THE ABSOLUTE CONTINUITY OF TOEPLITZ’S MATRICES

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1. Introduction. Suppose \( W \) is a real \( L^2(-\pi, \pi) \) function that is bounded below but not equivalent to a constant function. The Toeplitz matrix associated with \( W \) is \( T_0 = [w_{j-k}], j, k = 0, 1, 2, \cdots \), where

\[
(1.1) \quad w_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\phi) e^{-i n\phi} d\phi, \quad n = 0, \pm 1, \pm 2, \cdots
\]

The hermitian matrix \( T_0 \) gives rise to a semi-bounded transformation \( T_1 \) on complex sequential Hilbert space \( l_2 \), and thus the Friedrichs extension \( T \) of \( T_1 \) is a self-adjoint operator. \( T = T(W(\phi)) \) is the Toeplitz operator associated with \( W \).

In [5], [6] Hartman and Wintner show that the case in which \( W \) is not semi-bounded (which we prudently avoid here) presents special difficulty. However for semi-bounded \( W \) they prove that

(i) the spectrum of \( T \) fills the interval \([\text{ess inf } W, \text{ess sup } W]\),

and

(ii) \( T \) has no point spectrum.

Thus the spectral measure ([4], p. 58) \( E(\cdot) \) of \( T \) is such that \( \langle E(\cdot)F', F' \rangle \) is a nonatomic Borel measure for each \( F' \in \ell^2 \). If \( \langle E(\cdot)F', F' \rangle \) is AC (absolutely continuous with respect to Lebesgue measure) for each \( F' \in \ell^2 \), then we say that \( T \) is AC.

Our investigation continues work of C. R. Putnam [11]. He proves that \( T \) is AC in each of the following cases:

(i) \( W(\phi) = 2 \cos n\phi, \quad n = 1, 2, \cdots \)

(ii) \( W(\phi) = 2 \sin n\phi, \quad n = 1, 2, \cdots \)

(iii) Let \( a_{jk} = w_{k-j} \) for \( k - j \geq 1 \) and \( a_{jk} = 0 \) otherwise.

Further suppose that the \( \{w_n\} \) are real, that \( A_0 = [a_{jk}] \) is bounded, and that 0 is not an eigenvalue of the Hankel matrix \( [w_{j+k+1}], j, k = 0, 1, 2, \cdots \).

For case (i) Putnam gives a more complete spectral analysis. He applies the perturbation theory propounded in [13] to prove the following result:

1.2 \( T(2 \cos n\phi) \) is unitarily equivalent to \( 2T_n(\frac{1}{2} T(2 \cos \phi)) \). Here \( T_n \) is the \( n \)th degree Tchebichef polynomial, \( n = 1, 2, \cdots \).

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In §§ 2 and 3 we prove that every Toeplitz operator is AC. The method of proof first involves deriving a generating formula for the resolvent of $T$. This formula appears in the work [2] of Calderon, Spitzer, and Widom. However, we shall offer a different derivation, one that points out an interesting connection between $T$ and the Szegö kernel function. Next we shall apply a result from the Aronszajn-Donoghue [1] theory of exponential representations of analytic functions, and consequently deduce that $T$ is absolutely continuous. We conclude with § 4 where 1.2 is generalized. We elaborate on Putnam's method. By severely restricting $W$ we are able to employ Kato's generalization [7], [8] of [13] to exhibit a multiplication operator $M_{AC}$ on an $L^2$ space such that $T$ is unitarily equivalent to $M_{AC}$.

2. $T$ and the Szegö kernel function. We first set down some notation. We shall ambiguously employ "$F$" to denote

(a) the element $\{f_n\}_0^\infty$ of $l^2$;
(b) the element $F(e^{i\phi})$ of $L^2(-\pi, \pi)$ that has the Fourier series $\sum_{n=0}^\infty f_n e^{in\phi}$; and
(c) the holomorphic function $F(u) = \sum_{n=0}^\infty f_n u^n$, $|u| < 1$.

Let $\langle , \rangle$ be the $l^2$ inner product and suppose $*$ is the symbol of complex conjugation, used so $F^*(e^{i\phi}) \sim \sum_{n=0}^\infty f_n^* e^{in\phi}$ and $[F(e^{i\phi})]^* \sim \sum_{n=0}^\infty f_n^* e^{-in\phi}$. Then

$$(2.1) \quad \langle F, G \rangle = \sum_{n=0}^\infty f_n g_n^* = \frac{1}{2\pi} \int_{-\pi}^\pi F(e^{i\phi}) [G(e^{i\phi})]^* d\phi .$$

We suppose that $u, v$ are complex numbers such that $|u| < 1$, $|v| < 1$, and define $U = \{u^n\}_0^\infty \in l^2$, $V = \{v^n\}_0^\infty \in l^2$. Note that $U(e^{i\phi}) = (1 - ue^{i\phi})^{-1}$ and $V^*(e^{i\phi}) = (1 - v e^{i\phi})^{-1}$.

Select $\lambda$ so that $1 + \lambda \leq \text{ess inf } W$. Let $l^{2,\lambda}$ be the inner product space formed of elements $F \in l^2$ such that

$$[F, F] = \frac{1}{2\pi} \int_{-\pi}^\pi |F(e^{i\phi})|^2 (W(\phi) - \lambda) d\phi < \infty .$$

Since

$$[F, F] \geq \langle F, F \rangle$$

it follows that $l^{2,\lambda}$ is a (complete) Hilbert space. Define the linear functional $L_v$ on $l^{2,\lambda}$ by $L_v(F) = \langle F, V^* \rangle$. $L_v$ is bounded since

$$|L_v(F)|^2 \leq \langle F, F \rangle \langle V^*, V^* \rangle \leq [F, F] \langle V^*, V^* \rangle .$$

Hence by the Frechet-Riesz representation theorem ([12], p. 61) there exists a unique element $K_v \in l^{2,\lambda}$ such that $[F, K_v] = L_v(F)$. Thus

2.2

$$F(v) = \langle F, V^* \rangle = [F, K_v]$$
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\phi})[K_{\phi}(e^{i\phi})]^*(W(\phi) - \lambda) \, d\phi
\]
for all \( v, |v| < 1 \).

It follows from 2.2 that \( K_{\phi}(u) = \langle K_{\phi}, U^* \rangle \) is the Szegö kernel function associated with the Hilbert space of holomorphic functions \( F \) such that \([F, F]\) is finite. From ([3], p. 51);

\[
K_{\phi}(u) = (1 - uv^*)^{-1} [g(v)]^* g(u),
\]
where

\[
g(u) = \exp -\frac{1}{4\pi} \int_{-\pi}^{\pi} \log (W(\phi) - \lambda)(e^{i\phi} + u)(e^{i\phi} - u)^{-1} \, d\phi.
\]

We next turn our attention to the Toeplitz matrix \( T_0 \). We define the transformation \( T_1 \) to be the restriction of \( T_0 \) to the subset \( \mathcal{D}_1 \) of \( \ell^2 \) consisting of elements \( F \) that have only a finite number of non-zero components. Then if \( F \in \mathcal{D}_1 \) and \( \delta \) is the Kronecker symbol,

\[
\langle (T - \lambda)F, F \rangle = \sum (w_{k-j} - \lambda \delta_{j,k}) f_k f_k^* = [F, F].
\]

Since \([F, F] \geq \langle F, F \rangle\) we are in a situation to which the Friedrichs extension theory is applicable ([12], p. 328–333). Upon applying this theory we note that:

(a) There exists a unique self-adjoint operator \( T \) that is an extension of \( T_1 \) and whose domain \( \mathcal{D} \) is contained in \( \ell^{2,\lambda} \). \( \mathcal{D} \) is independent of the choice of \( \lambda + 1 \leq \text{ess inf } W \). Notice that \( T \) is a quite convenient self-adjoint extension of \( T_1 \) since it preserves the analytic nicety 2.5 for all \( F \in \mathcal{D} \).

(b) \((T - \lambda)^{-1}\) is a bounded positive definite operator that maps \( \ell^2 \) into \( \ell^{2,\lambda} \), and furthermore

\[
\langle F, G \rangle = [F, (T - \lambda)^{-1}G] \text{ for all } G \in \ell^2 \text{ and } F \in \ell^{2,\lambda}.
\]

**Theorem 1.** Suppose \( \lambda + 1 \leq \text{ess inf } W \). Then \((T - \lambda)^{-1}\) exists, is bounded, and \(\langle (T - \lambda)^{-1}V^*, U^* \rangle = K_\phi(u)\).

**Proof.** Suppose \( F \in \ell^{2,\lambda} \). Then by 2.1 and 2.2, \(\langle F, V^* \rangle = \sum_{m=0}^{\infty} f_n v^* = F(v) = [F, K_\phi]\). But, by 2.6, \(\langle F, V^* \rangle = [F, (T - \lambda)^{-1}V^*]\). Thus \(K_\phi = (T - \lambda)^{-1}V^*\), and \(K_\phi(u) = \langle K_\phi, U^* \rangle = \langle (T - \lambda)^{-1}V^*, U^* \rangle\), as asserted.

As commented before, Theorem 1 can be derived from results in Calderon, Spitzer, and Widom’s paper [2].

**3. Exponential representation.** We list some of the results of the Aronszajn-Donoghue theory of exponential representations of holomorphic functions in

**Theorem 2.** Suppose \( R \) is a function holomorphic in the upper
half plane and there having a non-negative imaginary part. Then:

(i) ([1], p. 325). There exists a positive measure \( \mu \) and real numbers \( \alpha' \geq 0 \) and \( \beta \) such that

\[
R(\lambda) = \alpha' \lambda + \beta + \int_{-\infty}^{\infty} \left[ (t - \lambda)^{-1} - t(t^2 + 1)^{-1} \right] d\mu .
\]

\( \alpha', \beta, \mu \) are uniquely determined by \( R \), and \( (t^2 + 1)^{-1} \) is integrable with respect to \( \mu \). If \( |t| (t^2 + 1)^{-1} \) is integrable with respect to \( \mu \), then

\[
R(\lambda) = \alpha' \lambda + \beta' + \int_{-\infty}^{\infty} (t - \lambda)^{-1} d\mu ,
\]

where

\[
\beta' = \beta - \int_{-\infty}^{\infty} t(t^2 + 1)^{-1} d\mu .
\]

(ii) ([1], p. 331). There exists a Lebesgue measurable function \( \alpha \) with \( 0 \leq \alpha \leq 1 \) and a real number \( \sigma \) such that

\[
R(\lambda) = \exp \sigma \exp \int_{-\infty}^{\infty} \left[ (t - \lambda)^{-1} - t(t^2 + 1)^{-1} \right] \alpha(t) dt
\]

\( \alpha \) is determined by 3.3 modulo a set of Lebesgue measure zero.

(iii) ([1], p. 386). A sufficient condition for \( \mu \) to be \( \text{AC} \) is that for all real \( x \)

\[
\omega(x) = \lim \{ \omega(a, b) : a \uparrow x, b \downarrow x \} < 1 ,
\]

where

\[
\omega(a, b) = \sup \{ \alpha(d) - \alpha(c) : a < c < d < b \} .
\]

We next reframe 2.3 in a form suitable for application of the preceding theorem. Let \( \chi \) be the characteristic function of \( \{ \phi : W(\phi) \leq t, \ -\pi < \phi \leq \pi \} \). Put

\[
P(\phi, u, v) = \frac{1}{4\pi} \left[ (e^{i\phi} + u)(e^{i\phi} - u)^{-1} + (e^{-i\phi} + v^*)(e^{-i\phi} - v^*)^{-1} \right] ,
\]

so if

\[
v = re^{i\psi} ,
\]

\[
P(\phi, v, v) = \frac{1}{2\pi} (1 - r^2)(1 - 2r \cos (\phi - \psi) + r^2)^{-1}
\]

is the Poisson kernel. Let

\[
\sigma(u, v) = - \frac{1}{2} \int_{-\pi}^{\pi} \log [1 + (W(\phi))^2] P(\phi, u, v) d\phi ,
\]

and \( \alpha(t, u, v) = \int_{-\pi}^{\pi} \chi(\phi) P(\phi, u, v) d\phi \). Notice that \( \alpha(\cdot, u, v) \) is of bounded variation, with \( \alpha(t, u, v) = 0 \) or \( 1 \) according to whether \( t < \text{ess inf } W \) or \( t > \text{ess sup } W \) respectively. Also note that \( \alpha(\cdot, v, v) \) is monotone increasing with \( 0 \leq \alpha(t, v, v) \leq 1 \).

**Lemma 1.** If \( \Im \lambda \neq 0 \) or \( \lambda < \text{ess inf } W \), then
3.5 \( (1 - uv^*)(T - \lambda)^{-1}V^*, U^* \)
\[= \exp \sigma(u, v) \exp \int_{-\infty}^{\infty} \left[ (t - \lambda)^{-1} - t(t^2 + 1)^{-1} \right] \alpha(t, u, v) \, dt. \]

**Proof.** Temporarily assume that (*) \( \lambda + 1 \leq \text{ess inf } W \). By 2.3 and Theorem 1
\[ (1 - uv^*)(T - \lambda)^{-1}V^*, U^* \] \[= \exp \sigma(u, v) \exp \int_{-\pi}^{\pi} \log (W(\phi) - \lambda)P(\phi, u, v) \, d\phi \]
\[= \exp \sigma(u, v) \exp \int_{-\pi}^{\pi} \log [(W(\phi) - \lambda)((W(\phi))^2 + 1)^{-1/2}] P(\phi, u, v) \, d\phi \]
\[= \exp \sigma(u, v) \exp \int_{-\infty}^{\infty} \log [(t - \lambda)(t^2 + 1)^{-1/2}] \, d\alpha(t, u, v). \]

We integrate by parts to obtain 3.5 under assumption (*). An analytic continuation argument enables us to relax (*).

We now apply Theorem 2.

**Lemma 2.** Suppose \( |v| < 1 \). Then \( \langle E(\cdot) V^*, V^* \rangle \) is AC.

**Proof.** Consider \( R(\lambda) = (1 - |v|^2) \langle (T - \lambda)^{-1}V^*, V^* \rangle \). This is a holomorphic function of the type described in Theorem 2. 3.5 assures us that it has the exponential representation 3.3 with \( \alpha(t) = \alpha(t, u, v) \).

We shall show that \( \alpha \) satisfies 3.4 and from this it will follow that \( \mu(\cdot) = \langle E(\cdot) V^*, V^* \rangle \) is AC. Now,
\[ \omega(a, b) = \sup \left\{ \int_{-\pi}^{\pi} [\chi_a(\phi) - \chi_c(\phi)] P(\phi, v, v) \, d\phi : a < c < d < b \right\} \]
\[\leq \int_{-\pi}^{\pi} [\chi_b(\phi) - \chi_a(\phi)] P(\phi, v, v) \, d\phi \]

since \( P(\cdot, v, v) \) is positive. Thus
\[ \omega(x) \leq \int_{-\pi}^{\pi} [\chi_x(\phi) - \chi_{x-}(\phi)] P(\phi, v, v) \, d\phi = h(r, \psi), \] where \( v = re^{i\psi} \).

Since \( P(\phi, v, v) \) is the Poisson kernel, \( h \) is a non-negative harmonic function in \( |v| < 1 \). \( W \) is not equivalent to a constant, so \( h \) is not a constant function. Thus by the maximum principle, \( h(r, \psi) < 1 \) if \( r < 1 \). We invoke 3.4 to complete the proof.

Now we can settle

**Theorem 3.** \( T \) is AC.

**Proof.** From now on let \( \nu \) be real Lebesgue measure as restricted to the real Borel sets \( \mathcal{B} \). Assume \( \nu(\Delta) = 0 \). Lemma 2 assures us that if \( |v| < 1 \) then \( \langle E(\Delta)^V, V^* \rangle = 0 \). Suppose now that \( F \in l^2 \). We use the Schwarz inequality and the fact that \( E(\Delta) \) is a projection to
note that
\[ | \langle E(\Delta)V^*, F \rangle | \leq \| E(\Delta)V^* \| \| F \| = \left| \langle E(\Delta)V^*, E(\Delta)V^* \rangle \right|^{1/2} \| F \| = \left| \langle E(\Delta)V^*, V^* \rangle \right|^{1/2} \| F \| = 0. \]

Thus \( \langle E(\Delta)V^*, F \rangle = 0 \) for all \( v, |v| < 1 \). But the set \( \{ V^* : |v| < 1 \} \) is fundamental in \( L^2 \) since \( \langle G, V^* \rangle = \sum_{n=0}^{\infty} g_n v^n = 0 \) for all \( v, |v| < 1 \) implies that the \( g_n \) all vanish. Thus \( \langle E(\Delta)F, F \rangle = 0 \), and \( T \) is AC.

4. Spectral theory. Our principal goal now is to establish a spectral analysis for \( T \). More particularly, we wish to exhibit a multiplication operator \( M_{AC} \) on an \( L^2 \) space such that \( M_{AC} \) is unitarily equivalent to \( T \). However, we were able to achieve this goal only for a small class of \( T(W(\phi)) \). From now on we assume that \( W \) is even and AC, and that the derivative \( W' \) of \( W \) has an absolutely convergent Fourier series, so \( \sum_n |\omega_n| < \infty \). Our techniques follow those of Putnam [11], but whereas he uses the theory presented by this author in [13], we use T. Kato's generalization [7], [8] of [13]. See also [9] and [10].

We start by discussing some preliminary material that we include here for completeness. A countably-additive function \( E \) on \( \mathcal{B} \) to projection operators in a Hilbert space \( \mathcal{L} \) is AC if \( \nu(\Delta) = 0 \) implies \( E(\Delta) = 0 \). \( E \) is singular if there exists \( \beta \in \mathcal{B} \) such that \( \nu(\beta) = 0 \) but \( E(\Delta \cap \beta) = E(\Delta) \) for all \( \Delta \in \mathcal{B} \). It is easy to see that a self-adjoint operator \( M \) is AC if and only if its spectral measure is AC.

We shall now establish a Lebesgue decomposition theorem for spectral measures as a corollary of the classical version of that theorem.

**Lemma 3.** Suppose \( E(\cdot) \) is a spectral measure in a separable Hilbert space \( \mathcal{L} \). Then:

(i) There exists \( \gamma \in \mathcal{B} \) with \( \nu(\cdot - \gamma) = 0 \) and such that

4.1 \( E_{AC}(\cdot) = E(\cdot \cap \gamma) \) is an AC and

4.2 \( E_{\delta}(\cdot) = E(\cdot - \gamma) \) is a singular projection-valued measure.

(ii) If \( F, G \in \mathcal{L}, \Delta \in \mathcal{B} \), and \( E \) is the resolution of the identity associated with \( E(\cdot) \), then

\[ \langle E_{AC}(\Delta)F, G \rangle = \int_{\mathcal{A}} \, d \langle E_{\delta}F, G \rangle / \, dx \, dv. \]

(iii) The decomposition \( E(\cdot) = E_{AC}(\cdot) + E_{\delta}(\cdot) \) of \( E(\cdot) \) as the sum of an AC and singular measure is unique.

**Proof.** Suppose \( F, G \in \mathcal{L}, \Delta \in \mathcal{B} \). Then since \( \langle E(\cdot)F, G \rangle \) is of bounded variation it has a a derivative a.e. that is \( \nu \)-summable. Also

\[ \int_{\mathcal{A}} \, d \langle E_{\delta}F, F \rangle / \, dx \, dv \leq \int_{\mathcal{A}} \, d \langle E(\cdot)F, F \rangle = \leq E(\Delta)F, F \leq \| F \|^2, \]
so the first term above represents a bounded quadratic form. Thus by ([4], p. 33), \( b(F, G) = \int d\langle E_x F, G \rangle dx d\nu \) is a bounded bilinear functional, so there exists a bounded operator \( E_{AC}(\Delta) \) such that \( \langle E_{AC}(\Delta) F, G \rangle = b(F, G) \) for all \( F, G \). \( E_{AC}(\cdot) \) is clearly countably additive on \( \mathcal{B} \), and thus so is \( E_0(\cdot) = E(\cdot) - E_{AC}(\cdot) \).

Let \( \{F_j\}_{j=0}^{\infty} \) be a countable dense subset of \( \mathcal{L} \). By the classical version of the Lebesgue decomposition theorem as found in ([14], p. 119), corresponding to each pair \( j, k \) of non-negative integers there exists \( \beta_{j,k} \in B \) such that \( \nu(\beta_{j,k}) = 0 \) and

\[
\langle E(\Delta)F_j, F_k \rangle = \langle E(\Delta \cap \beta_{j,k})F_j, F_k \rangle + \langle E_{AC}(\Delta)F_j, F_k \rangle
\]

for all \( \Delta \in \mathcal{B} \). Let \( \beta \) be the union of all the \( \beta_{j,k} \), \( j, k = 0, 1, 2, \ldots \). Then \( \nu(\beta) = 0 \) and (*) holds with \( \beta_{j,k} \) replaced by \( \beta \). Now we pass from the dense subset to all of \( \mathcal{L} \). For all \( F, G \in \mathcal{L}, \Delta \in \mathcal{B} \)

\[
\langle E(\Delta)F, G \rangle = \langle E(\Delta \cap \beta)F, G \rangle + \langle E_{AC}(\Delta)F, G \rangle,
\]

where the decomposition of the left hand term into singular and AC parts is unique. Put \( \gamma = -\beta \). Then 4.2 holds and thus 4.1 is also true. (iii) follows from (**) .

It follows from lemma 3 that \( E_{AC}(\cdot) = E(\gamma)E(\cdot)E(\gamma) \) is a spectral measure in the Hilbert space \( E(\gamma)\mathcal{L} \). \( M_{AC} = E(\gamma)ME(\gamma) \) is the self-adjoint operator on \( E(\gamma)\mathcal{L} \) having this spectral measure. \( M_{AC} \) is obviously AC.

The following simple example will play a role in what happens later. Let \( W \) be as before, even, with \( \sum_n w_n < \infty \). Let \( M \) be the multiplication operator that maps any \( F \in L^2(0, \pi) = \mathcal{L} \) into \( W \cdot F \in \mathcal{L} \). Let \( \chi(\Delta) \) be the characteristic function of \( \{\phi : W(\phi) \in \Delta : 0 < \phi < \pi\} \). Since

\[
\langle MF, F \rangle = \frac{1}{\pi} \int_0^\pi W(\phi)|F(\phi)|^2 d\phi = \int_{-\infty}^{\infty} t \cdot \frac{1}{\pi} \int_0^\pi \chi(\Delta)(\phi) |F(\phi)|^2 d\phi
\]

it follows that the spectral measure \( E(\cdot) \) of \( M \) is defined by \( E(\Delta)F = \chi(\Delta) \cdot F \). Lemma 3 guarantees the existence of \( \gamma \in \mathcal{B} \) such the \( 1/\pi \int_0^\pi \chi(\gamma) \chi(\cdot)(\phi)|F(\phi)|^2 d\phi \) is AC for all \( F \in \mathcal{L} \), while \( E(\cdot - \gamma) \) is singular. \( E(\gamma)\mathcal{L} \) can be identified with the Hilbert space \( L^2(A) \), where \( F \in L^2(A) \) if and only if \( \|F\|_A < \infty \), where

\[
4.3 \quad \|F\|_A = \left[ \frac{1}{\pi} \int_0^\pi \chi(\gamma) |F(\phi)|^2 d\phi \right]^{1/2} = \left[ \frac{1}{\pi} \int_A |F(\phi)|^2 d\phi \right]^{1/2},
\]

and

\[
A = \{\phi : W(\phi) \in \gamma, 0 < \phi \leq \pi\}.
\]

Similarly \( M_{AC} \) can be considered to be the mapping that takes any \( F \in L^2(A) \) into \( W \cdot F \in L^2(A) \).

Another concept that we shall have cause to use is that of trace class. As is usual, a bounded operator on \( \mathcal{L} \) is identified with its
matrix representation. A matrix $H = [w_{j,k}]$, $i, j = 0, 1, 2, \cdots$ belongs to the Schmidt-Hilbert class $SH$ if $\sum_{j,k=0}^{\infty} |w_{j,k}|^2 < \infty$. $H$ belongs to the trace class $TC$ if $H \in SH$ and $\|H\|_1 < \infty$, where $\|H\|_1$ is the sum of the absolute values of the eigenvalues of $H$ repeated according to multiplicity.

As an example we treat the Hankel matrix $H = [w_{j+k+2}]$. As proved in [5], $H \in SH$ if and only if $\sum_{n=1}^{\infty} n |w_{n+1}|^2 < \infty$. This follows from the equality $\sum_{j,k=0}^{\infty} |w_{j+k+2}|^2 = \sum_{n=1}^{\infty} n |w_{n+1}|^2$, and gives a necessary condition that $H \in TC$. Now, define $H_n = [s_{j+k+2,n}]$. Then $H = \sum_{n=2}^{\infty} w_n H_n$. Since $\|H_n\|_1 \leq n$ it follows that $\|H\|_1 \leq \sum_{n=2}^{\infty} n |w_n| \|H_n\|_1 \leq \sum_{n=2}^{\infty} n |w_n|$. Thus a sufficient condition $H \in TC$ is that $W$ be $AC$ such that $W$ has an absolutely convergent Fourier series. This, of course, is part of our standing hypothesis on $W$ for this section. We do not know a useful necessary and sufficient condition for a Hankel matrix to belong to $TC$.

Hankel matrices enter into our picture, following an idea of Putnam's, via the following

**Lemma 4.** Let $H$ be as as in the above example. Let $S = [s_{j,k}]$, where $s_{j,k} = 2/\pi \int_0^\pi W(\phi) \sin(j+1)\phi \sin(k+1)\phi d\phi$, $j, k = 0, 1, 2, \cdots$. Then $T - S = H$.

**Proof.** $w_{j-k} - s_{j,k} = \frac{1}{\pi} \int_0^\pi W(\phi) \cos(j-k)\phi d\phi$

$= -\frac{2}{\pi} \int_0^\pi W(\phi) \sin(j+1)\phi \sin(k+1)\phi d\phi$

$= \frac{1}{\pi} \int_0^\pi W(\phi) \cos(j+k+2) d\phi$.

We can now state a specialization of Kato’s theorem in a form suitable for our application. It is understood that in the statement $T$ and $S$ need not necessarily be the operators we have already defined.

**Theorem 5.** (Kato). Suppose $T$ and $S$ are self-adjoint operators on a separable Hilbert space $\mathcal{L}$ such that $T - S = H \in TC$ and $T$ is AC. Let $\gamma$ and $E(\cdot) = E_{AC}(\cdot) + E_s(\cdot)$ be the Borel set and decomposition respectively guaranteed by Lemma 3. Then

(i) as $t \to \infty$, $\exp(itT) \exp(-itS) E(\gamma)$ converges strongly to an isometric mapping $U$ of $E(\gamma) \mathcal{L}$ onto $\mathcal{L}$.

(ii) $U^{-1}$ is the strong limit as $t \to \infty$ of $\exp(itS) \exp(-itT)$.

(iii) The self-adjoint operator $S_{AC} = E(\gamma) SE(\gamma)$ on $E(\gamma) \mathcal{L}$ is unitarily equivalent to $T$, with $I = US_{AC} U^{-1}$.

From this follows the following spectral analysis theorem for $T$.

**Theorem 6.** Suppose $W$ is a real even AC function on $(-\pi, \pi)$ whose derivative $W'$ has an absolutely convergent Fourier series. Then
the Toeplitz operator \( T(W(\phi)) \) is unitarily equivalent to the multiplication operator \( M_{AC}: f \rightarrow W \cdot f \) on \( L^1(\Lambda) \) (see 4.3).

**Proof.** The hypotheses of Theorem 5 are satisfied via Lemma 4, the discussion following Lemma 3, and Theorem 3. Thus \( T \) is unitarily equivalent to \( S_{AC} \). Since \( \{f_n\}_{n=0}^{\infty} \rightarrow 2^{1/2} \sum_{n=0}^{\infty} f_n \sin (n + 1)\phi \) is an isometry of \( l^2 \) onto \( L^1(0, \pi) \), it follows that \( S_{AC} \) is unitarily equivalent to \( M_{AC} \). Thus \( T \) is unitarily equivalent to \( M_{AC} \).

**COROLLARY 1.** Suppose \( W(\phi) = w_0 + 2 \sum_{n=0}^{m} w_n \cos n\phi \), where the \( w_n \) are real and \( m \) is a positive integer. Then \( T(W(\phi)) \) is unitarily equivalent to the multiplication operator \( M: f \rightarrow W \cdot f \) on \( L^1(0, \pi) \).

**Proof.** In this case \( M = M_{AC} \). (See Putnam [11], p. 522). Now use Theorem 6.

If \( W \) is AC and \( W'' \in L^1(0, \pi) \) then \( \sum_{n} |w_n| < \infty \). Hence a \( W \) satisfying Theorem 6 can have intervals of constancy. If such is the case, then \( M \) has an infinite number of eigenvectors. Thus one cannot validly replace "\( M_{AC} \)" and "\( L^1(\Lambda) \)" by "\( M \)" and "\( L^1(0, \pi) \)" respectively in the statement of Theorem 6, since \( T \) has no point spectra.

We can easily deduce 1.2 from Corollary 1. \( T(W(2 \cos n\phi)) \) is unitarily equivalent to multiplication by \( 2 \cos n\phi \) on \( L^1(0, \pi) \), \( n = 1, 2, \ldots \), and hence to \( 2 \cos (n \arccos \frac{1}{2} T(2 \cos \phi)) = 2 T_n(\frac{1}{2} T(2 \cos \phi)) \) on \( l^2 \).

It would be of great interest to evaluate the limits in Theorem 5 (ii) and (iii) so one could exhibit the unitary transformation of Theorem 6. One could then have a super-abundance of new unitary operators. We pose this as an unsolved problem.

5. **Appendix.** C. R. Putnam has extended the theory he set forth in [11] in his recent article "On Toeplitz matrices, absolute continuity and unitary equivalence", Pacific J. Math., 9 (1959), 837–846. He proves that \( T \) is AC provided that \( A_0 \) is bounded and \( M = M_{AC} \). whence, using [13], he proves Theorem 6 under the added hypothesis that \( M = M_{AC} \).

It is interesting to compare our proof that \( T \) is AC with Putnam’s weaker version of that result. He applies his abstract theory of commutators, while we exhibit the resolvent of \( T \) and employ the rather deep function-theoretic results of [1].

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