

Pacific Journal of Mathematics

ON JACOBI FUNCTIONS

EMMA LEHMER

ON JACOBI FUNCTIONS

EMMA LEHMER

The Jacobi functions¹ R_m are usually defined by

$$(1) \quad R_m = R_m^{(k)}(\alpha) = \sum_{s=1}^{p-2} \alpha^{\text{ind } s - (m+1)\text{ind } (s+1)}$$

where $\alpha = e^{2\pi i/k}$ and $\text{ind } s = \text{ind}_g s$ is taken with respect to some primitive root g of a prime $p = kn + 1$. Therefore R_m depends in general on the choice of primitive root and all the explicit results which have been given for special cases, as in [1], [2], [7] and others contain ambiguities of sign due to this indeterminacy. In a recent work on power character matrices [4] it became necessary to make the known results more explicit and to obtain some new ones. It is the purpose of this note to give explicit results in case 2 is *not* a k th power residue of p for $k = 3, 4, 5$ and 6 and for all m . The case in which 2 is a k th power residue of p still remains ambiguous.

We find it more convenient to use the character notation

$$(2) \quad \chi(h) = \chi_k(h) = \begin{cases} \alpha^{\text{ind } h} & \text{if } (h, p) = 1 \\ 0 & \text{otherwise} \end{cases}$$

in order to make use of all the multiplicative properties of the characters. In this notation R_m becomes

$$(3) \quad R_m = \sum_{s=1}^{p-2} \chi(s)[\chi(s+1)]^{-m-1}.$$

The following relations are well-known and can be easily derived from the definition as in [4].

$$(4) \quad R_m = \chi(-1)R_{k-1-m} = \chi(-1) \sum_{s=1}^{p-2} \chi(s)\chi^m(s+1)$$

$$(5) \quad R_{k-1} = -\chi(-1) = (-1)^{n+1}.$$

We shall need three other relations which we proceed to prove.

LEMMA 1. *If k is odd, then*

$$(6) \quad R_\lambda(\alpha) = R_1(\alpha^\lambda) \text{ for } k = 2\lambda + 1$$

Proof. Let $s\bar{s} \equiv 1 \pmod{p}$ Then

¹ Received November 17, 1959. The notation R_m is used here as in [2] instead of Jacobi's original ψ as in [1] and [4] to avoid conflict with Jacobsthal's ψ .

$$\begin{aligned}
 R_\lambda(\alpha) &= \sum_{s=1}^{p-2} \chi(s)\chi^\lambda(s+1) = \sum_{s=1}^{p-2} \chi^{k-1}(\bar{s})\chi^\lambda(s+1) \\
 &= \sum_{s=1}^{p-2} \chi^\lambda(\bar{s})\chi^\lambda(\bar{s}+1) = R_1(\alpha^\lambda) .
 \end{aligned}$$

LEMMA 2. *If k is even, then*

$$(7) \quad R_\mu = \chi(4)R_1, \text{ where } k = 2\mu .$$

Proof. Using (4)

$$\begin{aligned}
 R_\mu &= \chi(-1) \sum_{s=1}^{p-2} \chi(s)\chi^\mu(s+1) = \chi(-1) \sum_{s=1}^{p-2} \chi(s)\chi_2(s+1) \\
 &= \chi(-1) \left[\sum_{s=1}^{p-2} \chi(s)[1 + \chi_2(s+1)] - \sum_{s=1}^{p-2} \chi(s) \right] .
 \end{aligned}$$

If $s+1$ is not a square, then the expression in the square brackets vanishes. Letting $s+1 = t^2$ we obtain

$$R_\mu = \chi(-1) \left[\sum_{t=1}^{p-1} \chi(t^2 - 1) + \chi(-1) \right] = \chi(-1) \sum_{t=0}^{p-1} \chi(t^2 - 1) .$$

Now let $t = 2s + 1$, then

$$R_\mu = \chi(-4) \sum_{s=1}^{p-2} \chi(s)\chi(s+1) = \chi(4)R_1 .$$

LEMMA 3. *If k is oddly even, then*

$$(8) \quad R_{2\nu}^{(k)}(\alpha) = \chi_{2\nu+1}(4)R_\nu^{(2\nu+1)}(\beta) \text{ where } k = 4\nu + 2, \text{ and } \beta = \alpha^2 .$$

Proof.

$$\begin{aligned}
 R_{2\nu}^{(k)}(\alpha) &= \sum_{s=1}^{p-2} \chi(s)\chi^{2\nu+1}(s+1) = \sum_{s=1}^{p-2} \chi(\bar{s})\chi^{2\nu+1}(\bar{s}+1) \\
 &= \sum_{s=1}^{p-2} \chi^{4\nu+1}(s)\chi^{2\nu+2}(\bar{s}+1) = \sum_{s=1}^{p-2} \chi^{2\nu}(s)\chi^{2\nu+1}(s+1) \\
 &= \sum_{s=1}^{p-2} \chi_{2\nu+1}^\nu(s)\chi_2(s+1) \\
 &= \sum_{s=1}^{p-2} \chi_{2\nu+1}^\nu(s)[1 + \chi_2(s+1)] - \sum_{s=1}^{p-2} \chi_{2\nu+1}^\nu(s) .
 \end{aligned}$$

Letting $s+1 = t^2$ as before:

$$\begin{aligned}
 R_{2\nu}^{(k)}(\alpha) &= \sum_{t=0}^{p-2} \chi_{3\nu+1}^\nu(t^2 - 1) = \chi_{2\nu+1}(4) \sum_{s=1}^{p-2} \chi_{2\nu+1}^\nu(s)\chi_{2\nu+1}^\nu(s+1) \\
 &= \chi_{2\nu+1}(4)R_1^{(2\nu+1)}(\beta^\nu) \\
 &= \chi_{2\nu+1}(4)R_\nu^{(2\nu+1)}(\beta)
 \end{aligned}$$

by Lemma 1.

But by (4) and Lemma 2:

$$R_{2\nu+1} = \chi(-1)R_{2\nu} = \chi_k(4)R_1, \quad k = 4\nu + 2.$$

Hence by Lemma 3:

$$(9) \quad R_1^{(k)}(\alpha) = \chi_k(-4)R_\nu^{2\nu+1}(\beta) = \chi_k(-4)R_1^{2\nu+1}(\beta^\nu)$$

Armed with these relations we can express all the Jacobi functions for $k = 3, 4, 5$ and 6 in terms of the corresponding R_1 as follows.

$$k = 3, R_2 = -1$$

$$k = 4, R_3 = -\chi(-1), R_2 = \chi(-1)R_1 \text{ by (3)}$$

$$k = 5, R_4 = -1, R_3 = R_1 \text{ by (3) and } R_2 = R_1(\alpha^\nu) \text{ by Lemma 1.}$$

$$k = 6, R_5 = -\chi(-1). R_4 = \chi(-1)R_1 \text{ and } R_3 = \chi(-1)R_2 \text{ by (3).}$$

By Lemma 2, however, $R_3 = \chi(4)R_1$ and hence $R_2 = \chi(-4)R_1$. Moreover by (9) $R_1^{(6)} = \chi(-4)R_1^{(3)}$ so that it is sufficient to determine R_1 for $k = 3$ in order to determine all the R 's for $k = 3$ and $k = 6$.

We now proceed to expand R_1 in powers of α . If we write

$$R_1 = \chi(-1) \sum_{s=1}^{p-2} \chi(s)\chi(s+1) = \chi(-1) \sum_{\nu=0}^{k-1} a_\nu \alpha^\nu$$

then a_ν is the number of solutions of

$$s^2 + s = g^{kt+\nu} \quad (t = 0, 1, \dots, n-1)$$

and is given by

$$a_\nu = \sum_{\nu=0}^{k-1} \sum_{t=0}^{n-1} [1 + \chi_2(1 + 4g^{kt+\nu})].$$

Hence

$$\begin{aligned} R_1 &= \chi(-1) \sum_{\nu=0}^{k-1} \sum_{t=0}^{n-1} \chi_2(1 + 4g^{kt+\nu})\alpha^\nu \\ &= \frac{\chi(-1)}{k} \sum_{\nu=0}^{k-1} \sum_{x=1}^{p-1} \chi_2(1 + 4x^k g^\nu)\alpha^\nu \\ &= \frac{\chi(-1)}{k} \sum_{\nu=0}^{k-1} \alpha^\nu \sum_{x=0}^{p-1} \chi_2(4g^\nu)\chi_2(x^k + (\overline{4g^\nu})) \\ &= \frac{\chi(-1)}{k} \sum_{\nu=0}^{k-1} \chi_2(4g^\nu)\psi_k(\overline{4g^\nu}) \end{aligned}$$

where [5]

$$\psi_k(D) = \sum_{x=1}^{p-1} \chi_2(x^k + D) = \begin{cases} \left(\frac{D}{P}\right)\psi_k(\overline{D}) & \text{if } k \text{ is even} \\ \left(\frac{D}{P}\right)\varphi_k(\overline{D}) & \text{if } k \text{ is odd} \end{cases}$$

and

$$\varphi_k(D) = \sum_{x=1}^{p-1} \chi_2(x)\chi_2(x^k + D) = -\left(\frac{D}{P}\right)\varphi_k(\bar{D}), \quad k \text{ even}$$

is the well-known Jacobsthal [3] function. Hence

$$(10) \quad R_1 = \begin{cases} \frac{\chi(-1)}{k} \sum_{\nu=0}^{k-1} \psi_k(4g^\nu)\alpha^\nu & \text{if } k \text{ is even} \\ \frac{1}{k} \sum_{\nu=0}^{k-1} \varphi_k(4g^\nu)\alpha^\nu & \text{if } k \text{ is odd.} \end{cases}$$

Making use of the relations [5]

$$(11) \quad \varphi_k(m^k D) = \chi_2^{k+1}(m)\varphi_k(D)$$

$$(12) \quad \psi_k(m^k D) = \chi_2^k(m)\psi_k(D)$$

and

$$(13) \quad \psi_{2k}(D) = \psi_k(D) + \varphi_k(D)$$

we have for k even, substituting (13) into (10)

$$R_1 = \frac{\chi(-1)}{k} \left[\sum_{\nu=0}^{k-1} \psi_{k/2}(4g^\nu)\alpha^\nu + \sum_{\nu=0}^{k-1} \varphi_{k/2}(4g^\nu)\alpha^\nu \right].$$

By (11) and (12)

$$(14) \quad R_1 = \begin{cases} \frac{2\chi(-1)}{k} \sum_{\nu=0}^{k-1} \psi_{k/2}(4g^\nu)\alpha^\nu & \text{if } k/2 \text{ is odd} \\ \frac{2\chi(-1)}{k} \sum_{\nu=0}^{k-1} \varphi_{k/2}(4g^\nu)\alpha^\nu & \text{if } k/2 \text{ is even.} \end{cases}$$

Since the functions φ and ψ have been unequivocally determined by us in [5] and [6] for $k = 3, 4, 5$ and 6 in case 2 is not a k th power residue we can apply these results directly to the determination of the corresponding R_1 . For $k = 3$ let $p = A^2 + 3B^2 = 3n + 1, A \equiv B \equiv 1 \pmod{3}$.

By (10)

$$R_1 = \frac{1}{3} [\varphi_3(4) + \omega\varphi_3(4g) + \omega^2\varphi_3(4g^2)].$$

By [6]

$$\varphi_3(D) = \begin{cases} -(2A + 1) & \text{if } D \equiv u^3 \pmod{p} \\ A - 3B - 1 & \text{if } D \equiv 2u^3 \pmod{p} \\ A + 3B - 1 & \text{if } D \equiv 4u^3 \pmod{p}. \end{cases}$$

Hence

$$R_1 = \begin{cases} \frac{1}{3} [(A + 3B - 1) - (2A + 1)\omega + (A - 3B - 1)\omega^2] & \text{if } \text{ind } 2 \equiv 1 \pmod{3} \\ \frac{1}{3} [(A + 3B - 1) - (A - 3B - 1)\omega - (2A + 1)\omega^2] & \text{if } \text{ind } 2 \equiv 2 \pmod{3} \end{cases}$$

or

$$R_1 = \begin{cases} 2B + (B - A)\omega & \text{if } \text{ind } 2 \equiv 1(3) \text{ or if } \chi_3(2) = \omega \\ 2B + (B - A)\omega^2 & \text{if } \text{ind } 2 \equiv 2(3) \text{ or if } \chi_3(2) = \omega^2. \end{cases}$$

Hence if $\chi(2) \neq 1$, then

$$(15) \quad R_1 = 2B + (B - A)\chi_3(2), \quad A \equiv B \equiv 1 \pmod{3}.$$

If 2 is a cubic residue, $B \equiv 0 \pmod{3}$ and the sign of B is not determined. However

$$\begin{aligned} R_1 &= \frac{1}{3} [\varphi_3(1) + \varphi_3(g)\omega + \varphi_3(g^2)\omega^2] \\ &= \frac{1}{3} [-(2A + 1) + (A \pm 3B - 1)\omega + (A \mp 3B - 1)\omega^2] \\ &= -A \pm B(\omega - \omega^2) = (-A \pm B) \pm 2B\omega. \end{aligned}$$

For $k = 4, p = a^2 + b^2 = 4n + 1, a \equiv 1 \pmod{4}$ we obtain from (14)

$$R_1 = \frac{\chi_4(-1)}{2} [\varphi_2(4) + i\varphi_2(4g)].$$

We know that² [5]

$$\begin{aligned} \varphi_2(u^2) &= -\chi_2(u)2a \\ \varphi_2(2u^2) &= -\chi_2(u)2b \text{ if } \chi_2(2) = -1, [b/2 \equiv 1 \pmod{4}] \\ \varphi_2(\sqrt{2}u^2) &= -\chi_2(u)2b \text{ if } \chi_2(2) = +1, [b/4 \equiv (-1)^{n/2} \pmod{4}]. \end{aligned}$$

If $\chi_2(2) = -1$, then $\chi_4(-1) = -1$, and $\text{ind } 2 \equiv 1 \text{ or } 3 \pmod{4}$ so that

$$R_1 = \begin{cases} -(a + ib) & \text{if } \text{ind } 2 \equiv 1 \pmod{4} \\ -(a - ib) & \text{if } \text{ind } 2 \equiv 3 \pmod{4} \end{cases}$$

or

$$(16) \quad R_1 = -[a + b\chi_4(2)] \text{ if } \chi_2(2) = -1, [b/2 \equiv 1 \pmod{4}].$$

² There is a misprint in the corresponding formula (13) in [6] for $b/4 \equiv (-1)^n$ read $b/4 \equiv (-1)^{n/2}$. The same mistake is repeated four lines down.

If $\chi_2(2) = +1$, then $\chi_4(-1) = +1$. But $\chi_4(2) = -1$ and $\text{ind } \sqrt{2} \equiv 1$ or $3 \pmod{4}$. Hence

$$R_1 = \begin{cases} -a - bi & \text{if } \text{ind } \sqrt{2} \equiv 1 \pmod{4} \\ -a + bi & \text{if } \text{ind } \sqrt{2} \equiv 3 \pmod{4} \end{cases}$$

or

$$(17) \quad R_1 = -[a + b\chi_4(\sqrt{2})] \text{ if } \chi_2(2) = 1, [b/4 \equiv (-1)^{b/2} \pmod{4}].$$

If $\chi_4(2) = +1$, then $\chi_4(-1) = +1$, and

$$R_1 = -a \pm bi$$

but the sign of b remains undetermined.

For $k = 5$, we have by (10)

$$R_1 = \frac{1}{5} [\varphi_5(4) + \alpha\varphi_5(4g) + \alpha^2\varphi_5(4g^2) + \alpha^3\varphi_5(4g^3) + \alpha^4\varphi_5(4g^4)]$$

The φ 's have been determined previously [6] in terms of the partition

$$\begin{cases} 16p = x^2 + 50u^2 + 50v^2 + 125w^2 \\ xw = v^2 - u^2 - 4uv, x \equiv 1 \pmod{5} \end{cases}$$

to read

$$\begin{aligned} \varphi_5(4) &= x - 1 \\ \varphi_5(4g) &= \frac{1}{4} [-4 - x + 25w + 10(u + 2v)] \\ \varphi_5(4g^2) &= \frac{1}{4} [-4 - x - 25w + 10(2u - v)] \\ \varphi_5(4g^3) &= \frac{1}{4} [-4 - x - 25w - 10(2u - v)] \\ \varphi_5(4g^4) &= \frac{1}{4} [-4 - x + 25w - 10(u + 2v)]. \end{aligned}$$

This gives

$$\begin{aligned} R_1 &= \frac{1}{4} [x + \alpha(5w + 2u + 4v) + \alpha^2(-5w + 4u - 2v) \\ &\quad + \alpha^3(-5w - 4u + 2v) + \alpha^4(5w - 2u - 4v)]. \end{aligned}$$

In a previous paper [6] we have determined (x, u, v, w) uniquely in case $\text{ind } 2 \equiv 1 \pmod{5}$ by selecting u even and $v \equiv x + u \pmod{4}$. If $\text{ind } 2 \equiv m \pmod{5}$, the coefficient of $\alpha^{m\nu}$ becomes $\varphi(4g^\nu)$ or the coefficient of α^ν is $\varphi(4g^{\bar{m}\nu})$. This transformation is achieved if the solution:

$$(x, u, v, w) \text{ is replaced by } \begin{cases} (x, v, -u, -w) \text{ ind } 2 \equiv 2 \pmod{5} \\ (x, -v, u, -w) \text{ ind } 2 \equiv 3 \pmod{5} \\ (x, -u, -v, w) \text{ ind } 3 \equiv 4 \pmod{5} . \end{cases}$$

As before, if $\text{ind } 2 \equiv 0 \pmod{5}$, the indeterminacy remains.

REFERENCES

1. P. Bachmann, *Die Lehre von der Kreistheilung* (Leipzig, 1872).
2. L. E. Dickson, *Cyclotomy, higher congruences and Waring's problem*, Amer. Math. **57** (1935), 391-424.
3. E. Jacobsthal, *Anwendungen einer Formel aus der Theorie der quadratischen Reste*, Dissertation (Berlin 1906).
4. D. H. Lehmer, *Power Character Matrices*, Pacific J. Math. **10** (1960), pp. 895-907.
5. Emma Lehmer, *On the number of solutions of $u^k + D \equiv w^2 \pmod{p}$* , Pacific J. Math. **5** (1955), 103-118.
6. ———, *On Euler's criterion*, The Journ. of the Australian Math. Soc. **1** (1959), 64-70.
7. A. L. Whiteman, *The sixteenth power residue character of 2*, Canadian J. Math. **6** (1954), 364-373.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG

Stanford University
Stanford, California

F. H. BROWNELL

University of Washington
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California
Los Angeles 7, California

L. J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

E. HEWITT

A. HORN

L. NACHBIN

M. OHTSUKA

H. L. ROYDEN

M. M. SCHIFFER

E. SPANIER

E. G. STRAUS

F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 10, No. 3

November, 1960

Glen Earl Baxter, <i>An analytic problem whose solution follows from a simple algebraic identity</i>	731
Leonard D. Berkovitz and Melvin Dresher, <i>A multimove infinite game with linear payoff</i>	743
Earl Robert Berkson, <i>Sequel to a paper of A. E. Taylor</i>	767
Gerald Berman and Robert Jerome Silverman, <i>Embedding of algebraic systems</i>	777
Peter Crawley, <i>Lattices whose congruences form a boolean algebra</i>	787
Robert E. Edwards, <i>Integral bases in inductive limit spaces</i>	797
Daniel T. Finkbeiner, II, <i>Irreducible congruence relations on lattices</i>	813
William James Firey, <i>Isoperimetric ratios of Reuleaux polygons</i>	823
Delbert Ray Fulkerson, <i>Zero-one matrices with zero trace</i>	831
Leon W. Green, <i>A sphere characterization related to Blaschke's conjecture</i>	837
Israel (Yitzchak) Nathan Herstein and Erwin Kleinfeld, <i>Lie mappings in characteristic 2</i>	843
Charles Ray Hobby, <i>A characteristic subgroup of a p-group</i>	853
R. K. Juberg, <i>On the Dirichlet problem for certain higher order parabolic equations</i>	859
Melvin Katz, <i>Infinitely repeatable games</i>	879
Emma Lehmer, <i>On Jacobi functions</i>	887
D. H. Lehmer, <i>Power character matrices</i>	895
Henry B. Mann, <i>A refinement of the fundamental theorem on the density of the sum of two sets of integers</i>	909
Marvin David Marcus and Roy Westwick, <i>Linear maps on skew symmetric matrices: the invariance of elementary symmetric functions</i>	917
Richard Dean Mayer and Richard Scott Pierce, <i>Boolean algebras with ordered bases</i>	925
Trevor James McMinn, <i>On the line segments of a convex surface in E_3</i>	943
Frank Albert Raymond, <i>The end point compactification of manifolds</i>	947
Edgar Reich and S. E. Warschawski, <i>On canonical conformal maps of regions of arbitrary connectivity</i>	965
Marvin Rosenblum, <i>The absolute continuity of Toeplitz's matrices</i>	987
Lee Albert Rubel, <i>Maximal means and Tauberian theorems</i>	997
Helmut Heinrich Schaefer, <i>Some spectral properties of positive linear operators</i>	1009
Jeremiah Milton Stark, <i>Minimum problems in the theory of pseudo-conformal transformations and their application to estimation of the curvature of the invariant metric</i>	1021
Robert Steinberg, <i>The simplicity of certain groups</i>	1039
Hisahiro Tamano, <i>On paracompactness</i>	1043
Angus E. Taylor, <i>Mittag-Leffler expansions and spectral theory</i>	1049
Marion Franklin Tinsley, <i>Permanents of cyclic matrices</i>	1067
Charles J. Titus, <i>A theory of normal curves and some applications</i>	1083
Charles R. B. Wright, <i>On groups of exponent four with generators of order two</i>	1097