^{**}, (k = 1, \dots, n)$ , where the sequence  $(a_i^{**})$  is the diagonally restricted conjugate of  $(a_i)$ .*

Particular cases under which the rows and columns of the matrix can be rearranged so that the theorem applies are if the row sum upper bounds are constant, or if the column sum lower bounds are constant. In each of these cases, the obvious necessary conditions for feasibility turn out to be sufficient.

**3. Constant row or column sum bounds.** Unlike the sequence  $(a_i^*)$ , the sequence  $(a_i^{**})$  is not necessarily monotone (see Figure 2), but, as we shall prove in the following lemma, it is almost so, provided the  $a_i$  are monotone. Here the phrase “almost monotone” means that the sequence is either monotone or else it has at most one point of increase, and that increase is one. We shall make use of this fact in the proof of Corollary 1 below. First we state and prove the lemma.

LEMMA. *If  $a_1 \geq a_2 \geq \dots \geq a_n$ , then the diagonally restricted conjugate sequence  $(a_i^{**})$  is either monotone non-increasing or else, for some  $k = 1, \dots, n - 1$ ,*

$$a_1^{**} \geq \dots \geq a_k^{**}; a_{k+1}^{**} = a_k^{**} + 1 \geq a_{k+2}^{**} \geq \dots \geq a_n^{**}.$$

*Proof.* It follows from (8) that

$$|I_k| \geq |I_{k+1}| - 1$$

$$|J_k| \geq |J_{k+1}|$$

and hence from (9) that

$$a_k^{**} \geq a_{k+1}^{**} - 1$$

equality holding if and only if  $|I_k| = |I_{k+1}| - 1, |J_k| = |J_{k+1}|$ . Moreover, since  $a_1 \geq \dots \geq a_n$ , we have

$$a_k^{**} = a_{k+1}^{**} - 1$$

if and only if  $a_k \geq k$  and  $a_{k+1} < k$ . Thus, if there were two points of increase in the sequence  $(a_i^{**})$ , say  $k$  and  $l$  with  $k < l$ , then we should have

$$a_l \geq l > k > a_{k+1},$$

contradicting  $a_{k+1} \geq a_l$ . This completes the proof of the lemma.

COROLLARY 1. *There is an  $n \times n$  zero-one matrix with zero trace whose column sums are bounded below by  $b$ , and whose  $i^{\text{th}}$  row sum is bounded above by  $a_i$ , if and only if  $nb \leq \sum_1^n a_i^{**}$ , where  $(a_i^{**})$  is the diagonally restricted conjugate of a monotone non-increasing rearrangement of  $(a_i)$ .*

*Proof.* By effecting a rearrangement of rows and the same rearrangement of columns, we may assume  $a_1 \geq a_2 \geq \dots \geq a_n$  and hence apply the theorem. Thus, necessity being obvious, we need to show that the inequalities

$$kb \leq \sum_1^k a_i^{**} \quad (k = 1, \dots, n)$$

follow from  $nb \leq \sum_1^n a_i^{**}$ . This can be established by induction on  $n$ , as follows. For  $n = 1$ , there is nothing to prove. Assume the proposition for  $n - 1$  and consider the case for  $n$ . If  $b < a_n^{**}$ , then the lemma, together with the fact that we are dealing with integers, implies that  $b \leq a_i^{**}$  for all  $i$ . Consequently

$$kb \leq \sum_{i=1}^k a_i^{**} .$$

If, on the other hand,  $b \geq a_n^{**}$ , then

$$(n-1)b \leq \sum_{i=1}^{n-1} a_i^{**} ,$$

and the induction hypothesis applies.

**COROLLARY 2.** *There is an  $n \times n$  zero-one matrix with zero trace whose row sums are bounded above by  $a$ , and whose  $i^{\text{th}}$  column sum is bounded below by  $b_i$ , if and only if  $\sum_{i=1}^n b_i \leq na$ ,  $b_i \leq n-1$ .*

*Proof.* Again necessity is obvious. To prove sufficiency, we need to show that  $\sum_{i=1}^n b_i \leq na$  and  $b_i \leq n-1$  imply

$$\sum_{i=1}^k b_i \leq \sum_{i=1}^k a_i^{**} ,$$

where  $b_1 \geq \dots \geq b_n$ . If  $k \leq a$ , then  $a_i^{**} = n-1$  for  $i \leq k$ , and hence

$$\sum_{i=1}^k b_i \leq k(n-1) = \sum_{i=1}^k a_i^{**} .$$

If, on the other hand,  $k > a$ , then

$$a_i^{**} = \begin{cases} n-1 & \text{for } i \leq a \\ a & \text{for } i = a+1 \\ 0 & \text{for } i > a+1 , \end{cases}$$

and hence

$$\sum_{i=1}^k b_i \leq \sum_{i=1}^n b_i \leq na = \sum_{i=1}^k a_i^{**} .$$

## REFERENCES

1. L. R. Ford, Jr., and D. R. Fulkerson, *A simple algorithm for finding maximal network flows and an application to the Hitchcock problem*, Can. J. Math., **9** (1957), 210-218.
2. D. R. Fulkerson, *A network flow feasibility theorem and combinatorial applications* Can. S. Math., **11** (1959), 440-451.
3. D. Gale, *A theorem on flows in networks*, Pacific J. Math., **7** (1957), 1073-1082.
4. O. Ore, *Studies on directed graphs I*, Ann. of Math., **63** (1956), 383-406.
5. H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Can. J. Math., **9** (1957), 371-377.

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DAVID GILBARG  
Stanford University  
Stanford, California

F. H. BROWNELL  
University of Washington  
Seattle 5, Washington

A. L. WHITEMAN  
University of Southern California  
Los Angeles 7, California

L. J. PAIGE  
University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH  
T. M. CHERRY  
D. DERRY

E. HEWITT  
A. HORN  
L. NACHBIN

M. OHTSUKA  
H. L. ROYDEN  
M. M. SCHIFFER

E. SPANIER  
E. G. STRAUS  
F. WOLF

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE COLLEGE  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE COLLEGE  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
HUGHES AIRCRAFT COMPANY  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.



Glen Earl Baxter, <i>An analytic problem whose solution follows from a simple algebraic identity</i> .....	731
Leonard D. Berkovitz and Melvin Dresher, <i>A multimove infinite game with linear payoff</i> .....	743
Earl Robert Berkson, <i>Sequel to a paper of A. E. Taylor</i> .....	767
Gerald Berman and Robert Jerome Silverman, <i>Embedding of algebraic systems</i> .....	777
Peter Crawley, <i>Lattices whose congruences form a boolean algebra</i> .....	787
Robert E. Edwards, <i>Integral bases in inductive limit spaces</i> .....	797
Daniel T. Finkbeiner, II, <i>Irreducible congruence relations on lattices</i> .....	813
William James Firey, <i>Isoperimetric ratios of Reuleaux polygons</i> .....	823
Delbert Ray Fulkerson, <i>Zero-one matrices with zero trace</i> .....	831
Leon W. Green, <i>A sphere characterization related to Blaschke's conjecture</i> .....	837
Israel (Yitzchak) Nathan Herstein and Erwin Kleinfeld, <i>Lie mappings in characteristic 2</i> .....	843
Charles Ray Hobby, <i>A characteristic subgroup of a p-group</i> .....	853
R. K. Juberg, <i>On the Dirichlet problem for certain higher order parabolic equations</i> .....	859
Melvin Katz, <i>Infinitely repeatable games</i> .....	879
Emma Lehmer, <i>On Jacobi functions</i> .....	887
D. H. Lehmer, <i>Power character matrices</i> .....	895
Henry B. Mann, <i>A refinement of the fundamental theorem on the density of the sum of two sets of integers</i> .....	909
Marvin David Marcus and Roy Westwick, <i>Linear maps on skew symmetric matrices: the invariance of elementary symmetric functions</i> .....	917
Richard Dean Mayer and Richard Scott Pierce, <i>Boolean algebras with ordered bases</i> .....	925
Trevor James McMinn, <i>On the line segments of a convex surface in <math>E_3</math></i> .....	943
Frank Albert Raymond, <i>The end point compactification of manifolds</i> .....	947
Edgar Reich and S. E. Warschawski, <i>On canonical conformal maps of regions of arbitrary connectivity</i> .....	965
Marvin Rosenblum, <i>The absolute continuity of Toeplitz's matrices</i> .....	987
Lee Albert Rubel, <i>Maximal means and Tauberian theorems</i> .....	997
Helmut Heinrich Schaefer, <i>Some spectral properties of positive linear operators</i> .....	1009
Jeremiah Milton Stark, <i>Minimum problems in the theory of pseudo-conformal transformations and their application to estimation of the curvature of the invariant metric</i> .....	1021
Robert Steinberg, <i>The simplicity of certain groups</i> .....	1039
Hisahiro Tamano, <i>On paracompactness</i> .....	1043
Angus E. Taylor, <i>Mittag-Leffler expansions and spectral theory</i> .....	1049
Marion Franklin Tinsley, <i>Permanents of cyclic matrices</i> .....	1067
Charles J. Titus, <i>A theory of normal curves and some applications</i> .....	1083
Charles R. B. Wright, <i>On groups of exponent four with generators of order two</i> .....	1097