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**LINEAR MAPS ON SKEW SYMMETRIC MATRICES: THE  
INVARIANCE OF ELEMENTARY SYMMETRIC FUNCTIONS**

MARVIN DAVID MARCUS AND ROY WESTWICK

# LINEAR MAPS ON SKEW SYMMETRIC MATRICES: THE INVARIANCE OF ELEMENTARY SYMMETRIC FUNCTIONS

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**1. Introduction.** Let  $S_n$  be the space of  $n$ -square skew symmetric matrices over the field  $F$  of real numbers. Let  $E_{2k}(A)$  denote the sum of all  $2k$ -square principal subdeterminants of  $A \in S_n$  (the elementary symmetric function of degree  $2k$  of the eigenvalues of  $A$ ). It is classical that if  $U$  is an  $n$ -square real orthogonal matrix and  $A \in S_n$  then  $UAU' \in S_n$  and moreover for each  $k$

$$(1.1) \quad E_{2k}(UAU') = E_{2k}(A) .$$

The correspondence

$$(1.2) \quad A \rightarrow UAU'$$

for a fixed orthogonal  $U$  can then be regarded as a linear transformation on  $S_n$  onto itself that holds  $E_{2k}(A)$  invariant. The question we consider here is the following: to what extent does the fact that (1.1) holds for some  $k$  characterize the map (1.2). In other words, we obtain (Theorem 3) the complete structure of those linear maps  $T$  of  $S_n$  into itself that for some  $k > 1$  satisfy  $E_{2k}(T(A)) = E_{2k}(A)$  for each  $A \in S_n$ . Our results are made to depend on the structure of linear maps of the second Grassmann product space  $\Lambda^2 U$  of a vector space  $U$  over  $F$  into itself.

K. Morita [2] examined the structure of those maps  $T$  of  $S_n$  into itself that hold invariant the dominant singular value  $\alpha(A)$  of each  $A \in S_n$ . We recall that  $\alpha(A)$  is the largest eigenvalue of the non-negative Hermitian square root of  $A^*A$ . Morita shows that if  $\alpha(T(A)) = \alpha(A)$  for each  $A \in S_n$  then  $T$  has essentially the form given in our Theorem 3.

**2. Some definitions and preliminary results.** Let  $U$  be a finite dimensional vector space of dimension  $n$  over  $F$ . Let  $G_2(U)$  denote the space of all alternating bilinear functionals on the cartesian product  $U \times U$  to  $F$ . Then the dual space  $\Lambda^2 U$  of  $G_2(U)$  is called the second Grassmann product space of  $U$ . If  $x_1$  and  $x_2$  are any two vectors in  $U$  then  $f = x_1 \wedge x_2 \in \Lambda^2 U$  is defined by the equation

$$f(w) = w(x_1, x_2) , \quad w \in G_2(U) .$$

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Some elementary properties of  $x_1 \wedge x_2$  are:

(i)  $x_1 \wedge x_2 = 0$  if and only if  $x_1$  and  $x_2$  are linearly dependent.

(ii) if  $x_1 \wedge x_2 = y_1 \wedge y_2 \neq 0$  then  $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$  where  $\langle x_1, x_2 \rangle$  is the space spanned by  $x_1$  and  $x_2$ .

If  $A$  is a linear map of  $U$  into itself we define  $C_2(A)$ , the *second compound* of  $A$ , as a linear map of  $\Lambda^2 U$  into  $\Lambda^2 U$  by

$$(2.1) \quad C_2(A)x_1 \wedge x_2 = Ax_1 \wedge Ax_2 .$$

We remark that if  $x_1, \dots, x_n$  is a basis of  $U$  then  $x_i \wedge x_j, 1 \leq i < j \leq n$  is a basis of  $\Lambda^2 U$  and hence (2.1) defines  $C_2(A)$  by linear extension.

We first show that  $\Lambda^2 U$  is isomorphic in a natural way to  $S_n$  and under this isomorphism second compounds correspond to congruence transformations in  $S_n$ .

Specifically, let  $\alpha_1, \dots, \alpha_n$  be a basis of  $U$  and define  $\varphi$  by

$$(2.2) \quad \varphi(\alpha_i \wedge \alpha_j) = E_{ij} - E_{ji} \in S_n$$

where  $E_{ij}$  is the  $n$ -square matrix with 1 in position  $i, j$  and 0 elsewhere and extend  $\varphi$  linearly to all of  $\Lambda^2 U$ . It is obvious that  $\varphi$  is an isomorphism since  $E_{ij} - E_{ji}, 1 \leq i < j \leq n$  is a basis of  $S_n$ . Let  $T$  be a linear map of  $\Lambda^2 U$  into itself and define  $S$ , a linear map of  $S_n$  into itself, by

$$(2.3) \quad S(A) = \varphi T \varphi^{-1}(A), A \in S_n .$$

Let  $B$  be a linear map of  $U$  into itself. Then

**THEOREM 1.**  $T = C_2(B)$  if and only if  $S(A) = B_1 A B'_1$  where  $B_1$  is the matrix of  $B$  with respect to the ordered basis  $\alpha_1, \dots, \alpha_n$ .

*Proof.* Suppose  $T = C_2(B)$ . Then for  $i < j$

$$\begin{aligned} S(E_{ij} - E_{ji}) &= \varphi T \varphi^{-1}(E_{ij} - E_{ji}) \\ &= \varphi(B\alpha_i \wedge B\alpha_j) \\ &= \varphi\left(\sum_{k=1}^n b_{ki}\alpha_k \wedge \sum_{k=1}^n b_{kj}\alpha_k\right) \\ &= \sum_{s,t} b_{si}b_{tj}(E_{st} - E_{ts}) \\ &= B_1(E_{ij} - E_{ji})B'_1 . \end{aligned}$$

The implication in the other direction is similar.

Let  $L_{2r}$  denote the set of rank  $2r$  matrices in  $S_n$  and let  $\Omega_{2r}$  denote the set of vectors  $\sum_{i=1}^r x_i \wedge y_i$  in  $\Lambda^2 U$  where  $\dim \langle x_1, \dots, x_r, y_1, \dots, y_r \rangle = 2r$ .

**THEOREM 2.**  $\varphi(\Omega_{2r}) = L_{2r}$

*Proof.* Let

$$z = \sum_{i=1}^r x_i \wedge y_i \in \Omega_{2r} .$$

Choose a non-singular map  $B$  of  $U$  onto  $U$  such that  $B\alpha_{2j-1} = x_j$  and  $B\alpha_{2j} = y_j, j = 1, \dots, r$ . Then

$$z = C_2(B) \sum_{j=1}^r \alpha_{2j-1} \wedge \alpha_{2j} ,$$

so

$$(2.4) \quad \varphi(z) = \varphi C_2(B) \sum_{j=1}^r \alpha_{2j-1} \wedge \alpha_{2j} .$$

Let  $S(A) = B_1 A B_1'$  for  $A \in S_n$  where  $B_1$  is the matrix of  $B$  with respect to the ordered basis  $\alpha_1, \dots, \alpha_n$ . Then by Theorem 1,  $\varphi C_2(B) \varphi^{-1} = S$  and from (2.4) we have

$$\begin{aligned} \varphi(z) &= S\varphi \sum_{j=1}^r \alpha_{2j-1} \wedge \alpha_{2j} \\ &= S\left(\sum_{j=1}^r (E_{2j-1,2j} - E_{2j,2j-1})\right) \\ &= B_1\left(\sum_{j=1}^r (E_{2j-1,2j} - E_{2j,2j-1})\right)B_1' \in L_{2r} . \end{aligned}$$

The implication in the other direction is a reversal of this argument.

We see then that a map  $T$  of  $\Lambda^2 U$  into itself is a second compound of some linear map of  $U$  into itself if and only if  $\varphi T \varphi^{-1}$  is a congruence map of  $S_n$ ; and  $T(\Omega_{2r}) \subseteq \Omega_{2r}$  if and only if  $\varphi T \varphi^{-1}(L_{2r}) \subseteq L_{2r}$ .

**3.  $E_{2k}$  preservers.** Let  $S$  be a linear map of  $S_n$  into itself such that  $E_{2k}(S(A)) = E_{2k}(A)$  for all  $A \in S_n$ , where  $k$  is a fixed integer,  $k \geq 2$ . Then

LEMMA 1.  $S$  is non-singular.

*Proof.* Suppose  $S(A) = 0$ . Then

$$(3.1) \quad E_{2k}(A + X) = E_{2k}(S(A + X)) = E_{2k}(S(X)) = E_{2k}(X)$$

for all  $X \in S_n$ .

Obtain a real orthogonal  $P$  such that

$$(3.2) \quad PAP' = \sum_{i=1}^r \begin{pmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{pmatrix} \dot{+} 0_{n-2r}$$

where  $0_{n-2r}$  is an  $(n - 2r)$ -square matrix of zeros and  $\rho(A) = \text{rank } A = 2r$ .

Here  $\sum$  and  $\dot{+}$  indicate direct sum. Now if  $\rho(A) \geq 2k$  simply set  $X = 0$  and from (3.1) and (3.2) we see that

$$0 < E_{2k}(A) = E_k(\theta_1^2, \dots, \theta_r^2) = E_{2k}(0) = 0$$

a contradiction. On the other hand, if  $\rho(A) < 2k$  select  $X \in S_n$  such that

$$PXP' = 0_{2r} \dot{+} \sum_1^{(k-r)} (E_{12} - E_{21}) \dot{+} 0_{n-2k}$$

where  $E_{12}$  is a 2-square matrix. Then

$$E_{2k}(A + X) = E_{2k}(PAP' + PXP') = \prod_{j=1}^r \theta_j^2.$$

But  $E_{2k}(PXP') = E_{2k}(X) = 0$ , since  $k - r < k$ . Hence the proof is complete.

**LEMMA 2.** *If  $A \in S_n$  and  $\deg E_{2k}(xA + B) \leq 2$  for all  $B \in S_n$  and  $A \neq 0$  then  $\rho(A) = 2$ .*

*Proof.* Suppose  $\rho(A) = 2r$  and select a real orthogonal  $P$  such that  $PAP'$  has the form given in (3.2). Select  $B$  such that

$$PBP' = 0_{2r} \dot{+} \sum_2^{\lfloor \frac{n}{2} \rfloor - r} (E_{12} - E_{21}) \dot{+} C$$

where if  $n$  is even  $C$  doesn't appear and if  $n$  is odd  $C$  is a 1-square zero matrix.

Now if  $k \leq r$

$$E_{2k}(xA + B) = x^{2k} E_k(\theta_1^2, \dots, \theta_r^2) + \text{lower order terms in } x.$$

If  $k > r$

$$E_{2k}(xA + B) = \binom{\lfloor n/2 \rfloor - r}{k - r} \theta_1^2 \dots \theta_r^2 x^{2r} + \text{lower order terms in } x. \text{ Thus}$$

$$\deg E_{2k}(xA + B) \text{ is either } 2k \text{ or } 2r.$$

But this implies  $2r = 2$  and  $\rho(A) = 2$ .

**LEMMA 3.** *If  $E_{2k}(S(A)) = E_{2k}(A)$  for all  $A \in S_n$  then  $S(L_2) \subseteq L_2$ .*

*Proof.* Let  $p(x)$  be the polynomial  $E_{2k}(xA + B)$ . Then if  $\rho(A) = 2$  it is easy to check that  $\deg p(x) \leq 2$  for all  $B \in S_n$ . Hence  $\deg E_{2k}(xS(A) + S(B)) \leq 2$  for all  $B \in S_n$ . But  $S$  is non-singular by Lemma 1 and thus by Lemma 2,  $\rho(S(A)) = 2$ .

**THEOREM 3.** *If  $E_{2k}(S(A)) = E_{2k}(A)$  for all  $A \in S_n$ , where  $k$  is a fixed integer satisfying  $4 \leq 2k \leq n$  and  $n \geq 5$  then there exists a real matrix  $P$  such that*

$$(3.3) \quad S(A) = \alpha P A P' \text{ for all } A \in S_n$$

where  $\alpha P P' = I$  if  $2k < n$  and  $\alpha P P'$  is unimodular if  $2k = n$ . If  $2k = n = 4$  then either  $S$  has the form (3.3) or

$$(3.4) \quad S(A) = \alpha P \begin{pmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{pmatrix} P'$$

where  $A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$  and  $\alpha P P'$  is unimodular.

*Proof.* By Lemma 1,  $S^{-1}$  exists and we check that

$$E_{2k}(S^{-1}(A)) = E_{2k}(S S^{-1}(A)) = E_{2k}(A) ,$$

for any  $A \in S_n$ . Hence by Lemma 3

$$S^{-1}(L_2) \subseteq L_2 \text{ and thus } S(L_2) = L_2 .$$

Now define  $T$ , a mapping of  $\Lambda^2 U$  into itself, by (2.3)

$$T = \varphi^{-1} S \varphi .$$

By Theorem 2

$$\begin{aligned} T(\Omega_2) &= \varphi^{-1} S \varphi(\Omega_2) \\ &= \varphi^{-1} S(L_2) \\ &= \varphi^{-1}(L_2) \\ &= \Omega_2 . \end{aligned}$$

At this point we invoke a theorem of Chow [1, pp. 38]. Let  $T''$  be the mapping of 2-dimensional subspaces of  $U$  into themselves induced by  $T$ ; that is, let  $T''(\langle x, y \rangle) = \langle u, v \rangle$  whenever  $T(x \wedge y) = u \wedge v$ , (assuming of course that  $x$  and  $y$  are linearly independent). Then  $T''$  is well defined and it follows from the above that it is a one-to-one onto adjacency preserving transformation: if two 2-dimensional subspaces of  $U$  intersect in a subspace of dimension 1 then their images under  $T''$  intersect in a subspace of dimension 1. Therefore  $T''$  is induced either by a correlation or a collineation of the subspaces of  $U$ . If  $\dim U \geq 5$

$T''$  is induced by a collineation. If  $\dim U = 4$  and if  $T''$  is induced by a correlation then  $(TT_1)''$  is induced by a collineation. Here  $T_1$  maps  $\Lambda^2 U$  into itself and satisfies

$$(3.5) \quad \begin{aligned} T_1(x_i \wedge x_j) &= x_i \wedge x_m, \\ \{i, j, l, m\} &= \{1, 2, 3, 4\} \text{ and } i < j, l < m. \end{aligned}$$

Now, assuming  $T''$  is induced by a collineation we show that

$$(3.6) \quad T = \alpha C_2(P)$$

for some  $\alpha \in F$  and some linear transformation  $P: U \rightarrow U$ . The fundamental theorem of projective geometry states that there is a one-to-one semi-linear transformation  $Q: U \rightarrow U$  such that

$$(3.7) \quad T''(\langle x, y \rangle) = \langle Qx, Qy \rangle.$$

Let  $x_1, \dots, x_n$  be a basis of  $U$  and let  $Qx_i = y_i$ . Then

$$\begin{aligned} T(x_i \wedge x_j) &= \alpha_{ij} y_i \wedge y_j \quad \alpha_{ij} \in F. \\ 1 \leq i, j \leq n, \quad i \neq j. \end{aligned}$$

Then for  $s, k, t$  distinct integers in  $1, \dots, n$  and  $K \in F$ .

$$\begin{aligned} T((x_s + x_t) \wedge x_k) &= K(Q(x_s + x_t) \wedge Qx_k) \\ &= K(y_s + y_t) \wedge y_k, \end{aligned}$$

But

$$\begin{aligned} T((x_s + x_t) \wedge x_k) &= T(x_s \wedge x_k) + T(x_t \wedge x_k) \\ &= (\alpha_{sk} y_s + \alpha_{tk} y_t) \wedge y_k. \end{aligned}$$

Hence  $\alpha_{sk} = \alpha_{tk}$  and thus  $\alpha_{sk} = \alpha_{tk} = \alpha_{kt} = \alpha_{rt} = \alpha$  for any four distinct integers  $s, k, r, t$ . Hence

$$T(x_i \wedge x_j) = \alpha y_i \wedge y_j = \alpha C_2(P)x_i \wedge x_j,$$

where  $P: U \rightarrow U$  is a linear transformation with  $Px_j = y_j$ . Since  $\{x_i \wedge x_j \mid 1 \leq i < j \leq n\}$  is a basis of  $\Lambda^2 U$ ,  $T = \alpha C_2(P)$ .

Now by Theorem 1,

$$S(A) = \alpha PAP' \text{ for all } A \in S_n$$

for  $n \geq 5$  where  $P$  is an  $n$ -square non-singular matrix. If  $2k = n$  then clearly  $\alpha PP'$  is unimodular. Hence assume  $2k < n$ .

We next show that

$$\alpha PP' = I.$$

From the hypothesis,

$$E_{2k}(\alpha PAP') = E_{2k}(A), A \in S_n$$

and hence

$$\alpha^{2k} tr\{C_{2k}(PP')C_{2k}(A)\} = trC_{2k}(A).$$

By the polar factorization theorem let  $P = UB$ , where  $U$  is real orthogonal and  $B$  is positive definite symmetric. Let  $B = VDV'$ ,  $D$  diagonal with positive entries and  $V$  real orthogonal. Then since  $V'AV$  runs through all of  $S_n$  as  $A$  does we have

$$(3.9) \quad \alpha^{2k} tr\{C_{2k}(D^2)C_{2k}(A)\} = trC_{2k}(A).$$

We assert that any diagonal  $\binom{n}{2k}$ -square matrix is a linear combination of matrices  $C_{2k}(A)$  for  $A \in S_n$ . For, let  $1 \leq i_1, \dots, i_{2k} \leq n$ . Let  $A \in S_n$  and consider the  $2k$ -square principal submatrix  $B$  of  $A$  where

$$B_{\alpha\beta} = A_{i_\alpha i_\beta};$$

and suppose  $A$  has 0 entries outside of  $B$ . Then define  $B$  as follows:

$$\begin{aligned} B_{2k-\alpha, \alpha+1} &= -1, & \alpha &= 0, \dots, k-1 \\ B_{2k-\alpha, \alpha+1} &= 1, & \alpha &= k, \dots, 2k \end{aligned}$$

and  $B_{ij} = 0$  elsewhere. Then  $C_{2k}(A) = \pm E_{i_1 \dots i_{2k}}$ , where  $E_{i_1 \dots i_{2k}}$  is the  $\binom{n}{2k}$ -square matrix with the single non-zero entry 1 in the  $((i_1, \dots, i_{2k}), (i_1, \dots, i_{2k}))$  position ordered doubly lexicographically in the indices of the rows and columns of  $A$ . Returning to (3.9) we have

$$tr\{C_{2k}(\alpha D^2)X\} = trX$$

for all  $\binom{n}{2k}$ -square diagonal matrices  $X$  and hence  $C_{2k}(\alpha D^2) = I, \alpha D^2 = \pm I$ . From this we easily see that

$$\alpha PP' = I,$$

and (3.3) follows. The mapping  $T_1$  on  $\Lambda^2 U$  induces the map  $S^1$  on  $S_4$  where

$$S^1 \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{12} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{pmatrix}$$

This completes the proof.

We remark that Theorem 3 is no longer valid if  $k = 1$ : for consider the transformation which interchanges positions  $(i, j)$  and  $(j, i)$  in  $A$  for a fixed pair of integers  $1 \leq i < j \leq n$ . This clearly preserves  $E_2(A)$  but



does not have the form in Theorem 3. For example

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

is non-singular but interchanging the 1, 2 and 2, 1 entries results in a singular matrix.

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