

# Pacific Journal of Mathematics

**THE SIMPLICITY OF CERTAIN GROUPS**

ROBERT STEINBERG

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The purpose of this note is to give a proof of the simplicity of certain "Lie groups" considered in [2]. The main feature of the present development is the proof of Lemma 2 below: it is superior to the corresponding proof given in [2], because no assumption on the number of elements of the base field is required, and is very much shorter than the one given by Chevalley [1] for the direct analogues, over arbitrary fields, of the simple (complex) Lie groups. Thus it turns out that the groups  $E_6^1(q^2)$  with  $q \leq 4$ , and  $D_4^2(q^3)$  with  $q \leq 3$ , to which the proof in (2) is not applicable, are simple.

Assuming the notations of [1] and [2] to be in effect, we shall prove:

1. **THEOREM.** *If  $\hat{G}$  is one of the groups of type  $G^1, G^2$  or  $G^3$ , defined in [2], and the rank  $l$  of the corresponding Lie algebra is at least 3, then  $\hat{G}$  is simple.*

It will be noticed that the case  $A_2^1$  is excluded by the assumption on  $l$ . This is of necessity, since the simplicity of  $A_2^1$  is not universal, but depends on the base field. The same is true of groups of type  $A_1$ .

2. **MAIN LEMMA.** *Let  $\hat{G}$  be a group of type  $G$ , that is, one of the direct analogues of the ordinary simple Lie groups, or a group of type  $G^1, G^2$  or  $G^3$ , but assume  $\hat{G}$  is not of type  $A_1$  or  $A_2^1$ . Let  $\hat{U}$  be the nilpotent subgroup of  $\hat{G}$  corresponding to the positive roots of the underlying Lie algebra. Let  $H$  be a normal subgroup of  $\hat{G}$  such that  $|H| > 1$ . Then  $|H \cap \hat{U}| > 1$ .*

*Proof.* Assume first that  $G$  is of type  $G^1$ . By 7.2 of [2], there is  $x = uh\omega(w) \in H$  with  $u \in \mathcal{U}^1$ ,  $h \in \mathfrak{S}^1$ .

If  $w = 1$ , then [2, Lemma 8.5] yields the required conclusion.

If  $w \neq 1$ , consider first the case in which  $w = w_s$  with  $S$  a fundamental element of  $\Pi^1$ . Then there is a fundamental  $A \in \Pi^1$  such that  $B = wA > 0$  and  $wA \neq A$  (because  $A_1$  and  $A_2^1$  are excluded). Choose  $y \in \mathcal{U}_A^1$  so that  $y \neq 1$  and  $y \notin \mathcal{U}_s$ , the subgroup of  $\mathcal{U}$  generated by those  $x_r$  for which  $ht r \geq 2$ . Then we assert that the commutator  $z = (x, y)$  is in  $H \cap \mathcal{U}^1$  and that  $z \neq 1$ . In fact,  $z = uh\omega(w)y\omega(w)^{-1}h^{-1}u^{-1}y^{-1} = utu^{-1}y^{-1}$  with  $t \in \mathcal{U}_B^1$ ; hence  $z \in H \cap \mathcal{U}^1$ , and, since  $\mathcal{U}/\mathcal{U}_s$  is Abelian, we have  $z \equiv ty^{-1} \not\equiv 1 \pmod{\mathcal{U}_s}$ , by 4.3 of [2], whence  $z \neq 1$ .

Finally, consider the general case in which  $w \neq 1$ . Choose  $R \in \Pi^1$

so that  $-wR = S$  is fundamental in  $\Pi^1$ , and then  $y \in \mathbb{U}_s^1$  so that  $y \neq 1$ . Again form  $z = (x, y)$ . In the present case,  $\omega(w)y\omega(w)^{-1} \in \mathbb{U}_s^1 \mathfrak{S}^1 \omega(w_s) \mathbb{U}_s^1$  by 7.3 of [2], so that  $z$  is conjugate to an element  $x_1$  of the form  $u_1 h_1 \omega(w_s)$  with  $u_1 \in \mathbb{U}^1$ ,  $h_1 \in \mathfrak{S}^1$ . Clearly  $x_1 \neq 1$  and  $x_1 \in H$ . Thus the situation is that at the beginning of the preceding paragraph, and Lemma 2 is proved for groups of type  $G^1$ .

Now to get a proof for groups of type other than  $G^1$ , we need only delete all superscripts or replace them all by 2 or all by 3, depending on the group under consideration.

From this point on, we assume that  $\hat{G}$  is of type  $G^1$ , but not of type  $A_l^1$  ( $l$  even), and the ensuing discussion refers explicitly to this case. For groups of type  $A_l^1$  ( $l$  even),  $G^2$  or  $G^3$ , the changes to be made are quite clear: a prototype for these changes is the replacement of (\*) below by an appropriate analogue. For groups of type  $G$ , the rest of the proof of Theorem 1 is given in [1].

3. LEMMA. *If  $G^1$  is not of type  $A_l^1$  ( $l$  even) and  $H$  is a normal subgroup of  $G^1$  such that  $|H| > 1$ , then, for some  $R \in \Pi^1$ ,  $|H \cap \mathbb{U}_R^1| > 1$ .*

It is convenient to precede the proof of this lemma by some preparatory results.

4. LEMMA. *If  $s, a, s + a$  and  $t$  are roots such that  $\bar{a} \neq a$  and  $s + a = t + \bar{a}$ , then  $t = \bar{s}$ .*

*Proof.* We have  $s(a) < 0$  and  $s(\bar{a}) = (s + a)(\bar{a}) > 0$ . Hence  $\bar{s} \neq s$ , and a simple calculation shows that  $t - \bar{s} = s + a - \bar{s} - \bar{a}$  has length 0, since all roots have the same length and the only possible angles are the multiples of  $\pi/3$  and  $\pi/2$ . Hence  $t = \bar{s}$ .

Let us recall that, for each positive integer  $m$ ,  $\mathbb{U}_m$  denotes the subgroup of  $\mathbb{U}$  generated by those  $\mathfrak{X}_r$  for which  $ht r \geq m$ .

5. LEMMA. *Let  $s$  be a positive root,  $a$  a fundamental root, and  $S$  and  $A$  the elements of  $\Pi^1$  which contain them. Assume  $s(a) < 0$ ,  $x \in \mathbb{U}_s^1$ ,  $y \in \mathbb{U}_a^1$ , and set  $ht s = n$ . Then*

(a)  *$(x, y)$  is congruent, mod  $\mathbb{U}_{n+2}$ , to an element of  $\mathbb{U}^1$  whose representation 4.3 of [2] has all components other than those from  $\mathfrak{X}_{s+a}$  and  $\mathfrak{X}_{\bar{s}+\bar{a}}$  equal to 1, and*

(b) *if  $x$  is given and  $x \neq 1$ , then  $y$  can be chosen so that the  $\mathfrak{X}_{s+a}$  component is not 1.*

*Proof.* Assume first  $|S| = |A| = 2$ . Then  $(s, a) < 0$ , whence  $(s, \bar{a}) \geq 0$ , because the contrary assumption yields the false conclusion that  $s + \bar{s} + a + \bar{a}$  has length 0. Thus  $\mathfrak{X}_s$  and  $\mathfrak{X}_a$  commute elementwise with  $\mathfrak{X}_{\bar{s}}$  and  $\mathfrak{X}_{\bar{a}}$ , and 4.1 of [2] yields

$$(*) \quad (x_s(k)x_{\bar{s}}(\bar{k}), x_a(l)x_{\bar{a}}(\bar{l})) = x_{s+a}(N_{sa}kl)x_{\bar{s}+\bar{a}}(N_{sa}\bar{k}\bar{l}).$$

Thus (a) is true. If  $k \neq 0$ , we can choose  $l$  so that  $kl + \bar{k}\bar{l} \neq 0$ , and then coalesce the terms on the right of (\*) if  $\bar{s} + \bar{a} = s + a$ . Thus (b) is also true. If  $|S| = 1$  or  $|A| = 1$ , we replace (\*) in the above argument by an appropriate analogue (see 4.1 and 8.8 of [2]).

Let us recall that a root  $d$  is dominant if  $d(a) \geq 0$  for each fundamental root  $a$ . Since these inequalities define a fundamental region for  $W$ , and all roots are congruent under  $W$  in the present case, it follows that there is a unique dominant root  $d$ . If  $s$  is any other root, then  $(s, a) < 0$  for some fundamental root  $a$ , and then  $s + a$  is also a root. Thus the dominant root  $d$  may also be described as the unique root of maximum height; and one has  $\bar{d} = d$  and  $d > s$  for each root  $s \neq d$ .

We now turn to the proof of Lemma 3. Among all  $x \in H \cap \mathfrak{U}^1$  for which  $x \neq 1$ , choose one which maximizes the minimum  $S \in \Pi^1$  for which  $x_S \neq 1$  in the representation 4.5 of [2]. If this minimum is  $R$ , we show  $x = x_R$ . Assuming the contrary, one can write  $x = x_R x_T \cdots$  with  $x_T \neq 1$ . Set  $ht R = n$ . If  $r \in R$ , then  $r$  is not dominant, since  $R < T$ . Thus  $r(a) < 0$  for some fundamental root  $a$ , and  $r + a$  is a root. If  $a \in A \in \Pi^1$ , we conclude from Lemma 5 that there is  $y \in \mathfrak{U}_a^1$  such that  $(x_R, y)$  is congruent, mod  $\mathfrak{U}_{n+2}$ , to an element of  $\mathfrak{U}^1$  with the  $\mathfrak{X}_{r+a}$  component not 1. Since  $z = (x, y) \in H \cap \mathfrak{U}_{n+1}$ , and  $>$  respects heights, we need only show  $z \neq 1$  to reach a contradiction. We have  $(x, y) = (x_R, y)(x_T, y) \cdots \text{mod } \mathfrak{U}_{n+2}$ . Here the elements on the right are in  $\mathfrak{U}_{n+1}$ . By choice of  $y$ , the  $\mathfrak{X}_{r+a}$  component of  $(x_R, y)$  is not 1, and by Lemmas 4 and 5, the  $\mathfrak{X}_{r+a}$  component of each of  $(x_T, y) \cdots$  is 1. Thus we conclude from 4.3 of [2] and the fact that  $\mathfrak{U}_{n+1}/\mathfrak{U}_{n+2}$  is Abelian that  $(x, y) \not\equiv 1 \text{ mod } \mathfrak{U}_{n+2}$ . Therefore  $(x, y) \neq 1$ , and Lemma 3 is proved.

The proof of Theorem 1 can now be completed, just as in [2].

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# Pacific Journal of Mathematics

Vol. 10, No. 3

November, 1960

Glen Earl Baxter, <i>An analytic problem whose solution follows from a simple algebraic identity</i> .....	731
Leonard D. Berkovitz and Melvin Dresher, <i>A multimove infinite game with linear payoff</i> .....	743
Earl Robert Berkson, <i>Sequel to a paper of A. E. Taylor</i> .....	767
Gerald Berman and Robert Jerome Silverman, <i>Embedding of algebraic systems</i> .....	777
Peter Crawley, <i>Lattices whose congruences form a boolean algebra</i> .....	787
Robert E. Edwards, <i>Integral bases in inductive limit spaces</i> .....	797
Daniel T. Finkbeiner, II, <i>Irreducible congruence relations on lattices</i> .....	813
William James Firey, <i>Isoperimetric ratios of Reuleaux polygons</i> .....	823
Delbert Ray Fulkerson, <i>Zero-one matrices with zero trace</i> .....	831
Leon W. Green, <i>A sphere characterization related to Blaschke's conjecture</i> .....	837
Israel (Yitzchak) Nathan Herstein and Erwin Kleinfeld, <i>Lie mappings in characteristic 2</i> .....	843
Charles Ray Hobby, <i>A characteristic subgroup of a p-group</i> .....	853
R. K. Juberg, <i>On the Dirichlet problem for certain higher order parabolic equations</i> .....	859
Melvin Katz, <i>Infinitely repeatable games</i> .....	879
Emma Lehmer, <i>On Jacobi functions</i> .....	887
D. H. Lehmer, <i>Power character matrices</i> .....	895
Henry B. Mann, <i>A refinement of the fundamental theorem on the density of the sum of two sets of integers</i> .....	909
Marvin David Marcus and Roy Westwick, <i>Linear maps on skew symmetric matrices: the invariance of elementary symmetric functions</i> .....	917
Richard Dean Mayer and Richard Scott Pierce, <i>Boolean algebras with ordered bases</i> .....	925
Trevor James McMinn, <i>On the line segments of a convex surface in <math>E_3</math></i> .....	943
Frank Albert Raymond, <i>The end point compactification of manifolds</i> .....	947
Edgar Reich and S. E. Warschawski, <i>On canonical conformal maps of regions of arbitrary connectivity</i> .....	965
Marvin Rosenblum, <i>The absolute continuity of Toeplitz's matrices</i> .....	987
Lee Albert Rubel, <i>Maximal means and Tauberian theorems</i> .....	997
Helmut Heinrich Schaefer, <i>Some spectral properties of positive linear operators</i> .....	1009
Jeremiah Milton Stark, <i>Minimum problems in the theory of pseudo-conformal transformations and their application to estimation of the curvature of the invariant metric</i> .....	1021
Robert Steinberg, <i>The simplicity of certain groups</i> .....	1039
Hisahiro Tamano, <i>On paracompactness</i> .....	1043
Angus E. Taylor, <i>Mittag-Leffler expansions and spectral theory</i> .....	1049
Marion Franklin Tinsley, <i>Permanents of cyclic matrices</i> .....	1067
Charles J. Titus, <i>A theory of normal curves and some applications</i> .....	1083
Charles R. B. Wright, <i>On groups of exponent four with generators of order two</i> .....	1097