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1. Introduction. Suppose W is a real $L^2(-\pi, \pi)$ function that is bounded below but not equivalent to a constant function. The *Toeplitz matrix* associated with W is $T_0 = [w_{j-k}]$, $j, k = 0, 1, 2, \cdots$, where

$$(1.1) \qquad w_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\phi) e^{-in\phi}\, d\phi,\ n = 0, \pm 1, \pm 2, \cdots$$

The hermitian matrix T_0 gives rise to a semi-bounded transformation T_1 on complex sequential Hilbert space l_2, and thus the Friedrichs extension T of T_1 is a self-adjoint operator. $T = T(W(\phi))$ is the *Toeplitz operator* associated with W.

In [5], [6] Hartman and Wintner show that the case in which W is not semi-bounded (which we prudently avoid here) presents special difficulty. However for semi-bounded W they prove that

(i) the spectrum of T fills the interval
[ess inf W, ess sup W],
and

(ii) T has no point spectrum.

Thus the spectral measure ([4], p. 58) $E(\cdot)$ of T is such that $\langle E(\cdot)F, F\rangle$ is a nonatomic Borel measure for each $F \in l^2$. If $\langle E(\cdot)F, F\rangle$ is AC (absolutely continuous with respect to Lebesgue measure) for each $F \in l^2$, then we say that T is AC.

Our investigation continues work of C. R. Putnam [11]. He proves that T is AC in each of the following cases:

(i) $W(\phi) = 2\cos n\phi$, $n = 1, 2, \cdots$

(ii) $W(\phi) = 2\sin n\phi$, $n = 1, 2, \cdots$

(iii) Let $a_{jk} = w_{k-j}$ for $k - j \geqq 1$ and $a_{jk} = 0$ otherwise.

Further suppose that the $\{w_n\}$ are real, that $A_0 = [a_{jk}]$ is bounded, and that 0 is not an eigenvalue of the Hankel matrix $[w_{j+k+1}]$, $j, k = 0, 1, 2, \cdots$.

For case (i) Putnam gives a more complete spectral analysis. He applies the perturbation theory propounded in [13] to prove the following result:

1.2 $T(2\cos n\phi)$ is unitarily equivalent to $2T_n(\frac{1}{2}T(2\cos\phi))$. Here T_n is the nth degree Tchebichef polynomial, $n = 1, 2, \cdots$.

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In §§ 2 and 3 we prove that every Toeplitz operator is AC. The method of proof first involves deriving a generating function formula for the resolvent of $\boldsymbol{T}$. This formula appears in the work [2] of Calderon, Spitzer, and Widom. However, we shall offer a different derivation, one that points out an interesting connection between $\boldsymbol{T}$ and the Szegö kernel function. Next we shall apply a result from the Aronszajn-Donoghue [1] theory of exponential representations of analytic functions, and consequently deduce that $\boldsymbol{T}$ is absolutely continuous. We conclude with § 4 where 1.2 is generalized. We elaborate on Putnam's method. By severely restricting W we are able to employ Kato's generalization [7], [8] of [13] to exhibit a multiplication operator $\boldsymbol{M}_{\mathrm{AC}}$ on an L^2 space such that T is unitarily equivalent to $\boldsymbol{M}_{\mathrm{AC}}$.

2. $\boldsymbol{T}$ and the Szegö kernel function. We first set down some notation. We shall ambiguously employ "F" to denote

(a) the element $\{f_n\}_0^\infty$ of l^2;

(b) the element $F(e^{i\phi})$ of $L^2(-\pi, \pi)$ that has the Fourier series $\sum_{n=0}^\infty f_n e^{in\phi}$; and

(c) the holomorphic function $F(u) = \sum_{n=0}^\infty f_n u^n$, $|u| < 1$.

Let $\langle\ ,\ \rangle$ be the l^2 inner product and suppose $*$ is the symbol of complex conjugation, used so $F^*(e^{i\phi}) \sim \sum_{n=0}^\infty f_n^* e^{in\phi}$ and $[F(e^{i\phi})]^* \sim \sum_{n=0}^\infty f_n^* e^{-in\phi}$. Then

$$\langle F, G\rangle = \sum_{n=0}^{\infty} f_n g_n^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\phi})\,[G(e^{i\phi})]^*\,d\phi\ . \tag{2.1}$$

We suppose that u, v are complex numbers such that $|u| < 1$, $|v| < 1$, and define $U = \{u^n\}_0^\infty \in l^2$, $V = \{v^n\}_0^\infty \in l^2$. Note that $U(e^{i\phi}) = (1 - ue^{i\phi})^{-1}$ and $V^*(e^{i\phi}) = (1 - v^*e^{i\phi})^{-1}$.

Select λ so that $1 + \lambda \leqq \operatorname{ess\,inf} W$. Let $l^{2,\lambda}$ be the inner product space formd of elements $F \in l^2$ such that

$$[F, F] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(e^{i\phi})|^2 (W(\phi) - \lambda)\,d\phi < \infty\ .$$

Since

$$[F, F] \geqq \langle F, F\rangle$$

it follows that $l^{2,\lambda}$ is a (complete) Hilbert space. Define the linear functional L_v on $l^{2,\lambda}$ by $L_v(F) = \langle F, V^*\rangle$. L_v is bounded since

$$|L_v(F)|^2 \leqq \langle F, F\rangle\langle V^*, V^*\rangle \leqq [F, F]\langle V^*, V^*\rangle\ .$$

Hence by the Frechet-Riesz representation theorem ([12], p. 61) there exists a unique element $K_v \in l^{2,\lambda}$ such that $[F, K_v] = L_v(F)$. Thus

$$F(v) = \langle F, V^*\rangle = [F, K_v] \tag{2.2}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\phi})[K_v(e^{i\phi})]^*(W(\phi)-\lambda)\,d\phi$$

$$\text{for all } v,\ |v|<1.$$

It follows from 2.2 that $K_v(u) = \langle K_v, U^*\rangle$ is the Szegö kernel function associated with the Hilbert space of holomorphic functions F such that $[F, F]$ is finite. From ([3], p. 51);

$$2.3 \qquad K_v(u) = (1 - uv^*)^{-1}[g(v)]^*\, g(u),$$

where

$$2.4 \qquad g(u) = \exp -\frac{1}{4\pi}\int_{-\pi}^{\pi} \log(W(\phi)-\lambda)(e^{i\phi}+u)(e^{i\phi}-u)^{-1}\,d\phi\ .$$

We next turn our attention to the Toeplitz matrix $\boldsymbol{T}_0$. We define the transformation $\boldsymbol{T}_1$ to be the restriction of $\boldsymbol{T}_0$ to the subset $\mathscr{D}_1$ of l^2 consisting of elements F that have only a finite number of non-zero components. Then if $F \in \mathscr{D}_1$, and δ is the Kronecker symbol,

$$2.5 \qquad \langle(T-\lambda)F, F\rangle = \sum (w_{k-j} - \lambda\delta_{j,k})f_j f_k^* = [F, F]\ .$$

Since $[F, F] \geqq \langle F, F\rangle$ we are in a situation to which the Friedrichs extension theory is applicable ([12], p. 328–333). Upon applying this theory we note that:

(a) There exists a unique self-adjoint operator $\boldsymbol{T}$ that is an extension of $\boldsymbol{T}_1$ and whose domain $\mathscr{D}$ is contained in $l^{2,\lambda}$. $\mathscr{D}$ is a independent of the choice of $\lambda + 1 \leqq \operatorname{ess\,inf} W$. Notice that $\boldsymbol{T}$ is a quite convenient self-adjoint extension of $\boldsymbol{T}_1$ since it preserves the analytic nicety 2.5 for all $F \in \mathscr{D}$.

(b) $(\boldsymbol{T}-\lambda)^{-1}$ is a bounded positive definite operator that maps l^2 into $l^{2,\lambda}$, and furthermore

$$2.6 \qquad \langle F, G\rangle = [F, (\boldsymbol{T}-\lambda)^{-1}G] \text{ for all } G \in l^2 \text{ and } F \in l^{2,\lambda}\ .$$

THEOREM 1. *Suppose* $\lambda + 1 \leqq \operatorname{ess\,inf} W$. *Then* $(\boldsymbol{T}-\lambda)^{-1}$ *exists, is bounded, and* $\langle(\boldsymbol{T}-\lambda)^{-1}V^*, U^*\rangle = K_v(u)$.

Proof. Suppose $F \in l^{2,\lambda}$. Then by 2.1 and 2.2, $\langle F, V^*\rangle = \sum_{n=0}^{\infty} f_n v^n = F(v) = [F, K_v]$. But, by 2.6, $\langle F, V^*\rangle = [F, (\boldsymbol{T}-\lambda)^{-1}V^*]$. Thus $K_v = (\boldsymbol{T}-\lambda)^{-1}V^*$, and $K_v(u) = \langle K_v, U^*\rangle = \langle(\boldsymbol{T}-\lambda)^{-1}V^*, U^*\rangle$, as asserted.

As commented before, Theorem 1 can be derived from results in Calderon, Spitzer, and Widom's paper [2].

3. Exponential representation.

We list some of the results of the Aronszajn-Donoghue theory of exponential representations of holomorphic functions in

THEOREM 2. *Suppose R is a function holomorphic in the upper* *half plane and there having a non-negative imaginary part. Then*:

(i) ([1], p. 325). *There exists a positive measure* μ *and real numbers* $\alpha' \geqq 0$ *and* β *such that*

$$3.1 \qquad R(\lambda) = \alpha'\lambda + \beta + \int_{-\infty}^{\infty} [(t-\lambda)^{-1} - t(t^2+1)^{-1}]\, d\mu\ .$$

$\alpha'\ \beta, \mu$ *are uniquely determined by* R, *and* $(t^2+1)^{-1}$ *is integrable with respect to* μ. *If* $|\,t\,|(t^2+1)^{-1}$ *is integrable with respect to* μ, *then*

$$3.2 \qquad R(\lambda) = \alpha'\lambda + \beta' + \int_{-\infty}^{\infty} (t-\lambda)^{-1}\, d\mu, \ \textit{where}$$

$$\beta' = \beta - \int_{-\infty}^{\infty} t(t^2+1)^{-1}\, d\mu$$

(ii) ([1], p. 331). *There exists a Lebesgue measurable function* α *with* $0 \leqq \alpha \leqq 1$ *and a real number* σ *such that*

$$3.3 \qquad R(\lambda) = \exp\sigma \exp \int_{-\infty}^{\infty} [(t-\lambda)^{-1} - t(t^2+1)^{-1}]\, \alpha(t)\ dt$$

α *is determined by* 3.3 *modulo a set of Lebesgue measure zero.*

(iii) ([1], p. 386). *A sufficient condition for* μ *to be* AC *is that for all real* x

$$3.4 \qquad \omega(x) = \lim\ \{\omega(a, b) : a \uparrow x, b \downarrow x\} < 1,$$

where

$$\omega(a, b) = \sup\ \{\alpha(d) - \alpha(c) : a < c < d < b\}\ .$$

We next reframe 2.3 in a form suitable for application of the preceding theorem. Let χ_t be the characteristic function of $\{\phi : W(\phi) \leqq t, -\pi < \phi \leqq \pi\}$. Put

$$P(\phi, u, v) = \frac{1}{4\pi} [(e^{i\phi} + u)(e^{i\phi} - u)^{-1} + (e^{-i\phi} + v^*)(e^{-i\phi} - v^*)^{-1}]\ ,$$

so if

$$v = re^{i\psi},\ P(\phi, v, v) = \frac{1}{2\pi}(1-r^2)(1 - 2r\cos(\phi - \psi) + r^2)^{-1}$$

is the Poisson kernel. Let

$$\sigma(u, v) = -\frac{1}{2}\int_{-\pi}^{\pi} \log\ [1 + (W(\phi))^2]\, P(\phi, u, v)\, d\phi\ ,$$

and $\alpha(t, u, v) = \int_{-\pi}^{\pi} \chi_t(\phi)\, P(\phi, u, v)\, d\phi$. Notice that $\alpha(\cdot, u, v)$ is of bounded variation, with $\alpha(t, u, v) = 0$ or 1 according to whether $t <$ ess inf W or $t >$ ess sup W respectively. Also note that $\alpha(\cdot, v, v)$ is monotone increasing with $0 \leqq \alpha(t, v, v) \leqq 1$.

LEMMA 1. *If* $\mathfrak{Im}\ \lambda \neq 0$ *or* $\lambda <$ ess inf W, *then*

3.5 $$(1 - uv^*)\langle(\boldsymbol{T} - \lambda)^{-1}V^*, U^*\rangle$$
$$= \exp\ \sigma(u, v) \exp \int_{-\infty}^{\infty} [(t - \lambda)^{-1} - t(t^2 + 1)^{-1}]\, \alpha(t, u, v)\, dt\ .$$

Proof. Temporarily assume that
(*) $\lambda + 1 \leqq$ ess inf W. By 2.3 and Theorem 1

$$(1 - uv^*)\langle(\boldsymbol{T} - \lambda)^{-1}V^*, U^*\rangle = \exp - \int_{-\pi}^{\pi} \log\,(W(\phi) - \lambda)P(\phi, u, v)\, d\phi$$
$$= \exp \sigma(u, v) \exp - \int_{-\pi}^{\pi} \log\,[(W(\phi) - \lambda)((W(\phi))^2 + 1)^{-1/2}]\, P(\phi, u, v)\, d\phi$$
$$= \exp \sigma(u, v) \exp - \int_{-\infty}^{\infty} \log\,[(t - \lambda)(t^2 + 1)^{-1/2}]\, d_t\, \alpha(t, u, v)\,.$$

We integrate by parts to obtain 3.5 under assumption (*). An analytic continuation argument enables us to relax (*).

We now apply Theorem 2.

LEMMA 2. *Suppose* $|v| < 1$. *Then* $\langle\boldsymbol{E}(\cdot)V^*, V^*\rangle$ *is* AC.

Proof. Consider $R(\lambda) = (1 - |v|^2)\langle(\boldsymbol{T} - \lambda)^{-1}V^*, V^*\rangle$. This is a holomorphic function of the type described in Theorem 2. 3.5 assures us that it has the exponential representation 3.3 with $\alpha(t) = \alpha(t, v, v)$. We shall show that α satisfies 3.4 and from this it will follow that $\mu(\cdot) = \langle\boldsymbol{E}(\cdot)V^*, V^*\rangle$ is AC. Now,

$$\omega(a, b) = \sup\left\{\int_{-\pi}^{\pi} [\chi_d(\phi) - \chi_c(\phi)]\, P(\phi, v, v)\, d\phi : a < c < d < b\right\}$$
$$\leqq \int_{-\pi}^{\pi} [\chi_{b-}(\phi) - \chi_{a+}(\phi)]\, P(\phi, v, v)\, d\phi$$

since $P(\cdot, v, v)$ is positive. Thus

$$\omega(x) \leqq \int_{-\pi}^{\pi} [\chi_{x+}(\phi) - \chi_{x-}(\phi)]\, P(\phi, v, v)\, d\phi = h(r, \psi)\,, \quad \text{where } v = re^{i\psi}\,.$$

Since $P(\phi, v, v)$ is the Poisson kernel, h is a non-negative harmonic function in $|v| < 1$. W is not equivalent to a constant, so h is not a constant function. Thus by the maximum principle, $h(r, \psi) < 1$ if $r < 1$. We invoke 3.4 to complete the proof.

Now we can settle

THEOREM 3. $\boldsymbol{T}$ *is* AC.

Proof. From now on let ν be real Lebesgue measure as restricted to the real Borel sets $\mathscr{B}$. Assume $\nu(\varDelta) = 0$. Lemma 2 assures us that if $|v| < 1$. then $\langle\boldsymbol{E}(\varDelta)V^*, V^*\rangle = 0$. Suppose now that $F \in l^2$. We use the Schwarz inequality and the fact that $\boldsymbol{E}(\varDelta)$ is a projection to note that

$$|\langle \boldsymbol{E}(\Delta)V^*, F\rangle| \leqq \|\boldsymbol{E}(\Delta)V^*\| \, \|F\| = [\langle \boldsymbol{E}(\Delta)V^*, \boldsymbol{E}(\Delta)V^*\rangle]^{1/2} \|F\|$$
$$= [\langle \boldsymbol{E}(\Delta)V^*, V^*\rangle]^{1/2} \|F\| = 0 .$$

Thus $\langle \boldsymbol{E}(\Delta)V^*, F\rangle = 0$ for all $v, |v| < 1$. But the set $\{V^* : |v| < 1\}$ is fundamental in l^2 since $\langle G, V^*\rangle = \sum_{n=0}^{\infty} g_n v^n = 0$ for all $v, |v| < 1$ implies that the g_n all vanish. Thus $\langle \boldsymbol{E}(\Delta)F, F\rangle = 0$, and $\boldsymbol{T}$ is AC.

4. Spectral theory. Our principal goal now is to establish a spectral analysis for $\boldsymbol{T}$. More particularly, we wish to exhibit a multiplication operator $\boldsymbol{M}_{\mathrm{AC}}$ on an L^2 space such that $\boldsymbol{M}_{\mathrm{AC}}$ is unitarily equivalent to $\boldsymbol{T}$. However, we were able to achieve this goal only for a small class of $\boldsymbol{T}(W(\phi))$. From now on we assume that W is even and AC, and that the derivative W' of W has an absolutely convergent Fourier series, so $\sum_n |w_n| < \infty$. Our techniques follow those of Putnam [11], but whereas he uses the theory presented by this author in [13], we use T. Kato's generalization [7], [8] of [13]. See also [9] and [10].

We start by discussing some preliminary material that we include here for completeness. A countably-additive function $\boldsymbol{E}$ on $\mathscr{B}$ to projection operators in a Hilbert space $\mathscr{L}$ is AC if $\nu(\Delta) = 0$ implies $\boldsymbol{E}(\Delta) = 0$. $\boldsymbol{E}$ is *singular* if there exists $\beta \in \mathscr{B}$ such that $\nu(\beta) = 0$ but $E(\Delta \cap \beta) = E(\Delta)$ for all $\Delta \in \mathscr{B}$. It is easy to see that a self-adjoint operator $\boldsymbol{M}$ is AC if and only if its spectral measure is AC.

We shall now establish a Lebesgue decomposition theorem for spectral measures as a corollary of the classical version of that theorem.

LEMMA 3. *Suppose $\boldsymbol{E}(\cdot)$ is a spectral measure in a separable Hilbert space $\mathscr{L}$. Then:*

(i) *There exists $\gamma \in \mathscr{B}$ with $\nu(-\gamma) = 0$ and such that*

4.1 $\quad \boldsymbol{E}_{AC}(\cdot) = \boldsymbol{E}(\cdot \cap \gamma)$ *is an* AC *and*

4.2 $\quad \boldsymbol{E}_S(\cdot) = \boldsymbol{E}(\cdot - \gamma)$ *is a singular projection-valued measure.*

(ii) *If $F, G \in \mathscr{L}$, $\Delta \in \mathscr{B}$, and $\boldsymbol{E}.$ is the resolution of the identity associated with $\boldsymbol{E}(\cdot)$, then*

$$\langle \boldsymbol{E}_{\mathrm{AC}}(\Delta)F, G\rangle = \int_\Delta d\langle \boldsymbol{E}_x F, G\rangle / dx \, d\nu .$$

(iii) *The decomposition $\boldsymbol{E}(\cdot) = \boldsymbol{E}_{\mathrm{AC}}(\cdot) + \boldsymbol{E}_S(\cdot)$ of $\boldsymbol{E}(\cdot)$ as the sum of an* AC *and singular measure is unique.*

Proof. Suppose $F, G \in \mathscr{L}$, $\Delta \in \mathscr{B}$. Then since $\langle \boldsymbol{E}. F, G\rangle$ is of bounded variation it has a a derivative a.e. that is ν-summable. Also

$$\int_\Delta d\langle \boldsymbol{E}_x F, F\rangle / dx \, d\nu \leqq \int_\Delta d\langle \boldsymbol{E}(\cdot)F, F\rangle = \leqq \boldsymbol{E}(\Delta)F, F \leqq \|F\|^2 ,$$

so the first term above represents a bounded quadratic form. Thus by ([4], p. 33), $b(F, G) = \int_{\Delta} d\langle \boldsymbol{E}_x F, G\rangle / dx\, d\nu$ is a bounded bilinear functional, so there exists a bounded operator $\boldsymbol{E}_{\mathrm{AC}}(\Delta)$ such that $\langle \boldsymbol{E}_{\mathrm{AC}}(\Delta)F, G\rangle = b(F, G)$ for all F, G. $\boldsymbol{E}_{\mathrm{AC}}(\cdot)$ is clearly countably additive on $\mathscr{B}$, and thus so is $\boldsymbol{E}_S(\cdot) = \boldsymbol{E}(\cdot) - \boldsymbol{E}_{\mathrm{AC}}(\cdot)$.

Let $\{F_j\}_{j=0}^{\infty}$ be a countable dense subset of $\mathscr{L}$. By the classical version of the Lebesgue decomposition theorem as found in ([14], p. 119), corresponding to each pair j, k of non-negative integers there exists $\beta_{j,k} \in B$ such that $\nu(\beta_{j,k}) = 0$ and

$$(*) \qquad \langle \boldsymbol{E}(\Delta)F_j, F_k\rangle = \langle \boldsymbol{E}(\Delta \cap \beta_{j,k})F_j, F_k\rangle + \langle \boldsymbol{E}_{\mathrm{AC}}(\Delta)F_j, F_k\rangle$$

for all $\Delta \in \mathscr{B}$. Let β be the union of all the $\beta_{j,k}$, $j, k = 0, 1, 2, \cdots$. Then $\nu(\beta) = 0$ and (*) holds with $\beta_{j,k}$ replaced by β. Now we pass from the dense subset to all of $\mathscr{L}$. For all $F, G \in \mathscr{L}$, $\Delta \in \mathscr{B}$

$$(**) \qquad \langle \boldsymbol{E}(\Delta)F, G\rangle = \langle \boldsymbol{E}(\Delta \cap \beta)F, G\rangle + \langle \boldsymbol{E}_{\mathrm{AC}}(\Delta)F, G\rangle\,,$$

where the decomposition of the left hand term into singular and AC parts is unique. Put $\gamma = -\beta$. Then 4.2 holds and thus 4.1 is also true. (iii) follows from (**).

It follows from lemma 3 that $\boldsymbol{E}_{\mathrm{AC}}(\cdot) = \boldsymbol{E}(\gamma)\boldsymbol{E}(\cdot)\boldsymbol{E}(\gamma)$ is a spectral measure in the Hilbert space $\boldsymbol{E}(\gamma)\mathscr{L}$. $\boldsymbol{M}_{\mathrm{AC}} = \boldsymbol{E}(\gamma)\boldsymbol{M}\boldsymbol{E}(\gamma)$ is the self-adjoint operator on $\boldsymbol{E}(\gamma)\mathscr{L}$ having this spectral measure. $\boldsymbol{M}_{\mathrm{AC}}$ is obviously AC.

The following simple example will play a role in what happens later. Let W be as before, even, with $\sum_n |w_n| < \infty$. Let $\boldsymbol{M}$ be the multiplication operator that maps any $F \in L^2(0, \pi) = \mathscr{L}$ into $W\cdot F \in \mathscr{L}$. Let $\chi(\Delta)$ be the characteristic function of $\{\phi : W(\phi) \in \Delta : 0 \le \phi \le \pi\}$. Since

$$\langle \boldsymbol{M}F, F\rangle = \frac{1}{\pi}\int_0^{\pi} W(\phi)|F(\phi)|^2\, d\phi = \int_{-\infty}^{\infty} t\, d_t \frac{1}{\pi}\int_0^{\pi} \chi(\Delta)(\phi)|F(\phi)|^2\, d\phi$$

it follows that the spectral measure $\boldsymbol{E}(\cdot)$ of $\boldsymbol{M}$ is defined by $\boldsymbol{E}(\Delta)F = \chi(\Delta)\cdot F$. Lemma 3 guarantees the existence of $\gamma \in \mathscr{B}$ such the $1/\pi \int_0^{\pi} \chi(\gamma)\chi(\cdot)(\phi)|F(\phi)|^2\, d\phi$ is AC for all $F \in \mathscr{L}$, while $\boldsymbol{E}(\cdot - \gamma)$ is singular. $\boldsymbol{E}(\gamma)\mathscr{L}$ can be identified with the Hilbert space $L^2(A)$, where $F \in L^2(A)$ if and only if $\|F\|_A < \infty$, where

$$4.3 \qquad \|F\|_A = \left[\frac{1}{\pi}\int_0^{\pi} \chi(\gamma)\, |F(\phi)|^2\, d\phi\right]^{1/2} = \left[\frac{1}{\pi}\int_A |F(\phi)|^2\, d\phi\right]^{1/2},$$

and

$$A = \{\phi : W(\phi) \in \gamma,\ \ 0 \leqq \phi \leqq \pi\}\ .$$

Similarly $\boldsymbol{M}_{\mathrm{AC}}$ can be considered to be the mapping that takes any $F \in L^2(A)$ into $W\cdot F \in L^2(A)$.

Another concept that we shall have cause to use is that of trace class. As is usual, a bounded operator on l^2 is identified with its matrix representation. A matrix $H = [w_{j,k}]$, $i, j = 0, 1, 2, \cdots$ belongs to the *Schmidt-Hilbert class* SH if $\sum_{j,k=0}^{\infty}|w_{j,k}|^2 < \infty$. H belongs to the *trace class* TC if $H \in$ SH and $\| H \|_1 < \infty$, where $\| H \|_1$ is the sum of the absolute values of the eigenvalues of H repeated according to multiplicity.

As an example we treat the Hankel matrix $H = [w_{j+k+2}]$. As proved in [5], $H \in$ SH if and only if $\sum_{n=1}^{\infty} n \mid w_{n+1} \mid^2 < \infty$. This follows from the equality $\sum_{j,k=0}^{\infty}|w_{j+k+2}|^2 = \sum_{n=1}^{\infty} n|w_{n+1}|^2$, and gives a necessary condition that $H \in$ TC. Now, define $H_n = [\delta_{j+k+2,n}]$. Then $H = \sum_{n=2}^{\infty} w_n H_n$. Since $\|H_n\|_1 \leqq n$ it follows that $\|H\|_1 \leqq \sum_{n=2}^{\infty}|w_n| \, \|H_n\|_1 \leqq \sum_{n=2}^{\infty} n|w_n|$. Thus a sufficient condition $H \in$ TC is that W be AC such that W' has an absolutely convergent Fourier series. This, of course, is part of our standing hypothesis on W for this section. We do not know a useful necessary and sufficient condition for a Hankel matrix to belong to TC.

Hankel matrices enter into our picture, following an idea of Putnam's, via the following

LEMMA 4. *Let H be as as in the above example. Let $S = [s_{j,k}]$, where $s_{j,k} = 2/\pi \int_0^{\pi} W(\phi) \sin (j + 1)\phi \sin (k + 1)\phi \, d\phi$, $j, k = 0, 1, 2, \cdots$. Then $T - S = H$.*

Proof.
$$w_{j-k} - s_{j,k} = \frac{1}{\pi}\int_0^{\pi} W(\phi) \cos \, (j - k)\phi \, d\phi$$
$$- \frac{2}{\pi}\int_0^{\pi} W(\phi) \sin (j + 1)\phi \sin (k + 1)\phi \, d\phi \, .$$
$$= \frac{1}{\pi}\int_0^{\pi} W(\phi) \cos (j + k + 2) \, d\phi \, .$$

We can now state a specialization of Kato's theorem in a form suitable for our application. It is understood that in the statement T and S need not necessarily be the operators we have already defined.

THEOREM 5. (Kato). *Suppose T and S are self-adjoint operators on a separable Hilbert space $\mathscr{L}$ such that $T - S = H \in$ TC and T is* AC. *Let γ and $E(\cdot) = E_{AC}(\cdot) + E_S(\cdot)$ be the Borel set and decomposition respectively guaranteed by Lemma 3. Then*

(i) *as $t \to \infty$,* $\exp(itT) \exp(-itS) E(\gamma)$ *converges strongly to an isometric mapping U of $E(\gamma)\mathscr{L}$ onto $\mathscr{L}$.*

(ii) *U^{-1} is the strong limit as $t \to \infty$ of* $\exp(itS) \exp(-itT)$.

(iii) *The self-adjoint operator $S_{AC} = E(\gamma) S E(\gamma)$ on $E(\gamma)\mathscr{L}$ is unitarily equivalent to T,* with $I = U S_{AC} U^{-1}$.

From this follows the following spectral analysis theorem for T.

THEOREM 6. *Suppose W is a real even* AC *function on $(-\pi, \pi)$ whose derivative W' has an absolutely convergent Fourier series. Then* *the Toeplitz operator* $\boldsymbol{T}(W(\phi))$ *is unitarily equivalent to to the multiplication operator* $\boldsymbol{M}_{\mathrm{AC}} : f \to W \cdot f$ *on* $L^2(A)$ (*see* 4.3).

Proof. The hypotheses of Theorem 5 are satisfied via Lemma 4, the discussion following Lemma 3, and Theorem 3. Thus $\boldsymbol{T}$ is unitarily equivalent to $\boldsymbol{S}_{\mathrm{AC}}$. Since $\{f_n\}_0^\infty \to 2^{1/2} \sum_{n=0}^\infty f_n \sin(n+1)\phi$ is an isometry of l^2 onto $L^2(0, \pi)$, it follows that $\boldsymbol{S}_{\mathrm{AC}}$ is unitarily equivalent to $\boldsymbol{M}_{\mathrm{AC}}$. Thus $\boldsymbol{T}$ is unitarily equivalent to $\boldsymbol{M}_{\mathrm{AC}}$.

COROLLARY 1. *Suppose* $W(\phi) = w_0 + 2\sum_1^m w_n \cos n\phi$, *where the* w_n *are real and* m *is a positive integer. Then* $\boldsymbol{T}(W(\phi))$ *is unitarily equivalent to the multiplication operator* $\boldsymbol{M} : f \to W \cdot f$ *on* $L^2(0, \pi)$.

Proof. In this case $\boldsymbol{M} = \boldsymbol{M}_{\mathrm{AC}}$. (See Putnam [**11**], p. 522). Now use Theorem 6.

If W' is AC and $W'' \in L^2(0, \pi)$ then $\sum_n |w_n| < \infty$. Hence a W satisfying Theorem 6 can haye intervals of constancy. If such is the case, then $\boldsymbol{M}$ has an infinite number of eigenvectors. Thus one cannot validly replace "$\boldsymbol{M}_{\mathrm{AC}}$" and "$L^2(A)$" by "$\boldsymbol{M}$" and "$L^2(0, \pi)$" respectively in the statement of Theorem 6, since $\boldsymbol{T}$ has no point spectra.

We can easily deduce 1.2 from Corollary 1. $\boldsymbol{T}(W(2\cos n\phi))$ is unitarily equivalent to multiplication by $2\cos n\phi$ on $L^2(0, \pi)$, $n = 1, 2, \cdots$, and hence to $2\cos\,(n \text{ arc } \cos \frac{1}{2}\boldsymbol{T}(2\cos\phi)) = 2\, T_n(\frac{1}{2}\boldsymbol{T}(2\cos\phi))$ on l^2.

It would be of great interest to evaluate the limits in Theorem 5 (ii) and (iii) so one could exhibit the unitary transformation of Theorem 6. One could then have a super-abundance of new unitary operators. We pose this as an unsolved problem.

5. Appendix. C. R. Putnam has extended the theory he set forth in [**11**] in his recent article "*On Toeplitz matrices, absolute continuity and unitary equivalence*", Pacific J. Math., **9** (1959), 837–846. He proves that $\boldsymbol{T}$ is AC provided that $\boldsymbol{A}_0$ is bounded and $\boldsymbol{M} = \boldsymbol{M}_{\mathrm{AC}}$. whence, using [**13**], he proves Theorem 6 under the added hypothesis that $\boldsymbol{M} = \boldsymbol{M}_{\mathrm{AC}}$.

It is interesting to compare our proof that $\boldsymbol{T}$ is AC with Putnam's weaker version of that result. He applies his abstract theory of commutators, while we exhibit the resolvent of $\boldsymbol{T}$ and employ the rather deep function-theoretic results of [**1**].

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