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A CHARACTERISTIC SUBGROUP OF A p -GROUP

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If x, y are elements and H, K subsets of the p -group G , we shall denote by $[x, y]$ the element $y^{-p}x^{-p}(xy)^p$ of G , and by $[H, K]$ the subgroup of G generated by the set of all $[h, k]$ for h in H and k in K . We call a p -group G *p-abelian* if $(xy)^p = x^p y^p$ for all elements x, y of G . If we let $\theta(G) = [G, G]$ then $\theta(G)$ is a characteristic subgroup of G and $G/\theta(G)$ is *p-abelian*. In fact, $\theta(G)$ is the minimal normal subgroup N of G for which G/N is *p-abelian*. It is clear that $\theta(G)$ is contained in the derived group of G , and $G/\theta(G)$ is *regular* in the sense of P. Hall [3].

Theorem 1 lists some elementary properties of *p-abelian* groups. These properties are used to obtain a characterization of *p*-groups G (for $p \geq 3$) in which the subgroup generated by the p th powers of elements of G coincides with the Frattini subgroup of G (Theorems 2 and 3). A group G is said to be *metacyclic* if there exists a cyclic normal subgroup N with G/N cyclic. Theorem 4 states that a *p*-group G , for $p > 2$, is *metacyclic* if and only if $G/\theta(G)$ is *metacyclic*. Theorems on *metacyclic p*-groups due to Blackburn and Huppert are obtained as corollaries of Theorems 3 and 4.

The following notation is used: G is a *p*-group; $G^{(n)}$ is the n th derived group of G ; G_n is the n th element in the descending central series of G ; $P(G)$ is the subgroup of G generated by the set of all x^p for x belonging to G ; $\Phi(G)$ is the Frattini subgroup of G ; $\langle x, y, \dots \rangle$ is the subgroup generated by the elements x, y, \dots ; $Z(G)$ is the center of G ; $(h, k) = h^{-1}k^{-1}hk$; if H, K are subsets of G , then (H, K) is the subgroup generated by the set of all (h, k) for $h \in H$ and $k \in K$.

THEOREM 1. *If G is p-abelian, then*

$$(1.1) \quad P(G^{(1)}) = P(G)^{(1)},$$

$$(1.2) \quad P(G) \subseteq Z(G),$$

$$(1.3) \quad \Phi(G^{(1)}) = \Phi(G)^{(1)} = G^{(2)}.$$

Proof of (1.1). $\theta(G) = \langle 1 \rangle$ implies that $(xyx^{-1}y^{-1})^p = x^p y^p x^{-p} y^{-p}$ for all x, y in G . (1.1) follows immediately.

Proof of (1.2). Let x be an arbitrary element of G , and suppose the order of x is p^n . Let $u = x^{1+p+\dots+p^{n-1}}$. Then, for any y in G ,

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$$uy^pu^{-1} = (uyu^{-1})^p = u^py^pu^{-p},$$

where the last equality follows from $\theta(G) = \langle 1 \rangle$. Therefore $u^{1-p}y^pu^{p-1} = y^p$. But $u^{1-p} = x^{1-p^n} = x$, hence $xy^px^{-1} = y^p$, and (1.2) follows.

Proof of (1.3). It is easy to see that $\Phi(G) = P(G)G^{(1)}$, hence $\Phi(G)^{(1)} \supseteq P(G)^{(1)}G^{(2)}$. Thus, by (1.1), $\Phi(G)^{(1)} \supseteq P(G^{(1)})G^{(2)} = \Phi(G^{(1)}) \supseteq G^{(2)}$. It remains to show that $G^{(2)} \supseteq \Phi(G)^{(1)}$. But if x, y belong to $\Phi(G)$, we can write $x = x'u, y = y'v$ for x', y' in $P(G)$ and u, v in $G^{(1)}$ (since $\Phi(G) = P(G)G^{(1)}$). By (1.2), x' and y' belong to $Z(G)$, hence $xyx^{-1}y^{-1} = uvu^{-1}v^{-1}$ is an element of $G^{(2)}$. Thus $\Phi(G)^{(1)} \subseteq G^{(2)}$, and the proof is complete.

COROLLARY 1.1. $P(G^{(1)}) \subseteq \theta(G)$.

Proof. It suffices to show that $\theta(G) = \langle 1 \rangle$ implies $P(G^{(1)}) = \langle 1 \rangle$. But, if $\theta(G) = \langle 1 \rangle$, it follows from (1.1) and (1.2) that $P(G^{(1)}) = P(G)^{(1)}$ and $P(G) \subseteq Z(G)$. Thus $P(G^{(1)}) = \langle 1 \rangle$.

REMARK 1. P. Hall [3] has shown that

$$(xy)^p = x^py^pcd$$

whenever x, y belong to a p -group G , where c is a product of p th powers of elements of $\langle x, y \rangle^{(1)}$ and d is a product of elements contained in the p th element of the descending central series of $\langle x, y \rangle$. We have, as an immediate consequence, $\theta(G) \subseteq P(G^{(1)})G_p$.

We shall now investigate p -groups G for which $P(G) = \Phi(G)$. The following lemma will be useful.

LEMMA 1. *Suppose $p \neq 2$. If $P(G) = \Phi(G)$ and $P(G^{(1)}) = \langle 1 \rangle$, then $G_3 = \langle 1 \rangle$.*

Proof. If $x, y \in G$, then

$$\begin{aligned} (y^p, x) &= y^{-p}(x^{-1}y^px) = y^{-p}(x^{-1}yx)^p \\ &= y^{-p}\{y(y, x)\}^p \\ &= (y, x)^p[y, (y, x)] = [y, (y, x)], \end{aligned}$$

where the last equality follows from $P(G^{(1)}) = \langle 1 \rangle$. Therefore $G_3 \subseteq (G, P(G)) \subseteq [G, G^{(1)}] \subseteq [G, P(G)]$. We complete the proof by showing that $[G, P(G)] \subseteq G_4$.

We first observe that $(x, y^p) \in G_3$, hence

$$(xy^p)^p = x^py^{p^2}(x, y^p)^{(p-1)/2z}$$

for some $z \in G_4$. Since $p \neq 2$ and $P(G^{(1)}) = \langle 1 \rangle$, we have $[x, y^p] \in G_4$ for

every $x, y \in G$. It follows that $[G, P(G)] \subseteq G_4$.

THEOREM 2. *If $P(G) = \Phi(G)$, then $P(G^{(k)}) = \Phi(G^{(k)})$ for $k = 1, 2, \dots$.*

Proof. Suppose G is a group of minimal order for which $P(G) = \Phi(G)$ but $P(G^{(k)}) \neq \Phi(G^{(k)})$ for some $k \geq 1$. If $P(G^{(1)}) = \Phi(G^{(1)})$, then we must have $P(G^{(k)}) = \Phi(G^{(k)})$ for all $k \geq 1$ since the order of $G^{(1)}$ is less than the order of G . Thus $P(G^{(1)}) \neq \Phi(G^{(1)})$. We assert that $P(G^{(1)})$ must be $\langle 1 \rangle$. For, if $P(G^{(1)}) \neq \langle 1 \rangle$, we let $H = G/P(G^{(1)})$. Then it is easy to see that $P(H) = \Phi(H)$. Thus, since H has smaller order than G , $P(H^{(1)}) = \Phi(H^{(1)})$. Also, $P(H^{(1)}) = \langle 1 \rangle$. Therefore

$$\langle 1 \rangle = \Phi(H^{(1)}) = \Phi(G^{(1)}/P(G^{(1)})) = \Phi(G^{(1)})P(G^{(1)})/P(G^{(1)}).$$

That is, $P(G^{(1)}) \supseteq \Phi(G^{(1)})$, and hence $P(G^{(1)}) = \Phi(G^{(1)})$, which contradicts our assumption.

If $p = 2$ it follows from $P(G^{(1)}) = \langle 1 \rangle$ that $G^{(1)}$ is abelian. If $p \neq 2$, then by Lemma 1, $G_3 = \langle 1 \rangle$ and $G^{(1)}$ is again abelian. Therefore $P(G^{(1)}) = \Phi(G^{(1)})$, contrary to our choice of G .

COROLLARY 2.1. *If $p \neq 2$ and $P(G) = \Phi(G)$, then $P(G^{(1)}) = \Phi(G^{(1)}) = \theta(G) \supseteq G_3$.*

Proof. By Corollary 1.1, $P(G^{(1)}) \subseteq \theta(G)$. By Lemma 1, $G_3 \subseteq P(G^{(1)})$. Therefore $P(G^{(1)})G_p = P(G^{(1)})$ since $p \neq 2$. It follows from Remark 1 that $P(G^{(1)}) = \theta(G)$. By Theorem 2, $P(G^{(1)}) = \Phi(G^{(1)})$, and the proof is complete.

COROLLARY 2.2. *Let $p \neq 2$ and $P(G) = \Phi(G)$. Then $P(G^{(1)}) \subseteq G^{(2)}$ implies $G_3 = \langle 1 \rangle$, and hence $G^{(2)} = \langle 1 \rangle$.*

Proof. By Corollary 2.1, $G_3 \subseteq P(G^{(1)})$, thus $G_3 \subseteq G^{(2)}$. It is known [3, Theorem 2.54] that $G^{(2)} \subseteq G_4$. Therefore $G_3 = G_4 = G^{(2)} = \langle 1 \rangle$.

THEOREM 3. *Suppose $p \neq 2$ and let x_1, x_2, \dots, x_k be coset representatives of a minimal basis of the abelian group $G/G^{(1)}$. Then $P(G) = \Phi(G)$ if, and only if, there exist integers $n(i)$ such that*

$$G^{(1)} = \langle x_1^{p^{n(1)}}, x_2^{p^{n(2)}}, \dots, x_k^{p^{n(k)}} \rangle.$$

Proof. If such integers $n(i)$ exist, then $G^{(1)} \subseteq P(G)$ and it follows that $P(G) = \Phi(G)$.

Suppose $P(G) = \Phi(G)$, and let $H = G/\theta(G)$. Then $\theta(H) = \langle 1 \rangle$, and $H = \langle y_1, y_2, \dots, y_k \rangle$ where y_i is the image of x_i under the homomorphism

mapping G onto $G/\theta(G)$. Since $\theta(H) = \langle 1 \rangle$, $P(H) = \langle y_1^p, y_2^p, \dots, y_k^p \rangle$, and $P(H) \subseteq Z(H)$. Also, $P(H) = \Phi(H) \supseteq H^{(1)}$, hence every element of $H^{(1)}$ can be expressed in the form $y_1^{p^u} y_2^{p^v} \dots y_k^{p^w}$ for suitable integers u, v, \dots, w . Since the y_i are independent generators of H modulo $H^{(1)}$, it follows that there exist integers n_1, n_2, \dots, n_k such that $H^{(1)} = \langle y_1^{pn_1}, y_2^{pn_2}, \dots, y_k^{pn_k} \rangle$. By Corollary 2.1, $\Phi(G^{(1)}) = \theta(G)$, thus $H^{(1)} = G^{(1)}/\theta(G) = G^{(1)}/\Phi(G^{(1)})$. Thus we can use the Burnside Basis Theorem [6, page 111] to obtain $G^{(1)} = \langle x_1^{pn_1}, x_2^{pn_2}, \dots, x_k^{pn_k} \rangle$. The proof follows if we let $n(i)$ be the largest positive integer n for which p^n divides pn_i .

COROLLARY 3.1. *Suppose $p \neq 2$ and $P(G) = \Phi(G)$. If G can be generated by k elements, then $G^{(r)}$ can be generated by k elements for $r = 1, 2, 3, \dots$.*

Proof. Follows immediately from Theorems 2 and 3.

LEMMA 2. *If $p \neq 2$ and $G/\Phi(G^{(1)})G_3$ is metacyclic, then*

$$\Phi(G^{(1)})G_3 = \theta(G).$$

Proof. Since $p > 2$ it follows from Remark 1 that $\theta(G) \subseteq P(G^{(1)})G_3$ and hence $\theta(G) \subseteq \Phi(G^{(1)})G_3$. The lemma will follow if it is shown that $\Phi(G^{(1)})G_3 \subseteq \theta(G)$. We may assume $\theta(G) = \langle 1 \rangle$. Then, by Corollary 1.1, $\check{P}(G^{(1)}) = \langle 1 \rangle$, thus $\Phi(G^{(1)})G_3 = G_3$. If $G_3 \neq \langle 1 \rangle$ we may assume $G_3 = \langle z \rangle$, where z is an element of order p in $Z(G)$. Since G/G_3 is metacyclic, there exist elements a, b such that $G = \langle a, b \rangle$ and $G^{(1)}$ is generated modulo G_3 by a^{p^k} for some integer $k > 0$. By (1.2), a^{p^k} belongs to $Z(G)$. But then $G^{(1)} = \langle a^{p^k}, z \rangle \subseteq Z(G)$ and $G_3 = \langle 1 \rangle$.

Blackburn [1] showed that a p -group G is metacyclic if, and only if, $G/\Phi(G^{(1)})G_3$ is metacyclic. Our next theorem follows immediately from Lemma 2 and this result of Blackburn. We shall give a simple direct proof of Theorem 4, and obtain Blackburn's result for $p > 2$ as Corollary 4.2.

THEOREM 4. *Suppose $p > 2$. Then G is metacyclic if, and only if, $G/\theta(G)$ is metacyclic.*

Proof. Since any factor group of a metacyclic group is again metacyclic, we need only show that $G/\theta(G)$ metacyclic implies G is metacyclic.

Suppose G is a non-metacyclic group of minimal order for which $G/\theta(G)$ is metacyclic. Then $\theta(G) \neq \langle 1 \rangle$ and hence we can find an element z in $\theta(G)$ such that z has order p and belongs to $Z(G)$. If we let $H = G/\langle z \rangle$, then $H/\theta(H) = (G/\langle z \rangle)/(\theta(G)/\langle z \rangle) \cong G/\theta(G)$ is metacyclic, and

consequently H is itself metacyclic since H has smaller order than G . Thus we can find \bar{a}, \bar{b} in H such that $H = \langle \bar{a}, \bar{b} \rangle$ and $H^{(1)} = \langle \bar{a}^{p^k} \rangle$ for some $k > 0$. If we let a, b be coset representatives in G of \bar{a}, \bar{b} , then it follows from the Burnside Basis Theorem that $G = \langle a, b \rangle$ and hence $G^{(1)} = \langle a^{p^k}, z \rangle$. In particular, if we let $c = a^{-1}b^{-1}ab$, there exist integers, n and m such that $c = a^{np^k}z^m$. Since z belongs to $Z(G)$, it is clear that $a^{-1}c^{-1}ac = 1$, and

$$b^{-1}cb = b^{-1}a^{np^k}bz^m = (b^{-1}ab)^{np^k}z^m = (a^{1+np^k}z^m)^{np^k}z^m,$$

thus

$$c^{-1}b^{-1}cb = a^{n^2p^{2k}}z^{mnnp^k} = a^{n^2p^{2k}}$$

where the last equality follows from $z^p = 1$. Similarly, $b^{-1}a^{p^k}b = a^{p^k+np^{2k}}$. Thus G_3 , which is generated by $c^{-1}b^{-1}cb, a^{-1}c^{-1}ac$, and the various conjugates of these elements, is contained in $\langle a^{p^k} \rangle$. Since $P(G^{(1)}) \subseteq \langle a^{p^k} \rangle$, it follows from Remark 1 that $\theta(G) \subseteq \langle a^{p^k} \rangle$. But z belongs to $\theta(G)$, hence $G^{(1)} = \langle a^{p^k} \rangle$ and G is metacyclic.

REMARK 2. If $p = 2$, it follows from $\theta(G) = \langle 1 \rangle$ that $(xy)^2 = x^2y^2$ and hence $x^{-1}yxy^{-1} = 1$ for all x, y in G . Thus $\theta(G) = G^{(1)}$ and $G/\theta(G)$ is metacyclic whenever G can be generated by two elements. Since there exist non-metacyclic 2-groups having two generators we see that Theorem 4 is false for $p = 2$.

The following result was established by Huppert [5, Hauptsatz 1].

COROLLARY 4.1. *Suppose $p \neq 2$ and G can be generated by two elements. Then G is metacyclic if, and only if, $P(G) = \Phi(G)$.*

Proof. It is clear that $P(G) = \Phi(G)$ if G is metacyclic. Suppose $P(G) = \Phi(G)$. Since G can be generated by two elements, $G^{(1)}$ is cyclic modulo G_3 [3, Theorem 2.81]. We see from Theorem 3 that, if $G = \langle a, b \rangle$, then $G^{(1)} = \langle a^{p^n}, b^{p^m} \rangle$ for some integers m and n . It follows that one of a^{p^n}, b^{p^m} is mapped on a generator of $G^{(1)}/G_3$ by the natural homomorphism. Thus G/G_3 is metacyclic. By Corollary 2.1, $\theta(G) \supseteq G_3$, hence $G/\theta(G)$ is metacyclic. It follows from Theorem 4 that G is metacyclic.

The next corollary is an immediate consequence of Lemma 2 and Theorem 4.

COROLLARY 4.2. *If $p \neq 2$, then G is metacyclic if, and only if, $G/\Phi(G^{(1)})G_3$ is metacyclic.*

REMARK 3. We define $\theta_1(G) = \theta(G)$ and $\theta_n(G) = \theta(\theta_{n-1}(G))$ for $n > 1$. The series $\theta_1(G) \supset \theta_2(G) \supset \dots \supset \theta_\infty(G) = \langle 1 \rangle$ can be considered a generalization of the derived series of G . Corresponding generalizations of the

ascending and descending central series of G can be obtained as follows: let $\Gamma_1(G)$ be the subgroup of G generated by the set of all x in G such that $(xy)^p = x^p y^p$ for every element y of G , and define $\Gamma_n(G)$ for $n > 1$ as the subgroup of G mapped onto $\Gamma_1(G/\Gamma_{n-1}(G))$ by the natural homomorphism; let $\Psi_1(G) = G$, and $\Psi_n(G) = [G, \Psi_{n-1}(G)]$ for $n > 1$. These series have an important property in common with the ascending and descending central series. Namely, if we define the lengths $l(\Gamma)$ and $l(\Psi)$ of the Γ and Ψ series as, respectively, the smallest integers m and n for which $\Gamma_m(G) = G$ and $\Psi_{n+1}(G) = \langle 1 \rangle$, it is easy to see that $l(\Gamma) = l(\Psi)$.

The group $\Gamma_1(G)$ has been studied by Grun [2]. The groups $\theta_n(G)$ and $\Psi_m(G)$ have not appeared in the literature, however the following result is an immediate consequence of earlier work [4, Remark 1].

THEOREM 5. *A non-abelian group with cyclic center cannot be one of the subgroups $\theta_n(G)$ or $\Psi_m(G)$ (for $m > 1$) of a p -group G .*

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Glen Earl Baxter, <i>An analytic problem whose solution follows from a simple algebraic identity</i>	731
Leonard D. Berkovitz and Melvin Dresher, <i>A multimove infinite game with linear payoff</i>	743
Earl Robert Berkson, <i>Sequel to a paper of A. E. Taylor</i>	767
Gerald Berman and Robert Jerome Silverman, <i>Embedding of algebraic systems</i>	777
Peter Crawley, <i>Lattices whose congruences form a boolean algebra</i>	787
Robert E. Edwards, <i>Integral bases in inductive limit spaces</i>	797
Daniel T. Finkbeiner, II, <i>Irreducible congruence relations on lattices</i>	813
William James Firey, <i>Isoperimetric ratios of Reuleaux polygons</i>	823
Delbert Ray Fulkerson, <i>Zero-one matrices with zero trace</i>	831
Leon W. Green, <i>A sphere characterization related to Blaschke's conjecture</i>	837
Israel (Yitzchak) Nathan Herstein and Erwin Kleinfeld, <i>Lie mappings in characteristic 2</i>	843
Charles Ray Hobby, <i>A characteristic subgroup of a p-group</i>	853
R. K. Juberg, <i>On the Dirichlet problem for certain higher order parabolic equations</i>	859
Melvin Katz, <i>Infinitely repeatable games</i>	879
Emma Lehmer, <i>On Jacobi functions</i>	887
D. H. Lehmer, <i>Power character matrices</i>	895
Henry B. Mann, <i>A refinement of the fundamental theorem on the density of the sum of two sets of integers</i>	909
Marvin David Marcus and Roy Westwick, <i>Linear maps on skew symmetric matrices: the invariance of elementary symmetric functions</i>	917
Richard Dean Mayer and Richard Scott Pierce, <i>Boolean algebras with ordered bases</i>	925
Trevor James McMinn, <i>On the line segments of a convex surface in E_3</i>	943
Frank Albert Raymond, <i>The end point compactification of manifolds</i>	947
Edgar Reich and S. E. Warschawski, <i>On canonical conformal maps of regions of arbitrary connectivity</i>	965
Marvin Rosenblum, <i>The absolute continuity of Toeplitz's matrices</i>	987
Lee Albert Rubel, <i>Maximal means and Tauberian theorems</i>	997
Helmut Heinrich Schaefer, <i>Some spectral properties of positive linear operators</i>	1009
Jeremiah Milton Stark, <i>Minimum problems in the theory of pseudo-conformal transformations and their application to estimation of the curvature of the invariant metric</i>	1021
Robert Steinberg, <i>The simplicity of certain groups</i>	1039
Hisahiro Tamano, <i>On paracompactness</i>	1043
Angus E. Taylor, <i>Mittag-Leffler expansions and spectral theory</i>	1049
Marion Franklin Tinsley, <i>Permanents of cyclic matrices</i>	1067
Charles J. Titus, <i>A theory of normal curves and some applications</i>	1083
Charles R. B. Wright, <i>On groups of exponent four with generators of order two</i>	1097