

# Pacific Journal of Mathematics

**ON THE LINE SEGMENTS OF A CONVEX SURFACE IN  $E_3$**

TREVOR JAMES MCMINN

# ON THE LINE SEGMENTS OF A CONVEX SURFACE IN $E_3$

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**1. Introduction.** For integral  $n \geq 2$  let  $C$  be a bounded open convex subset of Euclidean  $n$ -space  $E_n$ , and let  $C'$  be the boundary (surface) of  $C$ . Let  $B_n$  be the closed unit ball in  $E_n$ , that is, the set of points  $x$  in  $E_n$  with  $\|x\| \leq 1$ , and let  $S_{(n-1)}$  be the boundary of  $B_n$ , that is, the set of points  $x$  in  $E_n$  with  $\|x\| = 1$ . Let  $D$  be the set of directions of straight line segments lying in  $C'$ , specifically, the set of points  $(a - b)/\|a - b\|$ , where  $a$  and  $b$  are distinct points of a line segment lying in  $C'$ . Thus  $D$  is contained in  $S_{(n-1)}$ .

V. L. Klee [2] has stated that  $D$  is an  $F_\sigma$  set and has raised two questions: Is  $D$  of first category in  $S_{(n-1)}$ ? Is  $D$  of  $(n - 1)$ -dimensional measure zero? Both of these questions are herein answered affirmatively for the case  $n = 3$ . The method employed unfortunately does not generalize to  $n > 3$ . (For  $n = 2$  the case is trivial, for then  $D$  is countable. The case is also trivial if  $C$  is of dimension less than  $n$ , for then the  $(n - 2)$ -dimensional measure of  $D$  cannot be greater than the  $(n - 2)$ -dimensional measure of  $S_{(n-2)}$  which is finite. The restriction to bounded sets is only a matter of convenience, for any answers to the questions posed are easily made to serve the unbounded case.)

In one sense though, for the case  $n = 3$ , we show somewhat more, namely, that  $D$  is contained in the union of the ranges of a countable family of Lipschitz functions each on  $B_1$  to  $S_2$ . By virtue of the Lipschitz nature of these functions, they possess total differentials (Lebesgue measure) almost everywhere [4; straight forward generalizations of Definition 1, V. 2.2, and Lemmas 1 and 2, V. 2.3, to cover the case of a Lipschitz function on a domain contained in  $E_1$  to  $E_3$ ] and their ranges are compact and have finite one dimensional measure [1]. The affirmative answers to Klee's questions for this case immediately follow from these last two properties.

**2. Preliminaries.** We assume henceforth that  $n = 3$ .

Let a *flat side* of  $C'$  be a two dimensional intersection of  $C'$  with a plane supporting  $C'$ . It is easy to check that the set of flat sides is countable. (Check, for instance, that relative to  $C'$ , the interior of each flat side is non-vacuous and, that no two such interiors intersect.) Thus the set of directions of line segments lying in flat sides is the union of a countable family of great circles lying on  $S_2$  and can certainly be represented as the union of the ranges of an appropriately

chosen countable family of Lipschitz functions on  $B_1$  to  $S_2$ .

We go on to show that the set of directions of line segments not lying in flat sides can be similarly represented.

Let  $\mathcal{L}$  be the set of closed line segments each of which is the middle third line segment contained in a maximal line segment of  $C'$  not lying in a flat side. Clearly  $\mathcal{L}$  is disjointed, for if any two members intersected they would be forced by the convexity of  $C$  to lie in a flat side determined by the plane containing the two line segments.

Now choose a point  $a$  in  $C$  and let  $2\delta$  be the distance from  $a$  to  $C'$ . Let  $\mathcal{H}$  be the family of open right circular cylinders of radius  $\delta$  extending infinitely in two directions whose axis is a line radiating out from  $a$  infinitely in two directions. Thus each member of  $\mathcal{H}$  intersects  $C'$  in a set open relative to  $C'$  and having two components. Let  $\mathcal{M}$  be the set of all these components corresponding to all cylinders of  $\mathcal{H}$ .

Since  $\mathcal{M}$  forms an open covering of the compact space  $C'$  we can reduce it to a finite subcovering  $\mathcal{M}'$ .

Now let  $\mathcal{P}$  be the family of planes each of which intersects  $C$  and perpendicularly intersects a coordinate axis in a point with rational coordinates. Let  $\mathcal{Q}$  be the family of pairs of distinct parallel members of  $\mathcal{P}$ .

Clearly every member of  $\mathcal{L}$  intersects at least one member of  $\mathcal{M}'$  and every such intersection intersects both planes of at least one pair in  $\mathcal{Q}$ .

Since  $\mathcal{M}'$  is finite and  $\mathcal{Q}$  is countable, we will have achieved our aim when we have shown that corresponding to each member  $m$  of  $\mathcal{M}'$  and each pair  $(P_1, P_2)$  of planes in  $\mathcal{Q}$  both intersecting  $m$  there exist two Lipschitz functions each on  $B_1$  to  $S_2$  whose ranges together contain the set of directions of the members of  $\mathcal{L}$  each of which intersects both  $m \cap P_1$  and  $m \cap P_2$ . With  $m, P_1$ , and  $P_2$  fixed and letting  $\mathcal{L}'$  be the set of members of  $\mathcal{L}$  each intersecting both  $m \cap P_1$  and  $m \cap P_2$ , we proceed to secure the required functions.

**3. The Lipschitz direction functions.** Let  $f$  be the set of all pairs  $(x, y)$  such that  $x \in \lambda \cap P_1$  and  $y \in \lambda \cap P_2$  for some  $\lambda \in \mathcal{L}'$ . Let  $A$  be the domain of  $f$ . Since  $\mathcal{L}'$  is disjointed and since  $\lambda \cap P_1$  and  $\lambda \cap P_2$  are singletons we infer that  $f$  is a function. The key to the construction of the required functions lies in the

**LEMMA.**  *$f$  is Lipschitz.*

Momentarily leaving aside its proof, we first show how it is used to obtain these functions.

Drawing upon the lemma, we apply a method due to McShane [3; or 4, V. 2.4, Lemma 1] to get a Lipschitz extension  $f^*$  of  $f$  on the

closure of  $P_1 \cap m$ , that is, a Lipschitz function  $f^*$  on the closure of  $P_1 \cap m$  to  $P_2$  that agrees with  $f$  on  $A$ .

We next let  $h$  be a function that assigns to each member  $x$  of the closure of  $P_1 \cap m$  one of the directions of the line connecting  $x$  to  $f^*(x)$ , specifically for  $x$  in the closure of  $P_1 \cap m$  we let

$$h(x) = \frac{f^*(x) - x}{\|f^*(x) - x\|}.$$

Upon checking that the difference of two Lipschitz functions is Lipschitz and that the ratio of a Lipschitz function whose values are bounded away from the origin (in our case bounded by the distance between  $P_1$  and  $P_2$ ) with its norm is Lipschitz, we infer that  $h$  is Lipschitz. It is easy to construct a Lipschitz homeomorphism  $g$  on  $B_1$  onto the closure of  $P_1 \cap m$ . So finally upon defining functions  $k$  and  $k'$  on  $B_1$  to  $S_2$  to be such that for  $x$  in  $B_1$

$$k(x) = h(g(x)), \quad k'(x) = -k(x),$$

and noting that the composition of Lipschitz functions is Lipschitz, we conclude that  $k$  and  $k'$  are Lipschitz and furthermore that their ranges together contain the set of directions of members of  $\mathcal{L}'$ . These are the functions we seek.

We now turn our attention to the lemma and close our discussion with its proof.

**4. Proof of the Lemma.** We show that  $f$  is Lipschitz by showing that it can be represented as the composition of Lipschitz functions. To do this let us project  $m$  perpendicularly onto a plane perpendicular to the axis of the cylinder in  $\mathcal{N}$  associated with  $m$ . Let  $m'$  be the projected set and let  $p$  be the projecting function. Thus  $p$  is on  $m$  onto  $m'$ . From the convexity of  $C$  and the nature of the cylinder determining  $m$  we readily check that  $p$  is a Lipschitz homeomorphism on  $m$  onto  $m'$  whose inverse is also Lipschitz. For  $x'$  in  $p(A)$  let  $f'(x') = p(f(p^{-1}(x')))$ . For  $x$  in  $A$  clearly  $f(x) = p^{-1}(f'(p(x)))$ . We have only to show that  $f'$  is Lipschitz.

Let  $\lambda_1$  and  $\lambda_2$  be two members of  $\mathcal{L}'$ . Let  $x_1 \in \lambda_1 \cap P_1$  and  $x_2 \in \lambda_2 \cap P_1$ . Let  $l_1$  and  $l_2$  be maximal line segments contained in  $C'$  containing respectively  $\lambda_1$  and  $\lambda_2$ . Let  $l_1'$  and  $l_2'$  be the respective perpendicular projections of  $l_1$  and  $l_2$  onto the plane of  $m'$ . Clearly  $l_1$  and  $l_2$  fail to intersect or intersect only in an end point of both  $l_1$  and  $l_2$ . Consequently the same is true of  $l_1'$  any  $l_2'$ . If  $l_1'$  and  $l_2'$  are parallel or, when extended, intersect on the side of  $P_2$  opposite from  $P_1$ , then clearly

$$(1) \quad \|p(f(x_1)) - p(f(x_2))\| \leq \|p(x_1) - p(x_2)\|.$$

If, on the other hand,  $l_1'$  and  $l_2'$ , when extended, intersect in a point  $b$ , on the same side of  $P_2$  that  $P_1$  lies on, then either an end point of  $l_1'$  lies at  $b$  or between  $b$  and  $P_1$ , or an end point of  $l_2'$  lies at  $b$  or between  $b$  and  $P_1$ . We may assume the first of these two main disjunctions without loss of generality. Now since the line segment connecting  $p(x_1)$  with  $p(f(x_1))$  is contained in the middle third segment of  $l_1'$ , we have

$$\|p(f(x_1)) - p(x_1)\| \leq \|p(x_1) - b\| .$$

and hence

$$(2) \quad \|p(f(x_1)) - b\| = \|p(f(x_1)) - p(x_1)\| + \|p(x_1) - b\| \leq 2\|p(x_1) - b\| .$$

As  $P_1$  and  $P_2$  are parallel, we may use a property of similar triangles to get

$$(3) \quad \frac{\|p(f(x_1)) - p(f(x_2))\|}{\|p(x_1) - p(x_2)\|} = \frac{\|p(f(x_1)) - b\|}{\|p(x_1) - b\|} .$$

Combining (2) and (3) we get

$$(4) \quad \|p(f(x_1)) - p(f(x_2))\| \leq 2\|p(x_1) - p(x_2)\| .$$

Since equations (1) and (4) show that for any  $x_1'$  and  $x_2'$  in the domain of  $f'$

$$\|f'(x_1') - f'(x_2')\| \leq 2\|x_1' - x_2'\| ,$$

and hence that  $f'$  is Lipschitz, our proof is complete.

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