Pacific Journal of Mathematics

ON JACOBI FUNCTIONS

EMMA LEHMER

Vol. 10, No. 3 November 1960

ON JACOBI FUNCTIONS

EMMA LEHMER

The Jacobi functions[1] R_m are usually defined by

$$R_m = R_m^{(k)}(\alpha) = \sum_{s=1}^{p-2} \alpha^{\text{ind } s - (m+1)\text{ind }(s+1)} \tag{1}$$

where $\alpha = e^{2\pi i/k}$ and ind $s = \text{ind}_g s$ is taken with respect to some primitive root g of a prime $p = kn + 1$. Therefore R_m depends in general on the choice of primitive root and all the explicit results which have been given for special cases, as in [**1**], [**2**], [**7**] and others contain ambiguities of sign due to this indeterminancy. In a recent work on power character matrices [**4**] it became necessary to make the known results more explicit and to obtain some new ones. It is the purpose of this note to give explicit results in case 2 is *not* a kth power residue of p for $k = 3, 4, 5$ and 6 and for all m. The case in which 2 is a kth power residue of p still remains ambiguous.

We find it more convenient to use the character notation

$$\chi(h) = \chi_k(h) = \begin{cases} \alpha^{\text{ind } h} & \text{if } (h, p) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

in order to make use of all the multiplicative properties of the characters. In this notation R_m becomes

$$R_m = \sum_{s=1}^{p-2} \chi(s)[\chi(s+1)]^{-m-1} . \tag{3}$$

The following relations are well-known and can be easily derived from the definition as in [**4**].

$$R_m = \chi(-1)R_{k-1-m} = \chi(-1) \sum_{s=1}^{p-2} \chi(s)\chi^m(s+1) \tag{4}$$

$$R_{k-1} = -\chi(-1) = (-1)^{n+1} . \tag{5}$$

We shall need three other relations which we proceed to prove.

LEMMA 1. *If k is odd, then*

$$R_\lambda(\alpha) = R_1(\alpha^\lambda) \text{ for } k = 2\lambda + 1 \tag{6}$$

Proof. Let $s\bar{s} \equiv 1 \pmod{p}$ Then

[1] Received November 17, 1959. The notation R_m is used here as in [**2**] instead of Jacobi's original ψ as in [**1**] and [**4**] to avoid conflict with Jacobsthal's ψ.

$$R_\lambda(\alpha) = \sum_{s=1}^{p-2} \chi(s)\chi^\lambda(s+1) = \sum_{s=1}^{p-2} \chi^{k-1}(\bar{s})\chi^\lambda(s+1)$$
$$= \sum_{s=1}^{p-2} \chi^\lambda(\bar{s})\chi^\lambda(\bar{s}+1) = R_1(\alpha^\lambda) .$$

LEMMA 2. *If k is even, then*

(7) $$R_\mu = \chi(4)R_1, \text{ where } k = 2\mu .$$

Proof. Using (4)

$$R_\mu = \chi(-1)\sum_{s=1}^{p-2} \chi(s)\chi^\mu(s+1) = \chi(-1)\sum_{s=1}^{p-2} \chi(s)\chi_2(s+1)$$
$$= \chi(-1)\left[\sum_{s=1}^{p-2} \chi(s)[1+\chi_2(s+1)] - \sum_{s=1}^{p-2}\chi(s)\right] .$$

If $s+1$ is not a square, then the expression in the square brackets vanishes. Letting $s+1 = t^2$ we obtain

$$R_\mu = \chi(-1)\left[\sum_{t=1}^{p-1} \chi(t^2-1) + \chi(-1)\right] = \chi(-1)\sum_{t=0}^{p-1}\chi(t^2-1) .$$

Now let $t = 2s+1$, then

$$R_\mu = \chi(-4)\sum_{s=1}^{p-2}\chi(s)\chi(s+1) = \chi(4)R_1 .$$

LEMMA 3. *If k is oddly even, then*

(8) $$R_{2\nu}^{(k)}(\alpha) = \chi_{2\nu+1}(4)R_\nu^{(2\nu+1)}(\beta) \text{ where } k = 4\nu+2, \text{ and } \beta = \alpha^2.$$

Proof.

$$R_{2\nu}^{(k)}(\alpha) = \sum_{s=1}^{p-2}\chi(s)\chi^{2\nu+1}(s+1) = \sum_{s=1}^{p-2}\chi(\bar{s})\chi^{2\nu+1}(\bar{s}+1)$$
$$= \sum_{s=1}^{p-2}\chi^{4\nu+1}(s)\chi^{2\nu+2}(\bar{s}+1) = \sum_{s=1}^{p-2}\chi^{2\nu}(s)\chi^{2\nu+1}(s+1)$$
$$= \sum_{s=1}^{p-2}\chi_{2\nu+1}^{\nu}(s)\chi_2(s+1)$$
$$= \sum_{s=1}^{p-2}\chi_{2\nu+1}^{\nu}(s)[1+\chi_2(s+1)] - \sum_{s=1}^{p-2}\chi_{2\nu+1}^{\nu}(s) .$$

Letting $s+1 = t^2$ as before:

$$R_{2\nu}^{(k)}(\alpha) = \sum_{t=0}^{p-2}\chi_{3\nu+1}^{\nu}(t^2-1) = \chi_{2\nu+1}(4)\sum_{s=1}^{p-2}\chi_{2\nu+1}^{\nu}(s)\chi_{2\nu+1}^{\nu}(s+1)$$
$$= \chi_{2\nu+1}(4)R_1^{(2\nu+1)}(\beta^\nu)$$
$$= \chi_{2\nu+1}(4)R_\nu^{(2\nu+1)}(\beta)$$

by Lemma 1.
But by (4) and Lemma 2:

$$R_{2\nu+1} = \chi(-1)R_{2\nu} = \chi_k(4)R_1 , \qquad k = 4\nu + 2 .$$

Hence by Lemma 3:

$$R_1^{(k)}(\alpha) = \chi_k(-4)R_\nu^{2\nu+1}(\beta) = \chi_k(-4)R_1^{2\nu+1}(\beta^\nu) \tag{9}$$

Armed with these relations we can express all the Jacobi functions for $k = 3, 4, 5$ and 6 in terms of the corresponding R_1 as follows.
$k = 3$, $R_2 = -1$
$k = 4$, $R_3 = -\chi(-1)$, $R_2 = \chi(-1)R_1$ by (3)
$k = 5$, $R_4 = -1$, $R_3 = R_1$ by (3) and $R_2 = R_1(\alpha^2)$ by Lemma 1.
$k = 6$, $R_5 = -\chi(-1)$. $R_4 = \chi(-1)R_1$ and $R_3 = \chi(-1)R_2$ by (3).
By Lemma 2, however, $R_3 = \chi(4)R_1$ and hence $R_2 = \chi(-4)R_1$. Moreover by (9) $R_1^{(6)} = \chi(-4)R_1^{(3)}$ so that it is sufficient to determine R_1 for $k = 3$ in order to determine all the R's for $k = 3$ and $k = 6$.

We now proceed to expand R_1 in powers of α. If we write

$$R_1 = \chi(-1)\sum_{s=1}^{p-2}\chi(s)\chi(s+1) = \chi(-1)\sum_{\nu=0}^{k-1}a_\nu\alpha^\nu$$

then a_ν is the number of solutions of

$$s^2 + s = g^{kt+\nu} \qquad (t = 0, 1, \cdots, n-1)$$

and is given by

$$a_\nu = \sum_{\nu=0}^{k-1}\sum_{t=0}^{n-1}[1 + \chi_2(1 + 4g^{kt+\nu})] .$$

Hence

$$\begin{aligned} R_1 &= \chi(-1)\sum_{\nu=0}^{k-1}\sum_{t=0}^{n-1}\chi_2(1 + 4g^{kt+\nu})\alpha^\nu \\ &= \frac{\chi(-1)}{k}\sum_{\nu=0}^{k-1}\sum_{x=1}^{p-1}\chi_2(1 + 4x^k g^\nu)\alpha^\nu \\ &= \frac{\chi(-1)}{k}\sum_{\nu=0}^{k-1}\alpha^\nu\sum_{x=0}^{p-1}\chi_2(4g^\nu)\chi_2(x^k + (\overline{4g^\nu})) \\ &= \frac{\chi(-1)}{k}\sum_{\nu=0}^{k-1}\chi_2(4g^\nu)\psi_k(\overline{4g^\nu}) \end{aligned}$$

where [5]

$$\psi_k(D) = \sum_{x=1}^{p-1}\chi_2(x^k + D) = \begin{cases} \left(\dfrac{D}{P}\right)\psi_k(\overline{D}) & \text{if } k \text{ is even} \\ \left(\dfrac{D}{P}\right)\varphi_k(\overline{D}) & \text{if } k \text{ is odd} \end{cases}$$

and

$$\varphi_k(D) = \sum_{x=1}^{p-1} \chi_2(x)\chi_2(x^k + D) = -\left(\frac{D}{P}\right)\varphi_k(\overline{D}),\ k \text{ even}$$

is the well-known Jacobsthal [3] function. Hence

$$(10) \qquad R_1 = \begin{cases} \dfrac{\chi(-1)}{k} \sum_{\nu=0}^{k-1} \psi_k(4g^\nu)\alpha^\nu & \text{if } k \text{ is even} \\ \dfrac{1}{k} \sum_{\nu=0}^{k-1} \varphi_k(4g^\nu)\alpha^\nu & \text{if } k \text{ is odd} . \end{cases}$$

Making use of the relations [5]

$$(11) \qquad \varphi_k(m^k D) = \chi_2^{k+1}(m)\varphi_k(D)$$

$$(12) \qquad \psi_k(m^k D) = \chi_2^k(m)\psi_k(D)$$

and

$$(13) \qquad \psi_{2k}(D) = \psi_k(D) + \varphi_k(D)$$

we have for k even, substituting (13) into (10)

$$R_1 = \frac{\chi(-1)}{k}\left[\sum_{\nu=0}^{k-1} \psi_{k/2}(4g^\nu)\alpha^\nu + \sum_{\nu=0}^{k-1} \varphi_{k/2}(4g^\nu)\alpha^\nu\right] .$$

By (11) and (12)

$$(14) \qquad R_1 = \begin{cases} \dfrac{2\chi(-1)}{k} \sum_{\nu=0}^{k-1} \psi_{k/2}(4g^\nu)\alpha^\nu & \text{if } k/2 \text{ is odd} \\ \dfrac{2\chi(-1)}{k} \sum^{k-1} \varphi_{k/2}(4g^\nu)\alpha^\nu & \text{if } k/2 \text{ is even} . \end{cases}$$

Since the functions φ and ψ have been unequivocally determined by us in [5] and [6] for $k = 3, 4, 5$ and 6 in case 2 is not a kth power residue we can apply these results directly to the determination of the corresponding R_1. For $k = 3$ let $p = A^2 + 3B^2 = 3n + 1$, $A \equiv B \equiv 1 \pmod 3$.
By (10)

$$R_1 = \frac{1}{3}\left[\varphi_3(4) + \omega\varphi_3(4g) + \omega^2\varphi_3(4g^2)\right] .$$

By [6]

$$\varphi_3(D) = \begin{cases} -(2A + 1) & \text{if } D \equiv u^3 \pmod p \\ A - 3B - 1 & \text{if } D \equiv 2u^3 \pmod p \\ A + 3B - 1 & \text{if } D \equiv 4u^3 \pmod p) . \end{cases}$$

Hence

$$R_1 = \begin{cases} \frac{1}{3}[(A+3B-1)-(2A+1)\omega+(A-3B-1)\omega^2] \text{ if ind } 2 \equiv 1 \pmod 3 \\ \frac{1}{3}[(A+3B-1)-(A-3B-1)\omega-(2A+1)\omega^2] \text{ if ind } 2 \equiv 2 \pmod 3 \end{cases}$$

or

$$R_1 = \begin{cases} 2B+(B-A)\omega \text{ if ind } 2 \equiv 1(3) \text{ or if } \chi_3(2)=\omega \\ 2B+(B-A)\omega^2 \text{ if ind } 2 \equiv 2(3) \text{ or if } \chi_3(2)=\omega^2 . \end{cases}$$

Hence if $\chi(2) \neq 1$, then

(15) $$R_1 = 2B + (B-A)\chi_3(2) , \qquad A \equiv B \equiv 1 \pmod 3 .$$

If 2 is a cubic residue, $B \equiv 0 \pmod 3$ and the sign of B is not determined. However

$$\begin{aligned} R_1 &= \frac{1}{3}[\varphi_3(1)+\varphi_3(g)\omega+\varphi_3(g^2)\omega^2] \\ &= \frac{1}{3}[-(2A+1)+(A \pm 3B-1)\omega+(A \mp 3B-1)\omega^2] \\ &= -A \pm B(\omega-\omega^2) = (-A \pm B) \pm 2B\omega . \end{aligned}$$

For $k=4$, $p=a^2+b^2=4n+1$, $a \equiv 1 \pmod 4$ we obtain from (14)

$$R_1 = \frac{\chi_4(-1)}{2}[\varphi_2(4)+i\varphi_2(4g)] .$$

We know that[2] [5]

$$\begin{aligned} \varphi_2(u^2) &= -\chi_2(u)2a \\ \varphi_2(2u^2) &= -\chi_2(u)2b \text{ if } \chi_2(2)=-1, [b/2 \equiv 1 \bmod 4)] \\ \varphi_2(\sqrt{2}u^2) &= -\chi_2(u)2b \text{ if } \chi_2(2)=+1, [b/4 \equiv (-1)^{n/2} \pmod 4)] . \end{aligned}$$

If $\chi_2(2)=-1$, then $\chi_4(-1)=-1$, and ind $2 \equiv 1$ or $3 \pmod 4$ so that

$$R_1 = \begin{cases} -(a+ib) \text{ if ind } 2 \equiv 1 \pmod 4 \\ -(a-ib) \text{ if ind } 2 \equiv 3 \pmod 4 \end{cases}$$

or

(16) $$R_1 = -[a+b\chi_4(2)] \text{ if } \chi_2(2)=-1, [b/2 \equiv 1 \pmod 4)] .$$

[2] There is a misprint in the corresponding formula (13) in [**6**] for $b/4 \equiv (-1)^n$ read $b/4 \equiv (-1)^{n/2}$. The same mistake is repeated four lines down.

If $\chi_2(2) = +1$, then $\chi_4(-1) = +1$. But $\chi_4(2) = -1$ and $\operatorname{ind}\sqrt{2} \equiv 1$ or 3 (mod 4). Hence

$$R_1 = \begin{cases} -a - bi \text{ if } \operatorname{ind}\sqrt{2} \equiv 1 \pmod 4 \\ -a + bi \text{ if } \operatorname{ind}\sqrt{2} \equiv 1 \pmod 4 \end{cases}$$

or

$$(17) \qquad R_1 = -[a + b\chi_4(\sqrt{2})] \text{ if } \chi_2(2) = 1, [b/4 \equiv (-1)^{b/2} \pmod 4] .$$

If $\chi_4(2) = +1$, then $\chi_4(-1) = +1$, and

$$R_1 = -a \pm bi$$

but the sign of b remains undetermined.
For $k = 5$, we have by (10)

$$R_1 = \frac{1}{5}[\varphi_5(4) + \alpha\varphi_5(4g) + \alpha^2\varphi_5(4g^2) + \alpha^3\varphi_5(4g^3) + \alpha^4\varphi_5(4g^4)]$$

The φ's have been determined previously [6] in terms of the partition

$$\begin{cases} 16p = x^2 + 50u^2 + 50v^2 + 125w^2 \\ xw = v^2 - u^2 - 4uv, \ x \equiv 1 \pmod 5 \end{cases}$$

to read

$$\varphi_5(4) = x - 1$$

$$\varphi_5(4g) = \frac{1}{4}[-4 - x + 25w + 10(u + 2v)]$$

$$\varphi_5(4g^2) = \frac{1}{4}[-4 - x - 25w + 10(2u - v)]$$

$$\varphi_5(4g^3) = \frac{1}{4}[-4 - x - 25w - 10(2u - v)]$$

$$\varphi_5(4g^4) = \frac{1}{4}[-4 - x + 25w - 10(u + 2v)] .$$

This gives

$$R_1 = \frac{1}{4}[x + \alpha(5w + 2u + 4v) + \alpha^2(-5w + 4u - 2v)$$
$$+ \alpha^3(-5w - 4u + 2v) + \alpha^4(5w - 2u - 4v)] .$$

In a previous paper [6] we have determined (x, u, v, w) uniquely in case $\operatorname{ind} 2 \equiv 1 \pmod 5$ by selecting u even and $v \equiv x + u \pmod 4$. If $\operatorname{ind} 2 \equiv m \pmod 5$, the coefficient of $\alpha^{m\nu}$ becomes $\varphi(4g^\nu)$ or the coefficient of α^ν is $\varphi(4g^{\bar{m}\nu})$. This transformation is achieved if the solution:

$$(x,\ u,\ v,\ w) \text{ is replaced by} \begin{cases} (x,\ v,\ -u,\ -w) & \operatorname{ind} 2 \equiv 2 \pmod 5 \\ (x,\ -v,\ u,\ -w) & \operatorname{ind} 2 \equiv 3 \pmod 5 \\ (x,\ -u,\ -v,\ w) & \operatorname{ind} 3 \equiv 4 \pmod 5 \,. \end{cases}$$

As before, if $\operatorname{ind} 2 \equiv 0 \pmod 5$, the indeterminancy remains.

REFERENCES

1. P. Bachmann, *Die Lehre von der Kreistheilung* (Leipzig, 1872).
2. L. E. Dickson, *Cyclotomy, higher congruences and Waring's problem*, Amer. Math. **57** (1935), 391–424.
3. E. Jacobsthal, *Anwendungen einer Formel aus der Theorie der quadratischen Reste*, Dissertation (Berlin 1906).
4. D. H. Lehmer, *Power Character Matrices*, Pacific J. Math. **10** (1960), pp. 895–907.
5. Emma Lehmer, *On the number of solutions of* $u^k + D \equiv w^2 \pmod p$, Pacific J. Math. **5** (1955), 103–118.
6. ———, *On Euler's criterion*, The Journ. of the Australian Math. Soc. **1** (1959), 64–70.
7. A. L. Whiteman, *The sixteenth power residue character of* 2, Canadian J. Math. **6** (1954), 364–373.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 10, No. 3 November, 1960