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1. Introduction. The notion of the normal curve as introduced by Whitney [12] is important partly for the following reasons: the normal curves are dense in a natural sense in the regular C' curves; the combinatorial topology embodied in the self-intersection numbers is well-defined. The self-intersection numbers and another minor condition determine a curve up to a sense preserving homeomorphism of the plane onto itself. This fact follows, e.g., directly from work of Adkisson and MacLane [1] and Gehman [3] but will not be verified here since no direct use will be made. It is mentioned, however, since it clearly indicates the amount of information in the intersection numbers. This matter is discussed in detail and used in an essential way in [10].

In the literature, from the point of view of combinatorial topology, there has been, after the beginnings, an emphasis on the study of curves as point sets in their own right and suprisingly little on their study as mappings directly. In this paper relations between the intersections are defined and various properties of curves are described in terms of these relations. Some applications are then given primarily to clarify the sense in which the normal curves are dense among the regular curves and to show how one can obtain theorems on representations of curves directly which then imply theorems on the extensions of these representations to, e.g., interior mappings on the disk. For conditions on the intersection numbers that there exist an extension by an interior mapping, see [9] and [10].

Lemmas 1, 3, 4, 5, 7 are all implicity in the work of Whitney [12] and [13] and are proved here in more or less detail for completeness and in some cases because the direct development, in the special situation here, is more concise.

2. Basic concepts and techniques. A complex valued function $\zeta(t)$ defined on an interval of the real line is called a *regular representation* of a curve if it possesses a continuous non-vanishing derivative $\zeta'(t)$. If, as will usually be the case, ζ has the period 2π then ζ is a regular repesentation of a closed curve and the interval $[0, 2\pi)$ may be identified with the unit circle oriented by increasing t.

An image point ζ_1 is called a simple crossing if there exist exactly two distinct numbers t' and t'' such that $\zeta(t') = \zeta(t'') = \zeta_1$ and such that

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the tangent vectors $\zeta'(t')$ and $\zeta'(t'')$ are independent. A regular representation is called *normal* (Whitney [13], p. 281) if it has a finite number of simple crossing points and is one-to-one everywhere else.

Let $\zeta(t) = \xi(t) + i\eta(t)$ be a normal representation of period 2π on the real t-line and restricted to the interval $S = [0, 2\pi)$. Let $\{\zeta_k\}, k = 1, 2, \cdots, n$, be the set of all simple crossing points. Let $\{t'_k\}, \{t''_k\}, k = 1, 2, \cdots, n$, be the set of numbers in S indexed so that $\zeta(t'_k) = \zeta(t''_k)$, $t'_k < t''_k$ and $0 < t'_1 < t'_2 < \cdots < t'_n < 2\pi$. From now on the simple crossing points ζ_k will be called *vertices*. Let the set of all numbers in S which are mapped onto vertices by ζ be denoted also by $\{s_k\}, k = 1, 2, \cdots, 2n$, indexed so that $0 < s_1 < s_2 < \cdots < s_{2n} < 2\pi$. Given s_k let s_k^* be yet another name for the number in $\{s_k\}$ such that $\zeta(s_k) = \zeta(s_k^*)$. Let ν_k be given by

$$-\nu_k = -\nu(s_k) = \operatorname{sgn} \begin{vmatrix} \xi'(s_k) & \xi'(s_k) \\ \eta'(s_k^*) & \eta'(s_k^*) \end{vmatrix}, k = 1, 2, \cdots, 2n :$$

and λ_k by $\nu(t'_k)$, i.e.,

$$-\lambda_k = -\nu(t'_k) = \operatorname{sgn} \begin{vmatrix} \xi'(t'_k) & \xi'(t'_k) \\ \eta'(t''_k) & \eta'(t''_k) \end{vmatrix}, k = 1, 2, \cdots, n .$$

Given any pair of distinct vertices ζ_j and ζ_k exactly one of the following five relations R holds:

$R1$ $j \subset k$, i.e., $t'_k \leqq t'_j < t''_j \leqq t''_k$,

$R2$ $j \supset k$, i.e., $t'_j \leqq t'_k < t''_k \leqq t''_j$,

$R3$ j links k on the right, i.e., $t'_k < t'_j < t''_k < t''_j$, denoted by $j \in R_k$,

$R4$ j links k on the left, i.e., $t'_j < t'_k < t''_j < t''_k$, denoted by $j \in L_k$,

$R5$ j and k are disjoint, i.e., either $t'_j < t''_j < t'_k < t''_k$ or

$$t'_k < t''_k < t'_j < t''_j, \text{ denoted by } j \mid k.$$

The sequence $\{s_k\}$, the function given by $\nu(s_k)$ and the star operation * determine what will be called the *intersection sequence* of the normal representation ζ. A vertex ζ_k will be signed positive [negative] if $\nu(t'_k) > 0 [\nu(t'_k) < 0]$. In a straight forward but mildly tedious way one can show that a knowledge of the signs of the vertices and the relations R for each pair of vertices is equivalent to a knowledge of the intersection sequence.

Let $\tilde{\zeta}$ and ζ be a pair of regular representations defined on a common interval I. The following norm will be used:

$$\|\zeta - \tilde{\zeta}\| = \sup_{t \in I} \{|\zeta(t) - \tilde{\zeta}(t)| + |\zeta'(t) - \tilde{\zeta}'(t)|\} .$$

LEMMA 1. *The intersection sequence of a normal representation is stable; i.e., given any normal representation ζ on an interval I there*

exists an $\varepsilon > 0$ *such that for every regular representation on* I *for which* $\|\zeta - \tilde{\zeta}\| < \varepsilon$, *it follows that* $\tilde{\zeta}$ *is normal and possesses the same intersection sequence as* ζ; *namely,* $\operatorname{sgn} \zeta_k = \operatorname{sgn} \tilde{\zeta}_k$ *for all* k *and* ζ_k *and* ζ_j *satisfy the same relation* R *as* $\tilde{\zeta}_k$ *and* $\tilde{\zeta}_j$ *for all* j *and* k.

Proof. Let $\gamma_j(t) = \alpha_j(t) + i\beta_j(t)$ and $\tilde{\gamma}_j(t) = \tilde{\alpha}_j(t) + i\tilde{\beta}_j(t)$ be a pair of regular representations defined on an interval I_j. To prove the lemma the following two facts (i) and (ii) will be used. Each follows in a known way; see e.g., Whitney, [13], p. 281, proof of Lemma 2.

(i) If $\gamma_j(t)$ and $\gamma_k(t)$ have no image points in common then there exists and $\varepsilon > 0$ such that, whenever $\|\gamma_j - \tilde{\gamma}_j\|$ and $\|\gamma_k - \tilde{\gamma}_k\|$ are each less than ε, $\tilde{\gamma}_j(t)$ and $\tilde{\gamma}_k(t)$ have no image points in common.

(ii) If $\gamma_j(t)$ and $\gamma_k(t)$ have exactly one point in common, say $\gamma_k(t_j) = \gamma_k(t_k)$, and if the tangent vectors $\gamma_j'(t_j)$ and $\gamma_k'(t_k)$ are independent then there exists and $\varepsilon > 0$ such that, whenever $\|\gamma_j - \tilde{\gamma}_j\|$ and $\|\gamma_k - \tilde{\gamma}_k\|$ are each less than ε, $\tilde{\gamma}_j$ and $\tilde{\gamma}_k$ have exactly one point in common, say $\tilde{\gamma}_j(\tilde{t}_j) = \tilde{\gamma}_k(\tilde{t}_k)$; the tangents $\tilde{\gamma}_j'(\tilde{t}_j)$ and $\tilde{\gamma}_k'(\tilde{t}_k)$ are independent; and

$$\operatorname{sgn} \begin{vmatrix} \tilde{\alpha}_j'(\tilde{t}_j) & \tilde{\beta}_j'(\tilde{t}_j) \\ \tilde{\alpha}_k'(\tilde{t}_k) & \tilde{\beta}_k'(\tilde{t}_k) \end{vmatrix} = \operatorname{sgn} \begin{vmatrix} \alpha_j'(t_j) & \beta_j'(t_j) \\ \alpha_k'(t_k) & \beta_k'(t_k) \end{vmatrix} .$$

Select numbers σ_k so that $0 = \sigma_0 < s_1 < \sigma_1 < s_2 < \cdots < s_{2n} < \sigma_{2n} = 2\pi$ and define the intervals $I_j = (\sigma_{j-1}, \sigma_j)$, $j = 1, 2, \cdots, 2n$. Let $\gamma_j(t) = \zeta | I_j$; i.e., $\gamma_j(t) = \zeta(t)$ on I_j. Now, given any distinct pair of representations γ_j and γ_k either (i) or (ii) applies. The smallest ε associated with a distinct pair of representations is chosen and the lemma follows.

Let $[\zeta]$ denote the point set consisting of the images of ζ.

LEMMA 2. *Let* ζ *be a normal representation of a closed curve* ($\zeta(t)$ *has period* 2π) *and let* $\gamma(u) = \alpha(u) + i\beta(u)$ *be a normal representation on* $[0, 1]$ *with* $\gamma_0 = \gamma(0)$ *and* $\gamma_1 = \gamma(1)$ *not on* $[\zeta]$. *Assume that* ζ *and* γ *intersect one another in a finite number of points* σ_σ, $\sigma = 1, 2, \cdots, p$, *which correspond to the parameter values* v_σ *and* u_σ *respectively*; *that the points* $\zeta(v_\sigma)$ *and* $\zeta(u_\sigma)$ *are not self intersection points of* ζ *and* γ *respectively*; *that the tangents* $\zeta'(v_\sigma)$ *and* $\gamma'(u_\sigma)$ *are independent. Let* $\omega_0[\omega_1]$ *denote the index* (winding number) *of the curve* ζ *about the point* $\gamma_0[\gamma_1]$. *Then,*

$$\omega_1 - \omega_0 = \sum_{\sigma=1}^{p} \operatorname{sgn} \begin{vmatrix} \xi'(v_\sigma) & \eta'(v_\sigma) \\ \alpha'(u_\sigma) & \beta'(u_\sigma) \end{vmatrix} .$$

Proof. Let the self intersection points of γ correspond to parameter values $\{w_\sigma\}$ where $0 < w_1 < w_2 < \cdots < w_m < 1$. Let $\{w_\sigma'\}$ be chosen so that $0 = w_0' < w_1 < w_1' < \cdots < w_m < w_m' = 1$ and so that $\gamma(w_\sigma')$ never

lies on $[\zeta]$. Let I_j denote the interval $[w'_{j-1}, w'_j]$, $j = 1, 2, \cdots, m$. Let $\tilde{\omega}_\sigma$ denote the index of ζ about the point $\zeta(w'_\sigma)$. Then, since $\gamma | I_j$ is an arc, one has by classical intersection theory

$$\tilde{\omega}_{j+1} - \tilde{\omega}_j = \sum_{w'_j < j_\sigma < m_{j+1}} \operatorname{sgn} \begin{vmatrix} \xi'(v_\sigma) & \eta'(v_\sigma) \\ \alpha'(u_\sigma) & \beta'(u_\sigma) \end{vmatrix} .$$

Thus, since $\tilde{\omega}_0 = \omega_0$ and $\tilde{\omega}_m = \omega_1$, one has

$$\omega_1 - \omega_0 = \sum_{j=1}^{m} (\tilde{\omega}_j - \tilde{\omega}_{j-1}) = \sum_{\sigma=1}^{p} \operatorname{sgn} \begin{vmatrix} \xi'(v_\sigma) & \eta'(v_\sigma) \\ \alpha'(u_\sigma) & \beta'(u_\sigma) \end{vmatrix}$$

as was to be shown.

LEMMA 3. (Whitney, [13], Theorem 2, p. 281).

Let ζ be a normal representation with $\zeta_0 = \zeta(0)$ on the outer boundary of $[\zeta]$. (Then in every neighborhood of ζ_0 there exist point about which ζ has index ± 1.) Then, the tangent winding number $\tau(\zeta)$ is given by

$$\tau(\tau) = \sum_{\sigma=1}^{p} \lambda_\sigma \pm 1$$

where the sign is chosen $\pm$ as the sign of the non-zero index of about points chose to ζ_0 is $\pm$.

Proof. This lemma is easily reduced to the Theorem of Whitney mentioned.

LEMMA 4. *Given a regular periodic representation ζ and an $\varepsilon > 0$ there exists a regular periodic representation $\tilde{\zeta}$ such that*

(i) $\|\zeta - \tilde{\zeta}\| < \varepsilon$ *and there exists a $\delta > 0$ such that for all t*

(ii) $\{\zeta(\tau) | t - \delta < \tau < t + \delta\} \cap \{\tilde{\zeta}(\tau) | t - \delta < \tau < t + \delta\}$

is empty.

Proof. Let $\zeta^*(t) \equiv \dfrac{1}{2h} \displaystyle\int_{t-h}^{t+h} \zeta(\tau) d\tau$ and define

$$\tilde{\zeta}(t) \equiv \zeta(t) - i\rho[\zeta^*(t)]' .$$

The lemma will be proved by selecting $\tilde{\zeta}(t)$ to be of this form. For part (i), note that $\|\zeta - \tilde{\zeta}\| = \rho \max |(\zeta^*)'| + |(\zeta^*)''|$. Since $(\zeta^*)'$ and $(\zeta^*)''$ are continuous periodic functions and therefore bounded one can select a ρ_0 so that $\|\zeta - \tilde{\zeta}\| < \varepsilon$ for $\rho < \rho_0$. For part (ii) consider $|\zeta - \tilde{\zeta}| = \rho |(\zeta^*)'|$. Since $(\zeta^*)'$ converges uniformly to ζ' as h tends to zero and since $|\zeta'|$ is bounded away from zero there exists an h small enough that $|(\zeta^*)'| = |\zeta - \tilde{\zeta}|$ is bounded away from zero. There exists

therefore a $\delta > 0$ so that $|\tilde{\zeta}(t') - \zeta(t'')| > 0$ whenever $t - \delta < t'$, $t'' < t + \delta$.

Finally, to show that $\tilde{\zeta}$ can be chosen to be a regular representation, consider $\tilde{\zeta}' = \zeta' - i\rho(\zeta^*)'$. Since $|\zeta'|$ is bounded away from zero and $|(\zeta^*)'|$ is bounded there is a ρ small enough so that $\rho < \rho_0$ and $\tilde{\zeta}$ is never zero.

The idea of the intersection sequence has already been introduced. However, one can easily construct simple examples of intersection sequences which cannot come from a closed curve. For example, the sequence $t_1' < t_2' < t_1'' < t_2''$, no matter what function $\nu(t_\sigma')$ is chosen cannot correspond to a closed curve in the plane. It does however correspond to a closed curve on the torus. See v. Sz. Nagy [7], p. 579. Thus these questions are apparently related to the homology properties of the imbedding manifold. The next theorem extends the results of Nagy to intersection sequences of closed curves in the plane. Denote the interval $t_k' \leqq t \leqq t_k''$ by T_k and let $\zeta \mid T_k$ denote the representation given by ζ restricted to the interval T_k.

THEOREM 1. *Let ζ be a normal representation of a closed curve with ζ_0 chosen on the outer boundary. Let ω_k be the index of the curve given by $\zeta \mid T_k$ about the point $\zeta(v)$ where $s_j = t_k' > v > s_{j-1}$, $s_0 = 0$. Then*

$$\sum_{\sigma \in I_k} \lambda_\sigma = \sum_{\sigma \in R_k} \lambda_\sigma = \omega_k \ .$$

Proof. The proof follows immediately by use of Lemma 2 and the section of $\gamma(t)$ as $\gamma(t) \equiv \zeta(t)$, $0 \leqq t \leqq v$. This gives

$$\omega_k = \sum_{\sigma \in I_k} \lambda_\sigma \ .$$

The fact that

$$\omega_k = \sum_{\sigma \in R_k} \lambda_\sigma$$

follows similarly by choosing $\gamma(t) \equiv \zeta(t)$, $0 \leqq t \leqq w$, $s_i = t_k'' < w < s_{i+1}$, and using $\nu(t_\sigma') = -\nu(t_\sigma'')$.

COROLLARY 1. *If $\zeta \mid T_k$ is a representation with $\zeta(u)$, $t_k' = s_j > u > s_{j-1}$, such that $\zeta(u)$ is in unbounded component of complement of $[\zeta \mid T_k]$, then*

$$\sum_{\sigma \in I_k} \lambda_\sigma = \sum_{\sigma \in R_k} \lambda_\sigma = 0 \ .$$

Proof. Use Theorem 1 and the fact the $\omega_k = 0$ in this case.

COROLLARY 2. (Gauss-Nagy [7], Theorem I, p. 580).

The number of s_σ between s_j and s_j^* for any normal closed curve ζ is always even.

Proof. If s_σ is between $t_k' = s_j$ and $s_j^* = t_k''$ then the vertex corresponding to s_σ either links ζ_k on the left, on the right or $\sigma \subset k$. Since $\sum_{\sigma \in Lk} \lambda_\sigma = \sum_{\sigma \in Rk} \lambda_\sigma$ the number of s_σ between t_k' and t_k'', namely,

$$\sum_{\sigma \in L_k} |\lambda_\sigma| + \sum_{\sigma \in L_k} |\lambda_\sigma| + 2(\text{no. of } \zeta_\sigma, \sigma \subset k)$$

is certainly even since $\sum_{\sigma \in Lk} |\lambda_\sigma|$ and $\sum_{\sigma \in Rk} |\lambda_\sigma|$ are both even or both odd.

THEOREM 2. *Let $\tilde{\zeta}$ be a normal representation for which $\tilde{\zeta}_0 = \tilde{\zeta}(0)$ is not a vertex. Let the indices with respect to $\tilde{\zeta}$ of points near $\tilde{\zeta}_0$ be denoted by $\tilde{\omega}_0^+$ and $\tilde{\omega}_0^-$ where $\tilde{\omega}_0^+ = \tilde{\omega}_0^- + 1$. Then*

$$\tau(\tilde{\zeta}) = \sum_{\sigma=1}^{n} \tilde{\lambda}_\sigma + (\tilde{\omega}_0^+ + \tilde{\omega}_0^-) .$$

Proof. From the Whitney theorem, here Lemma 3, one has with ζ_0 selected on the outer boundary that

$$\tau(\zeta) = \sum_{\sigma=1}^{n} \lambda_\sigma + (\omega_0^+ + \omega_0^-) .$$

The following identity follows directly from the definitions of the λ_σ, the relations R and Theorem 1:

$$\text{(C)} \qquad \sum_{\sigma=1}^{n} \lambda_\sigma = \sum_{\sigma | k} \lambda_\sigma + \sum_{\sigma \subset k} \lambda_\sigma + \sum_{\sigma \supset k} \lambda_\sigma + 2 \sum_{\sigma \in L_k} \lambda_\sigma - \lambda_k .$$

Let u be a number such that $\zeta(u) = \tilde{\zeta}(0)$ and $s_j = t_k' < u < s_{j+1}$. Now, by use of Lemmas 1, 2 and 4 one has, with ζ and $\tilde{\zeta}$ representing the same curve,

$$\tilde{\omega}_0^- - \omega_0^- = \sum_{s_1 \leqq s_\sigma \leqq t_k'} \nu_\sigma ;$$

but again from definitions of the λ_σ and the relations R one has

$$\sum_{s_1 \leqq s_\sigma \leqq t_k'} \nu_\sigma = \sum_{\sigma \in L_k} \lambda_\sigma + \sum_{\sigma \subset k} \lambda_\sigma .$$

It follows easily by use of Lemma 2 that

$$\tilde{\omega}_0^+ - \omega_0^+ = \tilde{\omega}_0^- - \omega_0^- .$$

Thus,

$$\text{(D)} \qquad (\tilde{\omega}_0^+ + \tilde{\omega}_0^-) = 2\left\{ \sum_{\sigma \in L_k} \lambda_\sigma + \sum_{\sigma \supset k} \lambda_\sigma \right\} + (\omega_0^+ + \omega_0^-) .$$

Now, again from definitions of the λ_σ and the relations R,

$$\text{(E)} \qquad \sum_{\sigma=1}^{n} \tilde{\lambda}_\sigma = \sum_{\sigma|k} \lambda_\sigma + \sum_{\sigma\subset k} \lambda_\sigma - \sum_{\sigma\supset k} \lambda_\sigma - \lambda_k \ .$$

Thus, by (D) and (E),

$$\sum_{\sigma=1}^{n} \tilde{\lambda}_\sigma + (\tilde{\omega}_0^+ + \tilde{\omega}_0^-) = \sum_{\sigma|k} \lambda_\sigma + \sum_{\sigma\subset k} \lambda_\sigma + \sum_{\sigma\supset k} \lambda_\sigma + 2\sum_{\sigma\in L_k} \lambda_\sigma - \lambda_k + (\omega_0^+ + \omega_0^-) \ .$$

But, using (C),

$$\sum_{\sigma=1}^{n} \tilde{\lambda}_\sigma + (\tilde{\omega}_0^+ + \tilde{\omega}_0^-) = \sum_{\sigma=1}^{n} \lambda_\sigma + (\omega_0^+ + \omega_0^-) = \tau(\zeta) \ ,$$

and the Theorem follows.

Let φ_k denote the angle, $-\pi < \varphi_k < \pi$, measured from the direction of $\zeta'(t_k'')$ to the direction of $\zeta'(t_k')$. Let ψ_k denote the change in the angle of $\zeta'(t)$ as t varies from t_k' to t_k''. Then, as usual, the concept of the tangent winding number can be extended to the closed curve represented by $\zeta \mid T_k$ by

$$\tau(\zeta \mid T_k) = \frac{1}{2\pi}(\psi_k + \varphi_k) \ .$$

THEOREM 3. *With ζ_0 chosen on the outer boundary of $[\zeta]$, ζ a normal representation, then*

$$\tau(\zeta \mid T_k) = \sum_{\sigma\subset k} \lambda_\sigma + 2\sum_{\sigma\in L_k} \lambda_\sigma \ .$$

Proof. Since ζ_0 is chosen on the outer boundary of $[\zeta]$ it follows that the index of $\zeta \mid T_k$ about the point ζ_0 is zero for all k. In order to compute $\tau(\zeta \mid T_k)$ Theorem 2 will be used and thus $\omega_0^+ \mid T_k$ and $\omega_0^- \mid T_k$ must be computed. Explicitly, choose u, $t_k' = s_j < u < s_{j+1}$, and let $\omega_0^+ \mid T_k$ and $\omega_0^- \mid T_k$ be the indices of $\zeta \mid T_k$ about points near $\zeta(u)$ with as before $\omega_0^+ \mid T_k = \omega_0^- \mid T_k + 1$. Some computation using Lemmas 1 and 4 as in proof of Theorem 2 gives

$$\omega_0^+ \mid T_k + \omega_0^- \mid T_k = 2\sum_{\sigma\in L_k} \lambda_\sigma + \lambda_\kappa \ .$$

By Theorem 2, as applied to $\zeta \mid T_k$,

$$\tau(\zeta \mid T_k) = \sum_{\sigma\subset k} \lambda_\sigma + 2\sum_{\sigma\in L_k} \lambda_\sigma = \sum_{\substack{\sigma\subset k \\ \sigma\neq k}} \zeta_\sigma + \omega_0^+ \mid T_k + \omega_0^- \mid T_k$$

as was to be shown.

A regular closed curve is said to be of *non-negative circulation*, Loewner [6], p. 316, if the index of ζ about any point not on $[\zeta]$ is non-negative. Part of the importance of this concept comes from the

following fact: Let $w = w(z)$ be an analytic function on $|z| < 1$ and continuous on $|z| \leqq 1$. Then, the curve represented by $\zeta(t) \equiv w(e^{it})$, $z = \rho e^{it}$, is of non-negative circulation. The proof of this statement follows easily from a lemma of Loewner, [6], Lemma 2, page 318, and is left to the interested reader.

THEOREM 4. *A closed curve given by a normal representation ζ, with ζ_0 on the outer boundary, is of non-negative circulation if and only if*

$$\sum_{\sigma \supset k} \lambda_\sigma + \sum_{\sigma \in R_k} \lambda_\sigma \geqq 0 \text{ for all } k ,$$

and $\omega_0^+ + \omega_0^- = 1$. (As usual, ω_0^+ and ω_0^- are the indices of $\zeta(t)$ about points near ζ_0).

Proof. The condition that $\omega_0^+ + \omega_0^- = 1$ is clearly necessary. Consider $\zeta(v)$ with $s_j < v < s_{j+1}$. Let ω_v^- denote the smaller of the indices of ζ about points near $\zeta(v)$. By Lemmas 1, 2, and 4, again as in proof of Theorem 1, one obtains, since $\omega_0^- = 0$, that

$$\omega_v^- = \sum_{s_1 \leqq s_\sigma \leqq s_j} \nu(s_\sigma) .$$

Now, $\omega_v^- \geqq 0$ for all j since ζ is of non-negative circulation. Also it is clear that given any point π not on $[\zeta]$ there is a j such that the index of ζ about π is either equal to ω_v^- or to $\omega_v^- + 1$. Thus, the condition

$$\sum_{s_1 \leqq s_\sigma \leqq s_j} \nu(s_\sigma) \geqq 0 \text{ for all } j$$

together with $\omega_0^+ + \omega_0^- = 1$ is necessary and sufficient. But given any j there is an k, and vice-versa, such that

$$\sum_{s_1 \leqq s_\sigma \leqq s_j} \nu(s_\sigma) = \sum_{\sigma \subset k} \lambda_\sigma + \sum_{\sigma \in R_k} \lambda_\sigma .$$

For given j choose k to be the largest integer such that $t'_k \leqq s_j$. Given k choose j to be the least integer such that $s_j \geqq t'_k$. Using the fact that $\nu(t') = -\nu(t'')$ the above equality holds and the proof of Theorem 4 is complete.

A sub-set Q of the vertices of a normal curve ζ will be called *exterior* provided that

$$\sum_{\sigma \in L_k} \lambda_\sigma = 0, \text{ all } \zeta_k \varepsilon Q .$$

Let u be a number such that $t'_k = s_j > u > s_{j-1}$ and let $\hat{\omega}_k$ denote the index of $\zeta \mid T_k$ about the point $\zeta(u)$. Then as established in proof of Theorem 3, one has for all k that

$$\hat{\omega}_k = \sum_{\sigma \in L_k} \lambda_\sigma = \sum_{\sigma \in R_k} \lambda_\sigma \ .$$

Thus, Q is exterior provided that

$$\hat{\omega}_k = 0, \qquad \text{all } \zeta_k \in Q \ .$$

There is a natural way to generate an exterior sub-set Q of vertices to a given normal curve represented by ζ with ζ_0 on the outer boundary of $[\zeta]$ which is as follows:

(i) $\zeta_{j_1}, \cdots, \zeta_{j_m} \in Q$ if $\hat{\omega}_{j_\sigma}$ are all zero, $j_1 = 1$ and $j_\sigma, \sigma > 1$, is the smallest index such that $j_\sigma > j_{\sigma-1}$ and $j_\sigma | j_{\sigma-1}$;

(ii) $\zeta_{k_1}, \cdots, \zeta_{k_m} \in Q$ if $\hat{\omega}_{k_\sigma}$ are all zero and there exists a $\zeta_j \in Q$ such that k_1 is the smallest index with $k_1 \subset j$ and $k_\sigma, \sigma > 1$, is the smallest index with $k_\sigma \subset j$, $k_\sigma > k_{\sigma-1}$ and $k_\sigma | k_{\sigma-1}$.

Subsets Q so generated will be called *properly exterior.*

THEOREM 5. *Let Q be a properly exterior sub-set of vertices of a normal representation ζ then*

$$\tau(\zeta) = \sum_{\sigma \in Q} \lambda_\sigma + (\omega_0^+ + \omega_0^-) \ .$$

Proof. From the generation process it follows that each vertex $k \notin Q$ must link some vertex $j \in Q$ on the right; i.e., $k \in R_j$. Thus, to each vertex $k \notin Q$, one can associate a unique vertex $j \in Q$ by the following device: let $j \in Q$ be such that $k \in R_j$ and j is minimal on Q with respect to the order relation Rl. Let K_j be the set of vertices $k \notin Q$ which are associated by this device with the same vertex j. An argument, as in proof of Theorem 1, gives the fact that

$$\sum_{\sigma \in K_j \cap R_j} \lambda_\sigma = 0 \ , \qquad j \in Q \ .$$

Thus,

$$\sum_{\sigma=1}^{m} \lambda_\sigma = \sum_{\sigma \in Q} \lambda_\sigma + \sum_{j \in Q} \Big(\sum_{\sigma \in K_j \cap R_j} \lambda_\sigma \Big) = \sum_{\sigma \in Q} \lambda_\sigma \ ,$$

and the proof follows by Theorem 2.

3. A criteria that a curve be simple.

THEOREM 6. *A closed curve given by a normal representation ζ, ζ_0 on outer boundary, is simple and positively oriented if and only if*

(i) $\tau(\zeta) \leqq 1$, *i.e., tangent winding number not greater than* 1,

(ii) $\tau(\zeta \mid T_k) \geqq 0$ *for all k, i.e., tangent winding number of each sub-loop is non-negative,*

(iii) ζ *is of non-negative circulation.*

Proof. The conditions are clearly necessary for (ii) and (iii) follow immediately and (i) by the "Umlaufshtz", Hopf [4], p. 53.

From the theorems in §2 the hypotheses are equivalent, with ζ_0 on the outer boundary, to

(i)' $\sum_{\sigma=1}^{n} \lambda_\sigma \leqq 0$,

(ii)' $\sum_{\sigma \subset k} \lambda_\sigma + 2 \sum_{\sigma \in L_k} \lambda_\sigma \geqq 0$ for all k,

(iii)' $\sum_{\sigma \supset k} \lambda_\sigma + \sum_{\sigma \in L_k} \lambda_\sigma \geqq 0$ for all k.

By applying (iii), to a vertex ζ_k with $k \in Q$, Q properly exterior and with k maximal on Q, one has

$$0 \leqq \lambda_k + \sum_{\sigma \in L_k} \lambda_\sigma = \lambda_k$$

and thus that $\lambda_k = 1$. Let C_k denote the set of vertices in Q covered by k, (k covers j if $k \neq j$ and if $k \supset \sigma \supset j$ implies σ is either k or j). Now, since $k \in Q$,

$$\sum_{\sigma \subset k} \lambda_\sigma = \lambda_k + \sum_{j \in C_k} \left(\sum_{\sigma \subset j} \lambda_\sigma \right),$$

and by (ii)'

$$\sum_{\sigma \subset j} \lambda_\sigma \geqq 0 .$$

Thus

$$\sum_{\sigma \subset k} \lambda_\sigma \geqq \lambda_k = 1 .$$

Finally, with M denoting the maximal elements in Q, using Theorem 5,

$$\tau - 1 = \sum_{\sigma=1}^{n} \lambda_\sigma = \sum_{k \in M} \left(\sum_{\sigma \subset k} \lambda_\sigma \right) \geqq (\text{no. of vertices in } M) .$$

Thus, since $\tau - 1 \leqq 0$, there are no vertices in M and therefore ζ itself has no vertices. The proof is complete.

NOTE. Actually the full strength of (ii) as well as (iii) were not used and the Theorem could be strengthened. However, in the hypotheses, (ii) and (iii) are convenient for various applications and it seems, at present, that estimations of the maximum index of ζ under (ii), (iii) with more general inequalities on τ are more interesting. Some estimates of this type will be published in a later paper. For an intuitive discussion of these kinds of "valence" estimates as applied to analytic functions see Umezawa [11].

4. **Applications to regular curves.** In order, among other things, to illustrate how these ideas for normal curves can be applied by a limiting process to regular curves the following theorem will be proved.

THEOREM 7. *Let ζ be a regular representation of a closed curve which satisfies the following conditions*:

(i) $\tau(\zeta) \leqq 1$.

Let the angle variation of $\zeta'(t)$ on a sub-interval T on the circle on which ζ is defined be denoted by $\Phi(T)$; thus $2\pi\tau(\zeta)$ is the value of Φ on the whole circle.

(ii) $\Phi(T) > -\pi$ for all T.

(iii) There exists an $\varepsilon > 0$ such that for all regular $\tilde{\zeta}$ with $||\zeta - \tilde{\zeta}|| < \varepsilon$ it follows that ζ is of non-negative circulation.

Then ζ represents a simple positively oriented Jordan curve.

Some preliminary lemmas are required for the proof.

LEMMA 5 (Whitney). *Given a regular curve ζ there exist normal curves ζ^* such that $||\zeta^* - \zeta||$ is arbitrarily small.*

Proof. See Whitney, [13], page 281, Lemma 2.

LEMMA 6. *Let ζ be a normal representation of a closed curve with $\Phi(T) \geqq -\pi$ on every interval T on the circle then $\tau(\zeta \mid T_k) \geqq 0$ for all k.*

Proof. From the geometric definition of $\tau(\zeta \mid T_k)$ one has, as in introduction to Theorem 3, that

$$2\pi\tau(\zeta \mid T_k) = \Phi(T_k) + \varphi_k, \quad -\pi < \varphi_k < \pi .$$

Thus, $2\pi\tau(\zeta \mid T_k) > -2\pi$, and since $\tau(\zeta \mid T_k)$ is an integer the lemma follows.

LEMMA 7. *If ζ is a regular representation which is not simple then there exist normal representations ζ^*, with $||\zeta - \zeta^*||$ arbitrarily small, which are not simple.*

Proof. If there exist distinct values t' and t'' such that $\zeta(t') = \zeta(t'')$ then either the tangents $\zeta'(t')$ and $\zeta'(t'')$ are independent or they are not. If they are independent then, since the curve is regular, there exists a $\delta > 0$ such that ζ is one-to-one on $[t' - \delta, t' + \delta]$ and also on $[t'' - \delta, t'' + \delta]$. Furthermore, with δ sufficiently small there arcs intersect only at $\zeta(t') = \zeta(t'')$. By the density lemma, Lemma 5, ζ can be approximated arbitrarily closely by normal representation ζ^* and by the stability lemma, Lemma 1, the intersection will be preserved.

If $\zeta'(t')$ and $\zeta'(t'')$ are not independent then it is clear that there is

a regular curve arbitrarily close to ζ such that it intersects itself in at least one point with independent tangents and thus the method for the previous case can be applied. The proof is complete.

Proof of Theorem 7. By Lemma 5 there exist normal curves ζ^* such that $||\zeta - \zeta^*||$ is arbitrarily small. But when $||\zeta - \zeta^*||$ is small enough one has that $\tau(\zeta^*) \leqq 1$ and $\Phi(T) > -\pi$ on ζ^* since these conditions are easily seen to be stable. Furthermore, ζ^* is of non-negative circulation by the hypothesis (iii). Thus, by Theorem 6, ζ^* is simple and positively oriented when $||\zeta - \zeta^*||$ is sufficiently small. But if ζ is not simple Lemma 7 implies that there exist normal ζ^*, with $||\zeta - \zeta^*||$ arbitrarily small, which are not simple. This contradiction completes the proof.

5. Application to the theory of interior mappings. Regular representations which satisfy the condition $\Phi(T) > -\pi$ for all intervals T, are called *close to convex*, [5], p. 169.

The following theorem proved for interior mappings gives a result of Kaplan, [5], Theorem 1, p. 170, when the interior mapping is an analytic function. There is related work in these directions by Paatero [8], Umezawa [11] in the case of analytic functions. Their results depend strongly on the fact that the mappings are analytic and have no obvious analogue even when the interior mappings are solutions to elliptic systems of partial differential equations.

THEOREM 8. *Let ζ be a regular representation of a close to convex curve with $\tau(\zeta) = 1$ such that there exists a mapping f continuous on $|z| \leqq 1, C'$ and interior* (=light and open) *on $|z| < 1$ with $f(e^{it}) \equiv \zeta(t)$. Then the mapping f is a homeomorphism.*

Before beginning the proof consider the following lemmas.

LEMMA 8. (Loewner). *Let $f(z)$ be a mapping continuous on the annulus $0 < \rho_1 \leqq \rho \leqq \rho_2, z = \rho e^{it}$, continuously differentiable and possessing a non-negative Jacobian on $\rho_1 < \rho < \rho_2$. Then for every point P not on $[f(\rho_0 e^{it})] \cup [f(\rho_2 e^{it})]$ the index of $f(\rho_1 e^{it})$ about P is not greater than the index of $f(\rho_2 e^{it})$.*

Proof. See Loewner, [6], Lemma 2, p. 318.

LEMMA 9. *If f is a mapping continuous on $|z| \leqq 1, C'$ and interior on $|z| < 1$ and if $\zeta(t) = f(e^{it})$ is regular then there exists an $\varepsilon > 0$ such that any regular $\tilde{\zeta}$, with $||\zeta - \tilde{\zeta}|| < \varepsilon$, is of non-negative circulation.*

Proof. For any $k > 0$ the curve $f((1-k)e^{it})$ is of non-negative circulation as is easily seen by Lemma 8. As in the proof of Lemma 4 let

$$\zeta^* = \frac{1}{2h}\int_{t-h}^{t+h}\zeta(\tau)d\tau$$

but let $\tilde{\zeta} = \zeta + i\rho(\zeta^*)'$.

Consider $\tilde{\zeta}$ as a mapping from the annulus $1-k \leqq \rho \leqq 1$ in the ρt-polar coordinate plane. The polar coordinate Jacobian is non-negative with ρ and k sufficiently small so that Lemma 8 again applies and one has that for every point P in the plane, not on $[\zeta]\cup[\tilde{\zeta}]$, the index $\omega(p,\zeta)$ of ζ about P is not less that $\omega(p,\tilde{\zeta})$;

$$\omega(p,\zeta) \geqq \omega(p,\tilde{\zeta}) \text{ for all } P \notin [\zeta]\cup[\tilde{\zeta}] .$$

But given k the ρ can be chosen so small that, by the continuity of the degree functions on the space of regular representations,

$$\omega(p,\tilde{\zeta}) \geqq \omega(P, f((1-k)e^{it})) \text{ for all } P \notin [\tilde{\zeta}]\cup[f((1-k)e^{it}] .$$

But with ε chosen sufficiently small,

$$\omega(p,\tilde{\zeta}) \geqq \omega(p,\tilde{\zeta}) \text{ for all } P \notin [\tilde{\zeta}]\cup[\tilde{\zeta}] .$$

Thus,

$$\omega(p,\tilde{\zeta}) \geqq \omega(p,\tilde{\zeta}) \geqq \omega(p, f((1-k)e^{it})) \geqq 0$$

for all $p \notin [\tilde{\zeta}]\cup[\tilde{\zeta}]\cup[f((1-k)e^{it})]$. Since none of the sets of image points involved contain any open sets it follows by a simple limiting process that

$$\omega(p,\tilde{\zeta}) \geqq 0 \text{ for all } p \notin [\tilde{\zeta}] ,$$

or, in other words, that $\tilde{\zeta}$ is of non-negative circulation. The proof of the lemma is complete.

Proof of Theorem 8. By Lemma 9 the curve ζ satisfies the hypotheses of Theorem 7. Therefore, ζ represents a simple positively oriented Jordan curve. Therefore the mapping f is one-to-one on the boundary and as is well known, even for general interior mappings (here, Lemma 8 easily applies) the mapping f is a homeomorphism.

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