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Introduction. The purpose of this paper is:

1. To present a simple proof for the existence of a conformal map of an arbitrary bounded region onto a circular slit disk and—if the region possesses at least one isolated boundary components which is not a point—onto a radial slit disk;[1]

2. To remark on certain geometric properties and problems of uniqueness of the canonical slit domains thus obtained. The possibility of mapping regions of finite connectivity onto (circular and radial) slit disks was originally considered from the point of view of various variational problems by Koebe [8] and Rengel [14], Schiffer [18], Nehari [11]. Extension to regions of infinite connectivity were made by Grötzsch [3, 5], Schiffer [17] and more recently, within the framework of certain extremal problems on Riemann surfaces by Sario [16], Jurchescu [7], and in the book on Riemann Surfaces by Ahlfors and Sario [1], Chapter III, § 16D.[2]

In the present paper we consider, in connection with the circular slit mapping, an extremal problem which is suggested by the now common proof of Riemann's mapping theorem.[3] Let Ω be a bounded region which contains the origin, and let $\mathscr{F}$ denote the family of functions $f(z)$ with the properties:

1. $f(z)$ is analytic and univalent in Ω
2. $f(0) = 0, f'(0) > 0, |f(z)| \leq 1$ in Ω.

(Note that nothing regarding the boundary behavior of f is predicated in the definition of $\mathscr{F}$.) We shall show that the function $\phi \in \mathscr{F}$ which solves the extremal problem $\phi'(0) = \max_{f \in \mathscr{F}} f'(0)$, gives a mapping of Ω onto a circular slit disk. The proof that $\phi(\Omega)$ is a circular slit disk may be based on a well known inequality of Rengel [13]. We derive it, however, from a more general lemma (Lemma 1) which enables us, in addition, to discuss further properties of ϕ, in particular the uniqueness aspect of the solution. For a given Ω there may be more than one

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[1] A region is called a circular slit (radial slit) disk if a circle $|z| = r$ is one of the components of its boundary and if all other boundary components are circular arcs in $|z| < r$ concentric with $|z| = r$ (radial segments contained in $0 < |z| < r$), or points other than the origin.

[2] See also K. Oikawa's doctoral dissertation [12], Chapter III, in which he states that the circular slit disk mapping may be obtained from Sario [15].

[3] Cf. also Nehari [11], p. 352, for the case of finite connectivity.

extremal function ϕ, even in the case of finite connectivity. However, one achieves uniqueness of the solution of the extremal problem if one replaces the class $\mathscr{F}$ by the subclass $\mathscr{F}^* = \mathscr{F}^*(\Omega)$, whose elements f are further restricted by the requirement that they carry the outer boundary component of Ω into $|w| = 1$ (§§ 4 and 5).

To obtain an insight into all possible solutions ϕ of the extremal problem in $\mathscr{F}(\Omega)$ we consider the broader problem of the uniqueness of a properly normalized conformal map of Ω onto a circular slit unit disk. It is well known that the map f of a bounded region Ω, normalized by the condition that $f \in \mathscr{F}^*(\Omega)$, onto a circular slit unit disk Σ is not necessarily unique, unless Σ is further restricted. A characterization of such disks Σ onto which there exists only one normalized conformal map was given by Grötzsch [5], p. 239 (Normalbereiche). In this paper we obtain a characterization of these disks—which we detote as "extremal circular slit disks"—in terms of the extremal length $\lambda_q\{\gamma\}$ of all cycles $\gamma \subset \Sigma$ surrounding $z = 0$ and lying in an annulus $q < |z| < 1$ which contains all slits of Σ (Theorem 5). An extension of our fundamental Lemma 1 to extremal circular slit disks (Theorem 6) permits us to discuss certain features of the extremal function $\phi \in \mathscr{F}(\Omega)$.

The problem of mapping onto radial slit disks is treated by considering the $\min_{f \in \mathscr{F}^*} |f'(0)|$, where however Ω and $\mathscr{F}^*$ are subjected to an additional restriction. Similar results regarding the uniqueness of the extremal function and properties of the extremal domain are obtained here.

1. Mapping onto a circular slit disk. We begin with the principal lemma which is related to results of Grunsky [6] and Spencer [19].

LEMMA 1. *Let Σ be a circular slit unit disk centered at $z = 0$ such that the circular projection E of the boundary points of Σ onto a fixed radius*[4] *has linear measure zero. Suppose that $w = f(z)$ is analytic and univalent in Σ, that $f(0) = 0$, and that $f(z)$ maps Σ onto a bounded region Δ in such a manner that $|z| = 1$ corresponds to the outer boundary component of Δ. Let $F(z) = \log(f(z)/z)$, $F(0) = \log f'(0)$ ($F(z)$ is single-valued in Σ once a branch of $\log f'(0)$ is chosen). Let A denote the area of Δ. Then*

$$(1) \qquad \exp\left[\frac{1}{2\pi}\iint_{\Sigma} |F'(z)|^2 dxdy\right] |f'(0)| \leq \sqrt{\frac{A}{\pi}}\,.$$

In particular,

$$(1') \qquad |f'(0)| \leq \sqrt{\frac{A}{\pi}}$$

[4] i.e. E is the set of points on a fixed radius at which the circles $|z| = r$ containing a boundary point of Σ intersect this radius.

and equality implies $F'(z) \equiv 0$, i.e. $f(z) \equiv az$, *where* a *is a constant.*

REMARKS. If Σ has only a finite number of slits and if $|f(z)| \leq M$, (1′) implies Rengel's inequality [13], $|f'(0)| \leq M$, and the equality holds only if $f(z) \equiv az$. If Σ is the disk $|z| < 1$, then (1′) becomes Bieberbach's area theorem.

Proof. It is easily seen that the set E is closed. Consequently, given any $\varepsilon > 0$, there exist a finite number of intervals, $[r_\nu, r'_\nu]$ on $(0, 1]$, $r_\nu < r'_\nu < r_{\nu+1}$, $\nu = 1, 2, \cdots, n-1$, such that $E \subset \bigcup_{\nu=1}^{n} [r_\nu, r'_\nu]$ and that $\sum_{\nu=1}^{n} (r'_\nu - r_\nu) < \varepsilon$. We may assume that no boundary points of Σ are on $|z| = r_\nu$ and $|z| = r'_\nu$ $(\nu = 1, 2, \cdots, n-1)$ and on $|z| = r_n$, and that $r'_n = 1$.

Denote by D_ν the annulus $r'_\nu < |z| < r_{\nu+1}$ $(1 \leq \nu \leq n-1)$ and by D_0 the circle $|z| < r_1$. $F(z)$ is single-valued and analytic in each $\overline{D}_\nu$. Assume first that $f'(0) = 1$, $F'(0) = 0$.

Let $z = re^{i\theta}$, $F(z) = u + iv$, $f(z) = \rho e^{i\psi}$. Then[5] in each D_ν, $\nu = 1, 2, \cdots, n-1$,

$$\iint_{D_\nu} |F'(z)|^2 dxdy = \int_{C_\nu} u dv = \int_{C_\nu} \log \frac{\rho}{r} (d\psi - d\theta)$$

where C_ν is the boundary of D_ν. Furthermore, we have for $D_0 (C_0 = \{|z| = r_1\})$

$$\iint_{D_0} |F'(z)|^2 dxdy = \int_{C_0} \log \frac{\rho}{r} (d\psi - d\theta) .$$

Next we note that for $\nu = 0, 1, \cdots, n-1$

$$0 = \Im \left\{ \int_{C_\nu} \frac{F(z)}{z} dz \right\} = \int_{C_\nu} \left[\log \frac{\rho}{r} d\theta + (\psi - \theta) \frac{dr}{r} \right.$$

$$= \int_{C_\nu} [\log \rho d\theta - \log r d\psi] .$$

For $\nu = 0$ this follows from the fact that $F(0) = 0$, for $\nu > 0$ this is a consequence of the regularity of $F(z)/z$ in $\overline{D}_\nu$. Thus we obtain

$$\sum_{\nu=1}^{n-1} \iint_{D_\nu} |F'(z)|^2 dxdy = \sum_{\nu=0}^{n-1} \int_{C_\nu} \{\log \rho d\psi - 2 \log r d\psi + \log r d\theta\} .$$

We can evaluate the sum of the last two terms:

[5] The following argument is similar to a technique used by Golusin [2], pp. 149–150.

$$(2)\qquad \begin{cases} \sum_{\nu=0}^{n-1}\int_{C_\nu}\log r\,d\theta \\ \quad=\sum_{\nu=1}^{n-1}\left\{\log r_\nu\int_{|z|=r_\nu}d\theta-\log r'_\nu\int_{|z|=r'_\nu}d\theta\right\}+\log r_n\int_{|z|=r_n}d\theta \\ \quad=-2\pi\sum_{\nu=1}^{n-1}\log\frac{r'_\nu}{r_\nu}+2\pi\log r_n\,. \end{cases}$$

The assumption that under the mapping $w=f(z)$, the circle $|z|=1$ corresponds to the outer component of $f(\Sigma)$, implies that the images of $|z|=r_\nu$ and $|z|=r'_\nu$ are closed Jordan curves which contain $w=0$ in their interiors. Hence $\int_{|z|=r_\nu}d\psi=\int_{|z|=r'_\nu}d\psi=2\pi$. We obtain therefore in the same manner as above

$$(3)\qquad -\sum_{\nu=0}^{n-1}\int_{C_\nu}\log r\,d\psi=2\pi\sum_{\nu=1}^{n-1}\log\frac{r'_\nu}{r_\nu}-2\pi\log r_n\,.$$

Hence, combining (2) and (3) we find

$$\begin{aligned}\sum_{\nu=0}^{n-1}\left\{-2\int_{C_\nu}\log r\,d\psi+\int_{C_\nu}\log r\,d\theta\right\}&=2\pi\sum_{\nu=1}^{n-1}\log\frac{r'_\nu}{r_\nu}-2\pi\log r_n\\&\le 2\pi\sum_{\nu=1}^{n-1}\frac{r'_\nu-r_\nu}{r_\nu}+2\pi\frac{1-r_n}{r_n}\\&\le\frac{2\pi}{r_1}\sum_{\nu=1}^{n}(r'_\nu-r_\nu)<\frac{2\pi}{r_1}\varepsilon\end{aligned}$$

since $\sum_{\nu=1}^{n}(r'_\nu-r_\nu)<\varepsilon$. Thus, if $\tau=\Sigma-\bigcup_{\nu=0}^{n-1}D_\nu$ and $|z|\le d$ is a disk contained in Σ,

$$(4)\qquad \iint_{\Sigma}|F'(z)|^2dxdy\le\sum_{\nu=0}^{n-1}\int_{C_\nu}\log\rho\,d\psi+\frac{2\pi}{d}\varepsilon+\iint_{Z}|F'(z)|^2dxdy\,.$$

Now, if $\Gamma_\nu=f(C_\nu)$, $\Delta_\nu=f(D_\nu)$

$$\int_{C_\nu}\log\rho\,d\psi=\int_{\Gamma_\nu}\log\rho\,d\psi=\iint_{\Delta_\nu}\frac{du\,dv}{|w|^2}\qquad(\nu\ge 1)$$

and if we remove a sufficiently small disk k of radius μ about $w=0$,

$$\int_{C_0}\log\rho\,d\psi=\int_{\Gamma_0}\log\rho\,d\psi=\iint_{\Delta_0-k}\frac{du\,dv}{|w|^2}+2\pi\log\mu\,.$$

Hence, if $f(\Sigma)=\Delta$,

$$(5)\qquad \sum_{\nu=0}^{n-1}\int_{C_\nu}\log\rho\,d\psi\le\iint_{\Delta-k}\frac{du\,dv}{|w|^2}+2\pi\log\mu\,.$$

Let $K=\{w\mid |w|<R\}$ be a disk with area $\pi R^2=A$. Then

$$(6)\qquad \iint_{\Delta-k}\frac{dudv}{|w|^2}=\iint_{(\Delta-k)\cap K}\frac{dudv}{|w|^2}+\iint_{\Delta-k-K}\frac{dudv}{|w|^2}\leq\iint_{\mu\leq|w|\leq R}\frac{dudv}{|w|^2}$$
$$=2\pi(\log R-\log\mu)\,.$$

Thus we obtain from (4), (5) and (6)

$$\iint_{\Sigma}|F'(z)|^2dxdy\leq 2\pi\log\sqrt{\frac{A}{\pi}}+\frac{2\pi}{d}\varepsilon+\iint_{\tau}|F'(z)|^2dxdy\,.$$

Since the measure of τ is smaller than $2\pi\varepsilon$ and the last inequality is true for every $\varepsilon>0$, we obtain the conclusion for the case $f'(0)=1$. If $f'(0)\neq 1$, we need only to replace A by $A/|f'(0)|^2$ and the lemma follows.

Let Ω be a bounded region, let $z=0\in\Omega$, and let $\mathscr{F}$ denote the class of functions $f(z)$ with the properties:

1. $f(z)$ is analytic and univalent in Ω
2. $f(0)=0, f'(0)>0, |f(z)|\leq 1, z\in\Omega$.

Since $\mathscr{F}$ is normal there exists a function $\phi(z)\in\mathscr{F}$ such that $\phi'(0)\geq f'(0)$ for any $f\in\mathscr{F}$ (note that $\phi'(0)>0$ since $cz\in\mathscr{F}$ for some $c>0$, so that $\phi(z)\not\equiv$ constant).

THEOREM 1. *The function $w=\phi(z)$ maps Ω conformally onto a circular slit unit disk.*

Proof. Let $\Omega'=\phi(\Omega)$. Every point $w_0, |w_0|=1$, is a boundary point of Ω'. If not, there exists a region Δ_ε in $|w|<1$ bounded by $|w|=1$ and an arc $\{|w-w_0|=\varepsilon, |w|\leq 1\}$, such that $\Omega'\subset\Delta_\varepsilon$. If the function which maps Δ_ε onto $|w|<1$ and preserves the origin is composed with $\phi(z)$, then by the inequality (1') of Lemma 1, the result is a function $f(z)\in\mathscr{F}$ with $|f'(0)|>|\phi'(0)|$.

Let Λ be an arbitrary component of the boundary of Ω'.

Case 1. There exists a point $w_0\in\Lambda, |w_0|=1$. Then $\Delta_\Lambda=\{|w|<1\}-\Lambda$ is simply connected and $\Delta_\Lambda\supset\Omega'$. As above, it follows by Lemma 1 that $\Delta_\Lambda=\{|w|<1\}$ and hence $\Lambda=\{|w|=1\}$.

Case 2. Λ does not meet $\{|w|=1\}$. Then Λ is a circular slit concentric with $|w|=1$. Otherwise, we could map the doubly connected region $\Delta_\Lambda(\supset\Omega')$ onto the unit disk minus a concentric circular slit, preserving the origin and $|w|=1$, and the result is, again by (1') a function $f\in\mathscr{F}$ with $|f'(0)|>|\phi'(0)|$. This proves the theorem.

3. Some properties of $\phi(z)$. To study $\phi(z)$ further we derive several additional lemmas. Let D be a doubly connected circular slit region with outer boundary $|z|=1$ and slit $\sigma=\{z\,|\,z=re^{i\theta}, \theta_1\leq\theta\leq\theta_1+\alpha\}$.

Let D' be another region of this type in the w-plane with slit $\sigma' = \{w \mid w = \rho e^{i\theta}, \theta_1' \leq \theta \leq \theta_1' + \beta\}$. Let R be the annulus $\{\zeta \mid q < |\zeta| < 1\}$, $q > 0$.

LEMMA 2. *Suppose $w = f(z)$ provides a one-to-one conformal map of D onto D' such that $f(0) = 0$ and σ is mapped onto $|w| = 1$. (Hence σ' will correspond to $|z| = 1$.) Then*

$$|f'(0)| = \frac{\rho}{r}.$$

Proof. We may suppose σ to be symmetrical with respect to the real axis. The function $F(z) = \log[(f(z)/z)(z - r)]$ is regular in D once a branch of $F(0) = \log(-rf'(0))$ is chosen. Let $\sigma(\delta)$ be a curvilinear quadrilateral whose sides are $\{|z| = r \pm \delta, -\alpha/2 - \delta \leq \theta \leq \alpha/2 + \delta\}$ and $\{r - \delta \leq |z| \leq r + \delta, \theta = \pm(\alpha/2 + \delta)\}$. Let $C_\delta = \sigma(\delta) \cup \{|z| = 1\}$.

By Cauchy's formula

$$\log|rf'(0)| = \mathscr{R}\left\{\frac{1}{2\pi i}\int_{\sigma_\delta} \frac{F(z)}{z}dz\right\}.$$

Hence

$$\log|rf'(0)| = \frac{1}{2\pi}\int_0^{2\pi} (\log\rho + \log|e^{i\theta} - r|)d\theta - \mathscr{R}\frac{1}{2\pi i}\int_{\sigma_\delta} \frac{F(z)}{z}dz.$$

Now

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - r|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{e^{i\theta} - r}{1 - e^{i\theta}r}(1 - e^{i\theta}r)\right|d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|1 - e^{i\theta}r|\,d\theta = 0.$$

Furthermore, as is easily seen,

$$\mathscr{R}\left\{\frac{1}{2\pi i}\int_{\sigma_\delta} \frac{F(z)}{z}dz\right\} = o(1) \quad \text{as } \delta \to 0.$$

Thus $\log[r|f'(0)|] = \log\rho$, or $|f'(0)| = \rho/r$.

LEMMA 3. *Suppose $w = g(\zeta)$ provides a one-to-one conformal map of R onto D' in such a way that the outer boundaries of R and D' correspond. If $g(\zeta_0) = 0$ then $|\zeta_0| = \rho$.*

Proof. This lemma may be obtained from an explicit formula for $g(\zeta)$ as given by Komatu [10], but it may also be proved in an elementary way as follows.

Any branch of $F(\zeta) = 1/\zeta \log g(\zeta)/\zeta - \zeta_0$ is regular in R. Hence

$$\mathscr{R}\frac{1}{2\pi i}\int_{\partial R} F(\zeta)d\zeta = 0 .$$

We have

$$-\mathscr{R}\left[\frac{1}{2\pi i}\int_{\partial R}\frac{1}{\zeta}\log(\zeta - \zeta_0)d\zeta\right] = \mathscr{R}\left\{\frac{1}{2\pi i}\left[\int_{|\zeta|=q} - \int_{|\zeta|=1}\right]\right\}$$

$$= \mathscr{R}\log(-\zeta_0) + \frac{1}{2\pi}\int_0^{2\pi}\log|e^{i\theta} - \zeta_0|\,d\theta$$

$$= \log|\zeta_0|$$

and

$$\mathscr{R}\left\{\frac{1}{2\pi i}\int_{\partial R}\log g(\zeta)\frac{d\zeta}{\zeta}\right\} = -\log\rho .$$

Hence, $\rho = |\zeta_0|$, as was to be shown.

Now let μ_D be the harmonic measure of σ with respect to D, evaluated at the origin, i.e., if $\omega(z)$ is harmonic in D with boundary values $\omega(z) = 0, |z| = 1$, and $\omega(z) = 1, z \in \sigma$, then $\mu = \omega(0)$. Also let $q = q_D$ be the inner radius obtained when D is mapped onto an annulus R_D. (The numbers μ_D and q_D are completely determined by r and α.)

LEMMA 4. *Let $f(z)$ be defined as in Lemma 2. Then*

$$|f'(0)| = q_D^{1-2\mu_D}$$

Hence $|f'(0)| > 1$ if and only if $\mu_D > 1/2$.

Proof. Let D be mapped onto R_D in such a way that the outer boundaries of D and R_D correspond. If $\zeta = \zeta(z)$ is the mapping function.

$$\omega(z) = \frac{\log|\zeta(z)|}{\log q_D} .$$

Hence, if $\zeta_0 \in R$ is the point corresponding to the origin in D, we must have

$$\mu_D = \frac{\log|\zeta_0|}{\log q_D} .$$

By Lemma 3. $|\zeta_0| = r$. Hence $r = q_D^{\mu_D}$. Now $q_{D'} = q_D$, and $\mu_{D'} = 1 - \mu_D$ (due to the fact that the inner boundary of D corresponds to the outer boundary of D'). Therefore $\rho = q_D^{1-\mu_D}$. The assertion now follows by Lemma 2. We may note that if $\alpha \geq \pi$ this is sufficient to guarantee

that $\mu_D = 1/2\pi \int_0^{2\pi} \omega(re^{i\theta})\, d\theta > 1/2\pi \int_{\theta_1}^{\theta_1+\pi} d\theta = 1/2$. Hence $\alpha \geq \pi$ implies $|f'(0)| > 1$.

In view of Lemma 4 we can draw the following conclusion regarding the domain $\Omega' = \phi(\Omega)$ of Theorem 1. Let the arc Λ be an arbitrary inner component of $\partial\Omega'$ and μ_Λ the value of the harmonic measure of Λ with respect to $\Delta_\Lambda = \{|w| < 1\} - \Lambda$ at the origin. Then we have

THEOREM 2. *For any inner boundary component Λ of $\phi(\Omega)$, $\mu_\Lambda \leq 1/2$. In particular all inner boundary components of $\phi(\Omega)$ subtend angles less than π at the origin.*

Proof. If a boundary component Λ with $\mu_\Lambda > 1/2$ existed, form $\phi_1(z) = f(\phi(z))$, where $f(z)$ maps the doubly connected region $\Delta_\Lambda = \{|w| < 1\} - \Lambda$ conformally onto a circular slit unit disk such that $f(0) = 0$ and Λ corresponds to the unit circle. Then, by Lemma 4, $|\phi_1'(0)| > \phi'(0)|$, contrary to the definition of $\phi(z)$.

4. Mappings preserving the outer boundary component. The extremal function $\phi(z)$ of the family $\mathscr{F}$ obtained in Theorem 1 is not necessarily unique. For a region of *finite* connectivity $\phi(z)$ is uniquely determined if ϕ is subjected to the additional restriction that it carries a prescribed boundary component into the circle $|w| = 1$. In this case $\phi(z)$ is also the only mapping function of Ω onto a circular slit disk with $\phi(0) = 0$, $\phi'(0) > 0$. Lemma 4 shows for the case in which $\Omega = D$, a disk with only one slit σ, that there may exist two distinct function $\phi_1(z)$, $\phi_2(z)$ which map D onto a circular slit disk and for which $\phi_1'(0) = \phi_2'(0) = 1$. One of these carries $|z| = 1$ into the slit of Σ and the other into the outer boundary $|w| = 1$. This is the case when $\mu_D = 1/2$. These functions ϕ_1 and ϕ_2 are necessarily extremal functions of $\mathscr{F}$ in D.

We now prove,[6] for the case of an arbitrary bounded region Ω, the

[6] For the purpose of what follows it is convenient to use in place of the usual definition of a boundary component (viz. a boundary continuum which is not a proper subset of any connected subset of $\partial\Omega$) an alternate definition. Let $\{\Omega_n\}$ be a sequence of subregions of Ω such that (i) $\Omega_{n+1} \subset \Omega_n$, (ii) the relative boundary $\partial\Omega_n \cap \Omega$ consists of a single closed (rectifiable) Jordan curve, (iii) $\cap_{n=1}^{\infty} \Omega_n = 0$. Two sequences $\{\Omega_n\}$, $\{\Omega_n'\}$ are called equivalent if, for any n, there exist m such that $\Omega_m \subset \Omega_n'$ and $\Omega_m' \subset \Omega_n$. A boundary component of Ω is defined as an equivalence class of $\{\Omega_n\}$. Then it is known that, given any component B of $\partial\Omega$, there exists a sequence $\{\Omega_n\}$ such that $B = \cap_{n=1}^{\infty} \bar{\Omega}_n$ and conversely, given a sequence $\{\Omega_n\}$, the set $\cap_{n=1}^{\infty} \bar{\Omega}_n$ is a component B of $\partial\Omega$ in the usual sense.

Let $w = f(z)$ be a topological mapping of Ω onto a plane region Ω'. Then the present definition establishes a one-to-one correspondence between the boundary components of Ω and Ω'. (See Stoilow [**20**], pp. 85-87; Oikawa [**12**].) It also follows: If Ω is a region in $|z| < 1$, $z = 0 \in \Omega$, and if $|z| = 1$ is a boundary component of Ω, there exists a sequence of closed Jordan curves $C_n \subset \Omega$, such that C_n separates $z = 0$ from $|z| = 1$ and lies in the annulus $1 - \varepsilon_n < |z| < 1$ where $\lim_{n\to\infty} \varepsilon_n = 0$.

existence and uniqueness of the extremal function which preserves the outer boundary component.

THEOREM 3. *Let Ω be a bounded region which contains $z = 0$. Let $\mathscr{F}^*$ be the class of functions $f(z)$ with the properties:*

1. *$f(z)$ is analytic and univalent in Ω.*
2. *$f(0) = 0, f'(0) > 0, |f(z)| \leq 1, z \in \Omega$.*
3. *The outer boundary component of Ω corresponds to the outer boundary component of $f(\Omega)$.*

Then there exists a function $\phi(z) \in \mathscr{F}^*$ for which

$$\phi'(0) = \max_{f \in \mathscr{F}^*} f'(0) \tag{7}$$

and $\phi(z)$ maps Ω onto a circular slit unit disk. $\phi(z)$ is unique.

Proof. The existence of a function $\phi(z)$ which satisfies (7) and the properties 1 and 2 of the class $\mathscr{F}^*$, and which maps Ω onto a circular slit unit disk is seen as in the proof of Theorem 1. The fact that $\phi(z)$ also possesses property 3 of the class $\mathscr{F}^*$ follows from Lemma 5 below. The uniqueness will be proved in Theorem 4.

LEMMA 5. *Suppose $\{f_n(z)\}$ is a sequence of functions analytic and univalent in a bounded region Ω, which contains $z = 0$. Suppose that $f_n(0) = 0, |f_n(z)| \leq 1$, and that under the mapping $w = f_n(z)$ the outer boundary component B of Ω corresponds to the outer boundary component of $f_n(\Omega)$. If $\lim_{n\to\infty} f_n(z) = \phi(z) \not\equiv$ const. in Ω, uniformly on any compact subset, then, under the mapping $w = \phi(z)$, B corresponds to the outer boundary component of $\phi(\Omega)$.*

Proof. We may assume B to be the circle $|z| = 1$. There exists a sequence of closed rectifiable Jordan curves, C_k, such that C_k lies in the annulus $1 - \varepsilon_k < |z| < 1$, $\lim_{k\to\infty} \varepsilon_k = 0$, and such that C_k separates $z = 0$ from $|z| = 1$ (see[6]). If $\Gamma_k^{(n)} = f_n(C_k)$, then as is easily seen $\Gamma_k^{(n)}$ separates $w = 0$ from the outer component B_n' of $f_n(\Omega)$. Thus

$$1 = \frac{1}{2\pi i}\int_{\Gamma_k^{(n)}} \frac{dw}{w} = \frac{1}{2\pi i}\int_{C_k} \frac{f_n'(z)}{f_n(z)}\,dz\,.$$

In the last integral we let $n \to \infty$ and obtain, if $\phi(C_k) = \Gamma_k$,

$$1 = \frac{1}{2\pi i}\int_{C_k} \frac{\phi'(z)}{\phi(z)}\,dz = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{dw}{w}$$

which shows that $\Gamma_k = \phi(C_k)$ separates $w = 0$ from the outer boundary component B' of $\phi(\Omega)$.

Now, if Λ is any component of $\partial\Omega$ different from $|z| = 1$, then it is entirely contained in the interior $I(C_k)$ of C_k for sufficiently large k. Since $z = 0$ may then be connected with some point of Λ by an arc γ in $I(C_k) \cap \Omega$ (except for its endpoint on Λ), the boundary component Λ' corresponding to Λ under the mapping $w = \phi(z)$ must lie in $I(\Gamma_k)$. Thus Γ_k separates Λ' from B'. Hence no component of $\partial\Omega$ except for $\{|z| = 1\}$ can correspond to B'. This proves the lemma.

5. Uniqueness of the extremal function of $\mathscr{F}^*$. Let Ω be a bounded region which contains $z = 0$ and let $\{\Omega_n\}$ be an exhaustion of Ω, i.e., the regions Ω_n have the properties:

1. $z = 0 \in \Omega_n \subset \Omega_{n+1} \subset \Omega$.
2. Ω_n is bounded by n closed rectifiable Jordan curves $C_\nu^{(n)}$, $\nu = 1, 2, \cdots, n$, such that $C_n^{(n)}$ is the outer boundary component of Ω_n.
3. $\bigcup_{\nu=1}^{\infty} \Omega_\nu = \Omega$.

Let $\mathscr{F}^*$ be the class of functions defined in Theorem 3 and $\phi(z)$ an extremal function of $\mathscr{F}^*$. Let $\mathscr{F}_n^*$ be the corresponding class for Ω_n and $\phi_n(z)$ an extremal function of $\mathscr{F}_n^*$. Let Σ_n denote the circular slit disk $\phi_n(\Omega_n)$, and let Σ be $\phi(\Omega)$.

THEOREM 4. $\lim_{n\to\infty} \phi_n(z) = \phi(z)$ *uniformly in any compact set in* Ω. *Hence the solution of the extremal problem* (7) *is unique.*

Proof. The functions $\{\phi_n(z)\}$ form a normal family in any subregion of Ω. Let $\{\phi_{n_p}(z)\}$ be a convergent subsequence, $g(z) = \lim_{p\to\infty} \phi_{n_p}(z)$. If say, $\Omega \subset \{|w| < 1\}$ then $\phi_n'(0) \geq 1$ (the function z being a member of $\mathscr{F}_n^*$.) Hence also $g'(0) \geq 1$, so that $g(z) \not\equiv$ const. and hence univalent in Ω.

We note that $g(z) \in \mathscr{F}^*$. Conditions 1 and 2 are clearly satisfied. To verify Condition 3 we note first: If C is any closed rectifiable Jordan curve in Ω which contains $z = 0$ in its interior, then $\Gamma = g(C)$ contains $w = 0$ in its interior. For sufficiently large n, $C \subset \Omega_n$; consider the subregion of Ω_n which is bounded by C and the contours $C_\nu^{(n)}$ which are contained in $I(C)$. Then

$$1 = \frac{1}{2\pi i}\int_C \frac{\phi_n'(z)}{\phi_n(z)}dz - \frac{1}{2\pi i}\sum_\nu \int_{C_\nu^{(n)}} \frac{\phi_n'(z)}{\phi_n(z)}dz .$$

Since $\phi_n(C_\nu^{(n)})$ is a *slit* it follows that each integral of the last sum is 0. Letting $n \to \infty$ we obtain

$$1 = \frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w}$$

i.e., $w = 0 \in I(\Gamma)$.

Now it is seen as in the proof of Lemma 5 that any component Λ of $\partial\Omega$ which is different from the outer component B will be in the interior of a suitable closed Jordan curve C in Ω which separates $z=0$ from B. If Λ corresponds to Λ' then Λ' will be in the interior of $\Gamma = g(C)$, so that Γ separates Λ' from the outer boundary component of $g(\Omega)$.

Since $g(z) \in \mathscr{F}^*$ we have

$$g'(0) \leq \phi'(0) . \tag{8}$$

We apply now Lemma 1 with $f_n(w)=\phi[\phi_n^{-1}(w)]$, $F_n(z)=\log\{f_n(w)/w\} \equiv \log\{\phi(z)/\phi_n(z)\}$ $(z \in \Omega_n)$, where $F_n(0)=\log\phi'(0)-\log\phi_n'(0)$ is real. We find

$$\exp\left[\frac{1}{2\pi}\iint_{\Omega_n} |F_n'(z)|^2 dxdy\right] \leq \frac{\phi_n'(0)}{\phi'(0)} .$$

Here we take $n=n_k$ and observe that by (8), $\lim_{k\to\infty}\phi_{n_k}'(0)/\phi'(0) \leq 1$. Let $F(z)=\log\{\phi(z)/g(z)\}$, $\Im m F(0)=0$. Then, as $k\to\infty$, we have in any closed subdomain $S \subset \Omega$,

$$\frac{1}{2\pi}\iint_S |F'(z)|^2 dxdy = 0$$

and therefore $g(z) \equiv \phi(z)$.

Since any convergent subsequence of $\{\phi_n\}$ has the limit $\phi(z)$, the whole sequence converges.

COROLLARY. *The domain Σ onto which the extremal function ϕ of $\mathscr{F}^*$ maps Ω has Lebesgue area π. Hence $\partial\Sigma$ has measure 0.*

Proof. By (1'), applied to $f=f_n(w)$ we obtain

$$|f_n'(0)|^2 = \left[\frac{\phi'(0)}{\phi_n'(0)}\right]^2 \leq \frac{A_n}{\pi} = \frac{1}{\pi}\iint_{\Omega_n} |\phi'(z)|^2 dxdy .$$

For $n=n_k$ we find from (8), letting $k\to\infty$,

$$\pi \leq \lim_{k\to\infty}\iint_{\Omega_{n_k}} |\phi'(z)|^2 dxdy = \iint_{\Omega} |\phi'(z)|^2 dxdy .$$

Since the right side does not exceed π the conclusion follows.

6. Extremal slit disks. It is well known that the conformal transformation of a bounded region Ω onto a circular slit disk by means of a function $f \in \mathscr{F}^* = \mathscr{F}^*(\Omega)$ need not be unique.[7] To characterize a

[7] For example, if Ω is a circular slit unit disk with positive (two-dimensional) slit measure, the identity and the extremal function for $\mathscr{F}^*(\Omega)$ provide two distinct mappings of class $\mathscr{F}^*(\Omega)$ onto circular slit disks.

class of disks onto which there is only one (normalized) conformal map possible we introduce the following.

DEFINITION. *Σ is called an extremal circular slit unit disk* (e.s.d.) *if the extremal function $\psi(w)$ of $\mathscr{F}^*(\Sigma)$,* i.e., *the solution of problem* (7) *over Σ, is the identity, $\psi(w) \equiv w$.*[8]

Two consequences of this definition are immediate:

1. *The set S of the slits of an* e.s.d. *Σ has two-dimensional measure* 0. This follows from the Corollary of Theorem 4, applied to the extremal function $\psi(w)$ of $\mathscr{F}^*(\Sigma)$.

2. *If the circular projection E of the slits of a circular slit disk Σ onto a fixed radius has linear measure zero, then Σ is an* e.s.d. For in this case we can apply Lemma 1 to the extremal function $\psi(w)$ of $\mathscr{F}^*(\Sigma)$ and find $\psi'(0) \leq 1$. Since the function $w \in \mathscr{F}^*(\Sigma)$ we have $\psi'(0) \geq 1$ and hence $\psi'(0) = 1$ and, because of the uniqueness of the extremal function, $\psi(w) \equiv w$.

One can characterize an extremal slit disk in terms of extremal length. Suppose the annulus A_q: $0 < q < |z| < 1$ contains all slits of a circular slit unit disk Σ. Let $S_q = A_q \cap \Sigma$ and let $\lambda_q\{\gamma\}$ denote the extremal length of the family of closed rectifiable curves in S_q which separate $|z| = 1$ from $|z| = q$. Denote by $\Lambda_q\{\gamma\}$ the extremal length of the corresponding family $\{\gamma\}$ in A_q, so that $\Lambda_q\{\gamma\} = 2\pi/\log(1/q)$. Then:

THEOREM 5. *A necessary and sufficient condition that Σ be an extremal circular slit disk is that for some $q < 1$, for which A_q contains all slits of Σ,*

$$\lambda_q\{\gamma\} = \Lambda_q\{\gamma\} = \frac{2\pi}{\log\frac{1}{q}} \tag{9}$$[9]

REMARK. If (9) holds for some q it is also true for all p, $0 < p \leq q$. This is easily seen by use of elementary properties of the extremal length, since, for the family $\{\gamma\}$ in the annulus $p < |z| < q$, the extremal length $\Lambda\{\gamma\} = 2\pi/\log(q/p)$.

[8] This is the analogue to Koebe's "Minimaler Schlitzbereich" [**9**] in the case of the parallel slit mapping. Grötzsch [**5**] considered the corresponding domain for the mapping on a circular slit annulus and denoted it as "Normalbereich." He defined the Normalbereich by means of his "Extremalprinzip" [**5**], p. 239 which he formulated first for the parallel slit domain in [**4**]. Our characterization of an e.s.d. below is suggested by Grötzsch' Extremalenprinzip.

[9] Theorem 5 could also be obtained by use of inequalities due to Oikawa [**12**] and Jurchescu [**7**]. Cf. also Strebel [22], and Sakai [23], where similar, but not identical characterizations for mappings onto ***annuli*** are considered.

Proof. Suppose Σ is an extremal slit disk. Since $A_q \supset S_q$ we have

$$\lambda_q\{\gamma\} \geq \Lambda_q\{\gamma\} = \frac{2\pi}{\log\frac{1}{q}}.$$

Let $\{D_n\}$ be an exhaustion of Σ, where D_n is bounded by n rectifiable closed Jordan curves. We assume that the disk $|z| \leq q$ is contained in all D_n. Let $\phi_n(z)$ be the extremal function in $\mathscr{F}^*(D_n) = \mathscr{F}_n^*$; $\phi_n(z)$ maps D_n onto a circular slit unit disk Σ_n with $n-1$ slits. Let $q_n = \max_{|z|=q} |\phi_n(z)|$. Since Σ is an e.s.d., $\phi_n(z) \to z$, uniformly in any compact subset of Σ (Theorem 4) and, therefore, $q_n \to q$ as $n \to \infty$.

Let $S_q^{(n)} = A_q \cap D_n$. If $\lambda_q\{\gamma\}_n$ denotes the extremal length of the family $\{\gamma\}$ of $S_q^{(n)}$, then

$$\lambda_q\{\gamma\} \leq \lambda_q\{\gamma\}_n$$

since $S_q^{(n)} \subset S_q$. Now $\lambda_q\{\gamma\}_n$ may be calculated in the region $\phi_n(S_q^{(n)})$. But this region contains the annulus $q_n < |w| < 1$ minus the slits of Σ_n within this ring. If we denote the extremal length of the family $\{\gamma\}$ confined to this *slit annulus* by $\Lambda_{q_n}\{\gamma\}$ we have

$$\lambda_q\{\gamma\}_n \leq \Lambda_{q_n}\{\gamma\}.$$

From the fundamental properties of the extremal length it follows that the extremal length of the family $\{\gamma\}$ is an annulus slit along a *finite* number of slits is the same as that in the annulus itself. Hence

$$\Lambda_{q_n}\{\gamma\} = \frac{2\pi}{\log\frac{1}{q_n}}.$$

Thus we obtain

$$\frac{2\pi}{\log\frac{1}{q}} \leq \lambda_q\{\gamma\} \leq \frac{2\pi}{\log\frac{1}{q_n}}$$

and letting $n \to \infty$ we find (9).

Suppose now, conversely, that (9) is satisfied. Let $\phi(z)$ be the extremal function in $\mathscr{F}^*(\Sigma)$ and choose $\rho = |\phi'(z)/\phi(z)|$ as a metric in $S_q = A_q \cap \Sigma$. Then, from the definition of extremal length,

$$\lambda_q\{\gamma\} \geq \frac{(2\pi)^2}{\iint\limits_{S_q} \left|\frac{\phi'(z)}{\phi(z)}\right|^2 dxdy} \geq \frac{(2\pi)^2}{\iint\limits_{q'<|w|<1} \frac{dudv}{|w|^2}}$$

where $q' = \min_{|z|=q} |\phi(z)|$. Since $\iint_{q'<|w|<1} dudv/|w|^2 = \int_0^{2\pi}\int_{q'}^1 1/r\, drd\theta =$ $2\pi \log 1/q'$ we obtain

$$\lambda_q\{\gamma\} \geq \frac{2\pi}{\log\frac{1}{q'}} = \frac{2\pi}{\log\frac{1}{q} + \log\frac{q}{q'}}.$$

Suppose that Σ were not an e.s.d. Then $\phi'(0) > 1$. Since (by the Remark above) we may assume q arbitrarily small, we note that $\lim_{q\to 0} \log q/q' = \log 1/\phi'(0) < 0$, and hence, for a suitable $\delta > 0$,

$$\lambda_q\{\gamma\} \geq \frac{2\pi}{\log\frac{1}{q} - \delta} > \frac{2\pi}{\log\frac{1}{q}},$$

which contradicts the present hypothesis.

THEOREM 6. *Let Σ be a circular slit unit disk. Then the conclusion* (1) *of Lemma* 1 *holds for all $f \in \mathscr{F}^*(\Sigma)$ if and only if Σ is an extremal circular slit disk.*

Proof. (a) Suppose Σ is an e.s.d. Consider an exhaustion $\{D_n\}$ of Σ, where D_n is bounded by n closed rectifiable Jordan curves; $z = 0 \in D_n$. Let $\zeta = \phi_n(z)$ be the extremal function of $\mathscr{F}_n^* = \mathscr{F}^*(D_n)$; $\phi_n(z)$ maps D_n conformally onto a circular slit disk Σ_n with $n-1$ slits. We apply Lemma 1 to $f_n(\zeta) = f[\phi_n^{-1}(\zeta)]$ and

$$F_n(z) = \log\frac{f_n(\zeta)}{\zeta} = \log\frac{f(z)}{\phi_n(z)},$$

where $F_n(0) = \log f'(0) - \log\phi_n'(0)$ and $\log\phi_n'(0)$ is real. Let $A_n = \iint_{D_n} |f'(z)|^2 dxdy$; A_n is the area of $f(D_n)$. Then we obtain

$$\exp\left[\frac{1}{2\pi}\iint_{D_n} |F_n'(z)|^2 dxdy\right] \cdot \frac{|f'(0)|}{\phi_n'(0)} \leq \sqrt{\frac{A_n}{\pi}}.$$

Since Σ is an e.s.d., we have by Theorem 4, $\lim_{n\to\infty}\phi_n(z) = z$, uniformly in any compact subset of Σ. Furthermore, using the lower semicontinuity of the Dirichlet integral on the left and the fact that $\lim_{n\to\infty} A_n = A$, the area of $f(\Sigma)$, on the right, we obtain with $F(z) = \log f(z)/z$,

$$\exp\left[\frac{1}{2\pi}\iint_{\Sigma} |F'(z)|^2 dxdy\right] |f'(0)| \leq \sqrt{\frac{A}{\pi}}. \tag{1}$$

(b) Suppose conversely, that (1) is true for every $f \in \mathscr{F}^*(\Sigma)$. Then

we apply (1) to the extremal function ϕ in this class and find

$$\phi'(0) \leq 1 .$$

Since the function $f = z \in \mathscr{F}^*(\Sigma)$, we have $\phi'(0) \geq 1$, hence $\phi'(0) = 1$, and, therefore, in view of the uniqueness of the extremal function, $\phi(z) \equiv z$.

THEOREM 7. *Let Ω be a bounded region which contains $z = 0$. There is at most one conformal map, $f \in \mathscr{F}(\Omega)$, of Ω onto an extremal circular slit unit disk which carries a given component of $\partial\Omega$ into $|w| = 1$.*

Proof. Suppose there were two such mappings, f_1 and f_2, $f_1(\Omega) = \Sigma_1$, $f_2(\Omega) = \Sigma_2$. Then $g = f_1 \circ f_2^{-1} \in \mathscr{F}^*(\Sigma_2)$.

Since Σ_2 is an e.s.d. we have, by Theorem 6, $g'(0) \leq 1$. The same argument applies to $f_2 \circ f_1^{-1}$ and shows that $g'(0) \geq 1$. Hence, $g'(0) = 1$ and $g(w) \equiv w$ or $f_1 \equiv f_2$.

THEOREM 8. *If ϕ is an extremal function in $\mathscr{F}(\Omega)$, then $\phi(\Omega) = \Sigma$ is an extremal slit disk.*

Proof. Let $\psi(w)$ denote the extremal function of $\mathscr{F}^*(\Sigma)$. If Σ were not an e.s.d., then we would have $\psi'(0) > 1$. Then $f = \psi(\phi(z)) \in \mathscr{F}(\Omega)$, but $f'(0) = \psi'(0)\phi'(0) > \phi'(0)$, which contradicts the definition of ϕ.

REMARK. In view of Theorem 2 it is clear that the class of all possible domains $\phi(\Omega)$, produced by an extremal function ϕ of $\mathscr{F}(\Omega)$ is a proper subclass of the class of all extremal slit disks.

7. Mapping onto a radial slit disk. Suppose Ω is a bounded region which has a free boundary component B, not a single isolated point. We may assume that B is the circle $|z| = 1$ and that Ω is contained in $|z| < 1$ and contains an annulus $\rho < |z| < 1$. We suppose also that $z = 0 \in \Omega$.

Let $\mathfrak{S}$ denote the class of functions $g(z)$ with the properties:

1. $g(z)$ is analytic and univalent in Ω and on $|z| = 1$.
2. $g(0) = 0$, $g'(0) > 0$ and $|g(z)| \leq 1$, $z \in \Omega$.
3. $|g(z)| = 1$ for $|z| = 1$.

THEOREM 9. *There exists a unique function $\psi(z) \in \mathfrak{S}$ such that $\psi'(0) = \min_{g \in \mathfrak{S}} g'(0)$; $\psi(z)$ maps Ω conformally onto a radial slit disk of radius one such that $|z| = 1$ corresponds to $|w| = 1$.*[9]

[9] See Sario [**16**] where this mapping is obtained by a different extremal problem. An example by Oikawa [**12**] which is modeled after a construction of Strebel [**21**] indicates that the mapping of an *arbitrary* bounded region onto a radial slit disk is not always possible.

We shall need the following lemma which plays a similar role in the proof of this theorem as Lemma 1 in the case of the circular slit disk mapping.

LEMMA 6. *Let Σ be a radial slit unit disk centered at $z = 0$ such that the closure of the radial projection of the slits onto $|z| = 1$ has measure 0. Suppose $w = f(z)$ is analytic and univalent in Σ, that $f(0) = 0$ and that $f(z)$ maps Σ onto a bounded region Δ in such a manner that $|z| = 1$ corresponds to the outer boundary component of Δ. Let $m = \inf_{|z|\to 1} |f(z)|$, and let A denote the area of Δ. If $F(z) = \log(f(z)/z)$, $F(0) = \log f'(0)$, then*

$$\exp\left[\frac{1}{2\pi}\iint_{\Sigma} |F'(z)|^2 dxdy\right] \cdot m^2 \leq |f'(0)| \sqrt{\frac{A}{\pi}}\,. \tag{10}$$

In particular,

$$m^2 \leq |f'(0)| \sqrt{\frac{A}{\pi}} \tag{11}$$

and the equality sign implies that $F'(z) \equiv 0$, i.e., $f(z) \equiv az$, where a is a constant.

REMARK. If Σ has only a finite number of slits and $|f(z)| \leq M$, $z \in \Sigma$, then (10) implies Rengel's inequality $|f'(0)| \geq m^2/M$ [13]. Again, equality holds only if $f(z) \equiv az$.

Proof. Let E denote closure of the radial projection of the slits onto $|z| = 1$. E is a closed set of measure zero. Consequently, for every $\varepsilon > 0$—and we assume $\varepsilon < m$—there exist a finite number of arcs $\beta_\nu = \{z\,|\,|z| = 1,\ \theta_\nu \leq \theta \leq \theta'_\nu\}$, where $\theta_\nu < \theta'_\nu < \theta_{\nu+1}$, $\nu = 1, 2, \cdots, n$, and $\theta_{n+1} = \theta_1 + 2\pi$, such that

$$E \subset \bigcup_{\nu=1}^{n} \beta_\nu \quad \text{and} \quad \sum_{\nu=1}^{n} (\theta'_\nu - \theta_\nu) < \varepsilon\,. \tag{12}$$

We may assume that $\theta_\nu \notin E$, $\theta'_\nu \notin E$. Let $|z| \leq d$ be a disk contained in Σ and let $\delta > 0$ $(1 - \delta > d)$ be so small that, for $1 - \delta \leq |z| < 1$, $z \in \Sigma$, $|f(z)| \geq m - \varepsilon$.

We denote by c_ν the arc composed of the following three parts: the two segments $\{z\,|\,\theta = \theta_\nu,\ d \leq |z| \leq 1 - \delta\}$, $\{z\,|\,\theta = \theta'_\nu,\ d \leq |z| \leq 1 - \delta\}$ and the arc $\{z\,|\,|z| = d,\ \theta_\nu \leq \theta \leq \theta'_\nu\}$. Let $\gamma_\nu = \{z\,|\,|z| = 1 - \delta,\ \theta'_\nu \leq \theta \leq \theta_{\nu+1}\}$, $\nu = 1, 2, \cdots, n$. Then we define $C_\delta = (\bigcup_{\nu=1}^{n} c_\nu) \cup (\bigcup_{\nu=1}^{n} \gamma_\nu)$; C_δ is a closed Jordan curve in Σ which contains the origin in its interior D_δ, and $F(z)$ is regular in $D_\delta \cup C_\delta$. We assume again at first that $f'(0) = 1$, $F(0) = 0$. We write $F(z) = u + iv$, $f(z) = \rho e^{i\psi}$ and obtain

$$\iint_{D_\delta} |F'(z)|^2 dxdy = \int_{C_\delta} udv = \int_{C_\delta} \log \frac{\rho}{r}(d\psi - d\theta) .$$

Using the fact that $F(0) = 0$ and consequently

$$\Im\mathrm{m}\left\{\int_{C_\delta} \frac{F(z)}{z} dz\right\} = \int_{C_\delta} (\log r d\psi - \log \rho d\theta) = 0$$

we find

$$\iint_{D_\delta} |F'(z)|^2 dxdy = \int_{C_\delta} (\log \rho d\psi - 2 \log \rho d\theta + \log r d\theta) .$$

Since $\int_{C_\delta} \log r d\theta \leq 0$, we have

$$\iint_{D_\delta} |F'(z)|^2 dxdy + 2\int_{C_\delta} \log \rho d\theta \leq \int_{C_\delta} \log \rho d\psi . \tag{13}$$

Let $\rho_0 = \min_{|z|=a} |f(z)|$. Writing $\theta'_\nu - \theta_\nu = |\beta_\nu|$ and using (12) we obtain

$$\begin{aligned} \int_{C_\delta} \log \rho d\theta &\geq \log \rho_0 \sum_{\nu=1}^{n} |\beta_\nu| + \log (m - \varepsilon)\left[2\pi - \sum_{\nu=1}^{n} |\beta_\nu|\right] \\ &\geq 2\pi \log (m - \varepsilon) - \lambda\varepsilon \end{aligned} \tag{14}$$

where λ is a positive constant.

Let $\Gamma_\delta = f(C_\delta)$, $\Delta_\delta = f(D_\delta)$, k the disk $|w| \leq \mu$, where μ is so small that $k \subset \Delta_\delta$. We have

$$\int_{C_\delta} \log \rho d\psi = \int_{\Gamma_\delta} \log \rho d\psi = \iint_{\Delta_\delta - k} \frac{dudv}{|w|^2} + 2\pi \log \mu .$$

Just as in the proof of Lemma 1 we then obtain

$$\int_{C_\delta} \log \rho d\psi \leq 2\pi \log \sqrt{\frac{A}{\pi}} . \tag{15}$$

Thus we have from (13), (14), and (15)

$$\iint_{D_\delta} |F'(z)|^2 dxdy + 4\pi \log (m - \varepsilon) \leq 2\pi \log \sqrt{\frac{A}{\pi}} + 2\lambda\varepsilon .$$

Letting first $\delta \to 0$ and then observing that the resulting inequality holds for every ε, $0 < \varepsilon < m$, we obtain (10) when $f'(0) = 1$.

If $f'(0) \neq 1$, let $a = |f'(0)|$. Then we need only to replace $f(z)$ by $f(z)/a$, A by A/a^2 and m by m/a to obtain (10).

Proof of Theorem 9. Since $\mathfrak{S}$ is a normal family in Ω there exists a sequence $\{g_n(z)\}$, $g_n \in \mathfrak{S}$, which converges to a function $\psi(z)$ and for

which $\lim_{n\to\infty} g_n'(0) = \operatorname{Inf}_{g\in\mathfrak{S}} g'(0)$. Clearly, $\psi(z)$ is analytic in Ω, $\psi(0) = 0$, $|\psi(z)| \leq 1$. We show that $\psi(z) \not\equiv$ constant. Let P_ρ be the annulus $\rho < |z| < 1$ contained in Ω. Then for every r, $\rho < r < 1$, $|g_n(re^{i\theta})|$ assumes the value r. Otherwise the image of $|z| = r$ lies entirely either in the interior or in the exterior of $|z| = r$. In either case the ring onto which $w = g_n(z)$ maps P_r: $r < |z| < 1$ cannot have the module $\log(1/r)$ (which it must have). Because of the uniform convergence of $\{g_n(z)\}$ on $|z| = r$, $|\psi(re^{i\theta})|$ assume the value r somewhere on $|z| = r$. Hence $\psi \not\equiv$ const. Consequently ψ is univalent in Ω, $\psi'(0) > 0$, and $\psi(z) \neq 0$ in P_ρ since $\psi(0) = 0$.

Each $g \in \mathfrak{S}$ may be continued analytically across $|z| = 1$ by the relation

$$g(z) = \frac{1}{\bar{g}\left(\frac{1}{\bar{z}}\right)} \quad \left(1 < |z| < \frac{1}{\rho}\right).$$

If $\rho < r < 1$, the fact that $g_n(z)$ converges uniformly to $\psi(z)$ on $|z| = r$ and that $\psi(z) \neq 0$ implies that, uniformly on $|z| = 1/r$, $\lim_{n\to\infty} g_n(z) = 1/\overline{\psi}(1/\bar{z}) \equiv \psi(z)$.

The uniform convergence of g_n for $|z| = r$ and $|z| = 1/r$ implies that $g_n(z) \to \psi(z)$ uniformly in the ring $r < |z| < 1/r$ and therefore in particular on $|z| = 1$; hence $|\psi(z)| = 1$ for $|z| = 1$.

The function $\psi(z)$ maps, therefore, Ω onto a region Σ in $|w| < 1$ and $|z| = 1$ onto $|w| = 1$. We maintain that all other boundary components of Σ which are not points must be radial slits. For, if Λ is a boundary component of Σ in $|w| < 1$ which is not a slit, we map the doubly connected region Δ consisting of the complement of Λ with respect to $|w| < 1$ onto a radial slit unit disk. By Lemma 6 the mapping function $\zeta = h(w)$ has the property $|h'(0)| < 1$; hence $h[\psi(z)] \in \mathfrak{S}$ and $|h'(0)\psi'(0)| < |\psi'(0)|$ contrary to the construction of $\psi(z)$.

The uniqueness of $\psi(z)$ follows from Theorem 10 below.

8. Uniqueness of the extremal function of $\mathfrak{S}$. Suppose $\{\Omega_n\}$ denotes an exhaustion of the region Ω considered in § 7, of the following type:

1. $z = 0 \in \Omega_n \subset \Omega_{n+1} \subset \Omega$.
2. Each Ω_n is bounded by the circle $|z| = 1$ and $n - 1$ closed Jordan curves $C_\nu^{(n)}$, $\nu = 1, 2, \cdots, n-1$, contained in $|z| < 1$ and exterior each other.
3. $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$.

It follows, in particular, that the annulus P_ρ: $\rho < |z| < 1$, which is in Ω, is also contained in all Ω_n for sufficiently large n.

Let $\mathfrak{S}_n$ denote the family $\mathfrak{S}$ defined in Ω_n and let $\psi_n(z)$ be an extremal function in $\mathfrak{S}_n$, such that $\psi_n'(0) = \min_{g\in\mathfrak{S}_n} g'(0)$; $\psi_n(z)$ maps Ω_n

onto a radial slit disk Σ_n having $n-1$ slits. Let $\psi(z)$ be the function defined in Theorem 9 and let $\Sigma = \psi(\Omega)$.

THEOREM 10. *Uniformly in any compact subset of* Ω, $\lim_{n\to\infty} \psi_n(z) = \psi(z)$. *Hence the extremal function of Theorem* 9 *is unique.*

Proof. Since the functions $\{\psi_n(z)\}$ form a normal family in any fixed $\Omega_m (n \geq m)$, there exists a convergent subsequence $\{\psi_{n_k}\}$. Let $\gamma(z) = \lim_{k\to\infty} \psi_{n_k}(z)$. Just as in the proof of Theorem 9, by applying the argument to a fixed region Ω_m in place of Ω, one shows that $\gamma(z)$ is analytic on the unit circle and that $|\gamma(z)| = 1$ for $|z| = 1$. Thus $\gamma(z) \in \mathfrak{S}$, and we have, therefore,

$$\gamma'(0) \geq \psi'(0) \qquad (>0) . \tag{16}$$

We apply now Lemma 6 to $f_n(w) = \psi(\psi_n^{-1}(w))$,

$$F_n(z) = \log \frac{\psi(\psi_n^{-1}(w))}{w} \equiv \log \frac{\psi(z)}{\psi_n(z)} , \qquad z \in \Omega_n ,$$

where $F_n(0) = \log \psi'(0) \quad \log \psi_n'(0)$ is real. Noting that $m = 1, A \leq \pi$, we obtain

$$\exp\left[\frac{1}{2\pi}\right] \iint_{\Omega_n} |F_n'(z)|^2 dxdy \Big] \leq \frac{\psi'(0)}{\psi_n'(0)} .$$

Let $F(z) = \log\{\psi(z)/\gamma(z)\}$ where $F(0) = \log\{\psi'(0)/\gamma'(0)\}$ is real. Then we find for every compact subset $S \subset \Omega$, as $k \to \infty$, because of (16),

$$\frac{1}{2\pi} \iint_S |F'(z)|^2 dxdy = 0 .$$

Hence $F'(z) \equiv 0$ and consequently $\psi(z) \equiv \gamma(z)$.

Since every convergent subsequence of $\{\psi_n\}$ has the limit ψ the conclusion of the theorem follows.

COROLLARY. *The Lebesgue area of the domain* Σ *onto which the extremal function* $\psi(z)$ *of* $\mathfrak{S}$ *maps* Ω *is* π. *Hence the measure of* $\partial\Sigma$ *is zero.*

The proof is entirely analogous to that of the corollary to Theorem 4.

9. Extremal radial slit disks. Let Σ be a radial slit unit disk such that $|z| = 1$ is an isolated boundary component. Σ shall be called an *extremal radial slit unit disk* (e.r.s.d.) if the extremal function ψ of $\mathfrak{S}(\Sigma)$ is the identity, $\psi(w) \equiv w$.

Clearly, the slit measure of an e.r.s.d. is zero. Furthermore, if the closure of the radial projection of the slits of a radial slit unit disk Σ

has (linear) measure 0, then Σ is an e.r.s.d.

An e.r.s.d. may be characterized in terms of the extremal length of a suitable family $\{\gamma\}$. Let Σ be a radial slit unit disk *with isolated outer boundary*, let A_q: $q < |z| < 1$ be an annulus containing all slits of Σ, and let $S_q = A_q \cap \Sigma$. Suppose $\lambda_q^*\{\gamma\}$ denotes the extremal length of all rectifiable curves γ in S_q (except for the endpoints) which join $|z| = q$ to $|z| = 1$. Let $\Lambda_q^*\{\gamma\}$ be the extremal length of the corresponding family $\{\gamma\}$ in A_q.

THEOREM 11. *Necessary and sufficient that Σ be an* e.r.s.d. *is that $\lambda_q^*\{\gamma\} = \Lambda_q^*\{\gamma\} = 1/2\pi \log 1/q$, for some q, for which A_q contains all slits of Σ.*

The proof is entirely analogous to that of Theorem 5.[10]

Finally, one extends the validity of Lemma 6 to an extremal radial slit disk in a manner parallel to the extension of Lemma 1 to Theorem 6, and one proves by use of this extension:

THEOREM 12. *If $g \in \mathfrak{S} = \mathfrak{S}(\Omega)$ and if $\Sigma = g(\Omega)$ is an extremal radial slit unit disk, then g is the extremal function $\psi \in \mathfrak{S}$.*

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