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CRITERION FOR r TH POWER RESIDUACITY

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The Law of Quadratic Reciprocity in the rational integers states: If p, q are two distinct odd primes, then q is a square (mod p) if and only if $(-1)^{(p-1)/2}p$ is a square (mod q).

One of the classical generalizations of the law of reciprocity is of the following type. Let r be a fixed positive integer, $\phi(r)$ denotes the number of positive integers $\leq r$ which are relatively prime to r ; p, q are two distinct primes and $p \equiv 1 \pmod{r}$. Then can we find rational integers $a_1(p), a_2(p), \dots, a_h(p)$ determined by p , such that q is an r th power (mod p) if and only if $a_1(p), \dots, a_h(p)$ satisfy certain conditions (mod q).

The Law of Quadratic Reciprocity states that for $r = 2$, we may take $a_1(p) = (-1)^{(p-1)/2}p$.

Jacobi and Gauss solved this problem for $r = 3$ and $r = 4$, respectively. Mrs. E. Lehmer gave another solution recently [2].

In this paper I would like to develop the theory when r is a prime and $q \equiv 1 \pmod{r}$. I then show that q is an r th power (mod p) if and only if a certain linear combination of $a_1(p), \dots, a_{r-1}(p)$ is an r th power (mod q). $a_1(p), \dots, a_{r-1}(p)$ are determined by solving several simultaneous Diophantine equations. This determination appears mildly formidable and to make the actual numerical computations would certainly be so for a large r . (See Theorem B below.) Also given is a criterion for when r is an r th power (mod p) in terms of a linear combination of $a_1(p), \dots, a_{r-1}(p)$ (mod r^2). (See Theorem A below.)

It is possible by the methods developed in this paper to eliminate the conditions that r is a prime and $q \equiv 1 \pmod{r}$. This would complicate the paper a great deal, and the cases given clearly indicate the underlying theory.

Consider the following Diophantine equations in the rational integers:

$$(1) \quad r \sum_{j=1}^{r-1} X_j^2 - \left(\sum_{j=1}^{r-1} X_j \right)^2 = (r-1)p^{r-2}$$

$$(2) \quad \sum_i^{(1)} X_{j_1} X_{j_2} = \sum_i^{(1)} X_{j_1} X_{j_2} \quad i = 2, \dots, \frac{r-1}{2},$$

where $\sum_i^{(k)}$ denotes the sum over all $j_1, \dots, j_{k+1} = 1, 2, \dots, r-1$, with the condition $j_1 + \dots + j_k - kj_{k+1} \equiv i \pmod{r}$.

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$$(3) \quad 1 + \sum_{j=1}^{r-1} X_j \equiv \sum_{j=1}^{r-1} jX_j \equiv 0 \pmod{r}$$

(4) not all of the $X_j \equiv 0 \pmod{p}$ and

$$\sum_i^{(k)} X_{j_1} \cdots X_{j_{k+1}} - \sum_0^{(k)} X_{j_1} \cdots X_{j_{k+1}} \equiv 0 \pmod{p^{r-k-1}}$$

for $k = 2, \dots, r - 2; i = 1, 2, \dots, r - 1$.

We shall prove in § II that there exist exactly $r - 1$ distinct integral solutions of the equations (1) through (4). In particular let $\{X_j = a_j, j = 1, \dots, r - 1\}$ be a solution. Then we prove that the $a_j(p) = a_j$ satisfy our residuacity criterion, namely

THEOREM A. *r is an r th power \pmod{p} if and only if*

$$\sum_{j=1}^{r-1} ja_j + \frac{1}{2} ra_{r-1} \equiv 0 \pmod{r^2} .$$

THEOREM B. *If $q \equiv 1 \pmod{r}$ and h is any integer such that h^r is the least power of h which is $\equiv 1 \pmod{q}$, then q is an r th power \pmod{q} if and only if $\sum_{j=1}^{r-1} a_j h^j$ is an r th power \pmod{q} .*

At the end of § II various special cases are considered.

In particular, for $q = 2, r = 5$, then 2 is a quintic power \pmod{p} if and only if $a_j \equiv a_{5-j} \pmod{2}, j = 1, 2$.

For $q = 2, r = 7$, then 2 is a 7th power \pmod{p} if and only if $a_j \equiv 1 \pmod{2}, i = 1, \dots, 6$.

Let $r = 3$. Then the solutions to the Diophantine equations (1) to (4) are (a_1, a_2) and (a_2, a_1) , where

$$(5) \quad p = a_1^2 - a_1 a_2 + a_2^2, a_1 \equiv a_2 \equiv 1 \pmod{3} .$$

Multiplying (5) by 4 and grouping terms gives

$$4p = (a_1 + a_2)^2 + 3(a_1 - a_2)^2 .$$

Let $L = -a_1 - a_2, M = (a_1 - a_2)/3$. This gives the representation which Lehmer employs:

$$4p = L^2 + 27M^2, L \equiv 1 \pmod{3} .$$

Theorem A states that 3 is a cubic residue \pmod{p} if and only if $a_1 \equiv a_2 \pmod{9}$. This, in turn, is equivalent to M being divisible by 3, the condition quoted by Lehmer.

I. Notation. r denotes a prime number, ζ_r a primitive r th root of unity, Q the rational numbers, $Q(\zeta_r)$ the cyclotomic field over Q generated by ζ_r . For $j = 1, 2, \dots, r - 1, \sigma_j$ are the automorphisms of $Q(\zeta_r)/Q$

such that $\sigma_j(\zeta_r) = \zeta_r^j$. $\sigma^{-1}(\zeta_r) = \zeta_r^{j'}$, where $jj' \equiv 1 \pmod{r}$. p denotes a positive rational prime $\equiv 1 \pmod{r}$, and $\chi_p = \chi$ will be any primitive r th power character \pmod{p} .

$$g(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta_p^n$$

will be the Gaussian sum associated with χ_p . $\langle \alpha \rangle$ denotes the fractional part of α ; i.e., $\langle \alpha \rangle = \alpha - [\alpha]$.

- LEMMA 1. (i) $|g(\chi^k)|^2 = p$,
 (ii) $g(\chi)^k g(\chi^{-k}) \in Q(\zeta_r)$,
 (iii) $g(\chi)^r \in Q(\zeta_r)$, and
 (iv) $\sigma_k(g(\chi)^r) = g(\chi^k)^r$
 for $k = 1, 2, \dots, r - 1$.

Proof. (i) is the classical result about the absolute value of $g(\chi)$ and can easily be deduced from the definition of $g(\chi)$. (ii), (iii) and (iv) follow from Galois Theory using the relation $\sum_{n=1}^{p-1} \chi(n)\zeta_p^{nt} = \chi(t)^{-1}g(\chi)$ for any integer t prime to p .

LEMMA 2. *There exists a prime ideal \mathfrak{p} in $Q(\zeta_r)$ dividing p such that $(g(\chi^k)^r) = \sum_{j=1}^{r-1} \sigma_j^{-1} \mathfrak{p}^{r\langle kj/r \rangle}$.*

Conversely, given any prime ideal \mathfrak{p}_1 in $Q(\zeta_r)$ dividing p , there exists a k such that

$$(g(\chi^k)^r) = \sum_{j=1}^{r-1} \sigma_j^{-1} \mathfrak{p}_1^j.$$

Proof. Lemma 2 is a result of Stickelberger. For a proof see Davenport and Hasse [1]. See especially the elegant proof on page 181-2. In $Q(\zeta_r)$, the ideal $(r) = (1 - \zeta_r)^{r-1}$,

LEMMA 3. $(1 - \zeta_r^t)(1 - \zeta_r)^{-1} \equiv t \pmod{(1 - \zeta_r)}$ and $r(1 - \zeta_r^t)^{-r+1} \equiv -1 \pmod{(1 - \zeta_r)}$ for $(t, r) = 1$.

Proof. The first fact follows as

$$(1 - \zeta_r^t)(1 - \zeta_r)^{-1} = \sum_{j=0}^{t-1} \zeta_r^j \equiv \sum_{j=0}^{t-1} 1 \equiv t \pmod{(1 - \zeta_r)}.$$

The second follows from Wilson's Theorem as

$$\begin{aligned} r(1 - \zeta_r^t)^{-r+1} &= \left(\prod_{j=1}^{r-1} (1 - \zeta_r^{jt}) \right) (1 - \zeta_r^t)^{-r+1} \\ &= \prod_{j=1}^{r-1} (1 - \zeta_r^{jt})(1 - \zeta_r^t)^{-1} \equiv (r - 1)! \equiv -1 \pmod{(1 - \zeta_r)}. \end{aligned}$$

THEOREM 1. *For any t not divisible by r ,*

$$g(\chi^t)^r + 1 \equiv r(1 - \chi(r)^{-t}) \pmod{(1 - \zeta_r)^{r+1}},$$

and consequently, $\chi(r) = 1$ if and only if

$$g(\chi^t)^r + 1 \equiv 0 \pmod{(1 - \zeta_r)^{r+1}}.$$

Proof. As

$$g(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta_p^n,$$

the binomial theorem yields

$$\begin{aligned} -g(\chi)^r &= \left(-\sum_{n=1}^{p-1} \zeta_p^n + \sum_{n=1}^{p-1} (1 - \chi(n))\zeta_p^n \right)^r = \left(1 + \sum_n (1 - \chi(n))\zeta_p^n \right)^r \\ &\equiv 1 + r \sum_n (1 - \chi(n))\zeta_p^n + \sum_n (1 - \chi(n))^r \zeta_p^{rn} \pmod{(1 - \zeta_r)^{r+1}}, \end{aligned}$$

as all other terms are divisible by at least $r(1 - \zeta_r)^2$. By Lemma 3, if $\chi(n) \neq 1$, $(1 - \chi(n))^{r-1} \equiv -r \pmod{(1 - \zeta_r)^r}$, and clearly, if $\chi(n) = 1$,

$$(1 - \chi(n))^r \equiv -r(1 - \chi(n)) \pmod{(1 - \zeta_r)^{r+1}}.$$

Thus,

$$\begin{aligned} -g(\chi)^r &\equiv 1 + r \left(\sum_{n=1}^{p-1} (1 - \chi(n))\zeta_p^n - (1 - \chi(n))\zeta_p^{rn} \right) \\ &\equiv 1 + r \sum_n (1 - \chi(n))\zeta_p^n - (1 - \chi(n)\chi(r)^{-1})\zeta_p^{rn} \\ &\equiv 1 - r(1 - \chi(r)^{-1}) \sum_n \chi(n)\zeta_p^n \\ &\equiv 1 - r(1 - \chi(r)^{-1}) \sum_n \zeta_p^n \\ &\equiv 1 + r(1 - \chi(r)^{-1}) \pmod{(1 - \zeta_r)^{r+1}}. \end{aligned}$$

By (iv) of Lemma 1,

$$-g(\chi^t)^r = -\sigma_t(g(\chi)^r) \equiv 1 + r(1 - \chi(r)^{-t}) \pmod{(1 - \zeta_r)^{r+1}},$$

which completes the first statement of Theorem 1. The second statement in Theorem 1 then follows immediately.

Let q denote any positive rational prime other than r , f the least positive integer such that $q^f \equiv 1 \pmod{r}$, and $ef = r - 1$. Then in $Q(\zeta_r)$ the ideal $(q) = \mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_e$, where the \mathfrak{A}_j are prime ideals and

$$(6) \quad \text{Norm}_{Q(\zeta_r)/Q}(\mathfrak{A}_j) = q^f.$$

In the following let \mathfrak{A} be any of the e prime divisors \mathfrak{A}_j , $j = 1, \dots, e$.

THEOREM 2. *Let q , p , and r be distinct.*

Then

$$(7) \quad g(\chi)^{a^{f-1}} \equiv \chi(q)^{-f} \pmod{q} .$$

Consequently $\chi(q) = 1$ if and only if

$$(8) \quad g(\chi)^r \equiv \beta^r \pmod{\mathfrak{A}} \text{ for some } \beta \in Q(\zeta_r) .$$

Proof.
$$g(\chi)^{a^f} = \left(\sum_{n=1}^{p-1} \chi(n) \zeta_p^{na} \right)^{a^f}$$

$$\equiv \sum_{n=1}^{p-1} \chi(n)^{a^f} \zeta_p^{na^f} \pmod{q}$$

$$\equiv \sum_n \chi(n) \zeta_p^{na^f} \pmod{q}, \text{ as } r \mid q^f - 1 ,$$

$$\equiv \chi(q)^{-f} g(\chi) \pmod{q} .$$

Multiplying both sides of the above congruence by $\overline{g(\chi)}$, and noting (i) of Lemma 1, yields

$$p g(\chi)^{a^{f-1}} \equiv \chi(q)^{-f} p \pmod{q} \text{ or } g(\chi)^{a^{f-1}} \equiv \chi(q)^{-f} \pmod{q} ,$$

as p and q are distinct primes. Hence, we have proved (7).

Note that as $r \mid q^f - 1$, (7) becomes a congruence in $Q(\zeta_r)$. As $f \mid r - 1$, $(f, r) = 1$, we have by (7) that $\chi(q) = 1$ if and only if $g(\chi)^{a^{f-1}} \equiv 1 \pmod{\mathfrak{A}}$.

(Note that $1 - \zeta_r^t \not\equiv 0 \pmod{\mathfrak{A}}$ unless $\zeta_r^t = 1$.)

If $g(\chi)^r \equiv \beta^r \pmod{\mathfrak{A}}$ for some $\beta \in Q(\zeta_r)$, then

$$g(\chi)^{a^{f-1}} \equiv \beta^{a^{f-1}} \equiv 1 \pmod{\mathfrak{A}}$$

by (6).

Conversely, if $g(\chi)^{a^{f-1}} \equiv 1 \pmod{\mathfrak{A}}$ then $(g(\chi)^r)^{(a^{f-1})/r} \equiv 1 \pmod{\mathfrak{A}}$. By Lemma 1, $g(\chi)^r \in Q(\zeta_r)$. By (6) this implies $g(\chi)^r \equiv \beta^r \pmod{\mathfrak{A}}$. (Euler's Criterion for r th powers.)

In the above argument we must bear in mind that $g(\chi) \notin Q(\zeta_r)$.

II. In the last section we have developed a criterion for r th power residuacity in $Q(\zeta_r)$. From this we derive a criterion in the rational numbers Q , which is the purpose of Theorems A and B.

First let us assume that there is a rational integral solution $X_j = a_j$, of equations (1), (2), (3) and (4). In $Q(\zeta_r)$ define the algebraic integer $\alpha = \sum_{j=1}^{r-1} a_j \zeta_r^j$. We shall prove that α satisfies

$$(9) \quad |\sigma_k(\alpha)|^2 = p^{r-2}, \quad k = 1, 2, \dots, r-1 .$$

$$(10) \quad (p\alpha)^k \sigma_k(p\alpha)^{-1}$$

is also an algebraic integer in $Q(\zeta_r)$, for $k = 1, 2, \dots, r-1$.

To prove (9) we note that

$$\begin{aligned} |\alpha|^2 &= \left(\sum_j a_j \zeta_r^j \right) \left(\sum_i a_i \zeta_r^{r-i} \right) \\ &= \sum_{j,i} a_j a_i \zeta_r^{j-i} \\ &= \sum_{j=1}^{r-1} a_j^2 + \sum_{i=1}^{r-1} \left(\sum_i^{(1)} a_{j_1} a_{j_2} \right) \zeta_r^i. \end{aligned}$$

By (2) all of the coefficients of ζ_r^i are equal, since for any i , the sums corresponding to i and $r-i$ are identical. Thus

$$\begin{aligned} |\alpha|^2 &= \sum_j a_j^2 - \sum_{i=1}^{(1)} a_{j_1} a_{j_2} \\ &= \sum_j a_j^2 - (r-1)^{-1} \sum_{i=1}^{r-1} \sum_i^{(1)} a_{j_1} a_{j_2} \\ &= r(r-1)^{-1} \sum_j a_j^2 - (r-1)^{-1} \sum_{i=0}^{r-1} \sum_i^{(1)} a_{j_1} a_{j_2} \\ &= r(r-1)^{-1} \sum_{j=1}^{r-1} a_j^2 - (r-1)^{-1} \left(\sum_{j=1}^r a_j \right)^2 \\ &= p^{r-2} \end{aligned}$$

by (1). Similarly $|\sigma_k(\alpha)|^2 = p^{r-2}$. Thus (1) and (2) imply (9).

Let k be a fixed integer $2 \leq k \leq r-1$. Then

$$\begin{aligned} (11) \quad (p\alpha)^k \sigma_k(p\alpha)^{-1} &= p^{k-1} \alpha^k \sigma_k(\alpha)^{-1} \\ &= p^{k-1} \alpha^k \sigma_{-k}(\alpha) |\sigma_k(\alpha)|^{-2} \\ &= p^{-r+k+1} \alpha^k \sigma_{-k}(\alpha) \end{aligned}$$

by (10). Now

$$\begin{aligned} (12) \quad \alpha^k \sigma_{-k}(\alpha) &= \left(\sum a_j \zeta_r^j \right)^k \left(\sum a_j \zeta_r^{-jk} \right) \\ &= \sum_{i=0}^{r-1} \left(\sum_i^{(k)} a_{j_1} \cdots a_{j_{k+1}} \right) \zeta_r^i \\ &= \sum_{i=1}^{r-1} \left(\sum_i^{(k)} - \sum_0^{(k)} \right) \zeta_r^i. \end{aligned}$$

Condition (4) implies that each coefficient of ζ_r^i in (12) is divisible by p^{r-k-1} . Placing this information in (11) states that $(p\alpha)^k \sigma_k(p\alpha)^{-1}$ is an integer; thus proving (10).

(4) also tells us that p , but not p^2 , divides $p\alpha$, as not all the coefficients of ζ_r^j in $\alpha = \sum_{j=1}^{r-1} a_j \zeta_r^j$ are divisible by p .

If we restate the above facts in terms of ideals, we have that $(p\alpha)$ is an integral ideal in $Q(\zeta_r)$ divisible only by the prime ideals which divide p .

There exists one prime ideal, say \mathfrak{p} , dividing p , which divides $p\alpha$ but \mathfrak{p}^2 does not divide $p\alpha$. All other prime factors of p in $Q(\zeta_r)$ are of the form $\sigma_i^{-1}\mathfrak{p}$. Hence,

$$(13) \quad (p\alpha) = \sum_{i=1}^{r-1} \sigma_i^{-1} p^{d_i} \text{ where } d_1 = 1, d_i > 0 .$$

By (9)

$$\begin{aligned} (p\alpha)(\sigma_{-1}(p\alpha)) &= (p^2 | \alpha|^2) = p^r \\ &= \left(\prod_i \sigma_i^{-1} p^{d_i} \right) \left(\prod_i \sigma_{-1} \sigma_i^{-1} p^{d_i} \right) \\ &= \prod_i \sigma_i^{-1} p^{d_i + d_{r-i}} \end{aligned}$$

or

$$(14) \quad d_i + d_{r-i} = r .$$

By (10), $(p\alpha)^k \sigma_k(p\alpha)^{-1}$ is integral, or

$$\begin{aligned} (p\alpha)^k (\sigma_k(p\alpha))^{-1} &= \prod_i \sigma_i^{-1} p^{d_{ik}} \prod_i \sigma_k \sigma_i^{-1} p^{-d_i} \\ &= \prod_i \sigma_i^{-1} p^{d_{ik} - d_{ik}} \end{aligned}$$

is an integral ideal. (The index of d_{ik} is interpreted mod r .) Hence, $kd_i \geq d_{ik}$.

As $d_1 = 1, k \geq d_k$ for $k = 2, 3, \dots, r - 2$. By (14) this yields that $d_k = k$. By Lemma 2, we arrive at the fact that in terms of ideals

$$(15) \quad (p\alpha) = (g(\chi^t)^r) \text{ for some } 1 \leq t < r .$$

In proving (15) we have used (1), (2) and (4). We wish to prove that $p\alpha = g(\chi^t)^r$. To do this we now utilize (3). By (15) we have that for some unit $\eta \in Q(\zeta_r), g(\chi^t)^r = \eta p\alpha$, or

$$(16) \quad g(\chi^{tk})^r = \sigma_k(\eta p\alpha) = \sigma_k(\eta) \sigma_k(p\alpha) .$$

Taking the absolute value of both sides of (16) and utilizing (i) of Lemma 1 and (9) gives $p^r = |\sigma_k(\eta)|^2 p^r$, or $|\sigma_k(\eta)|^2 = 1$. By a Theorem of Dirichlet on units (See [3] Theorem IV 9, A pp. 174), any unit which has all of its conjugates with absolute value 1 is then a root of unity. As $\eta \in Q(\zeta_r), \eta = \pm \zeta_r^s$.

Now

$$\begin{aligned} \alpha &= \sum_{j=1}^r a_j \zeta_r^j = \sum_j a_j - \sum_j a_j (1 - \zeta_r^j) \\ &\equiv \sum_j a_j - \sum_j j a_j (1 - \zeta_r) \pmod{(1 - \zeta_r)^2} , \end{aligned}$$

by Lemma 3. As $p \equiv 1 \pmod{r}, p \equiv 1 \pmod{(1 - \zeta_r)^2}$. By (3),

$$1 + \sum_j a_j \equiv \sum_j j a_j \equiv 0 \pmod{r} .$$

Hence, $p\alpha \equiv -1 \pmod{(1 - \zeta_r)^2}$. By Theorem 1, $g(\chi^t)^r \equiv -1 \pmod{(1 - \zeta_r)^2}$. Therefore, $\eta \equiv 1 \pmod{(1 - \zeta_r)^2}$. But $\eta = \pm \zeta_r^s \equiv \pm(1 + s(1 - \zeta_r)) \pmod{(1 - \zeta_r)^2}$; i.e., $s \equiv 0 \pmod{r}$ and the + sign holds. Hence, $\eta = 1$.

Therefore, if the a_j are any integral solution of (1), (2), (3) and (4), there exists an integer $1 \leq t \leq r - 1$ such that

$$(17) \quad p \sum_{j=1}^{r-1} a_j \zeta_r^j = g(\chi^t)^r .$$

Conversely, given any integer $t, 1 \leq t \leq r - 1$, and writing

$$g(\chi^t)^r = p \sum_{j=1}^{r-1} a_j \zeta_r^j ,$$

we can prove that the a_j are rational integers which satisfy (1), (2), (3), and (4). The proof is merely reversing the above steps we used in proving (17). By Lemma 2 the prime factorizations of $(g(\chi^s)^r)$ and $(g(\chi^t)^r)$, $1 \leq s < t \leq r - 1$, are distinct, and thus $g(\chi^s)^r \neq g(\chi^t)^r$. Hence, we have shown that there are precisely $r - 1$ rational integral solutions of (1), (2), (3), and (4).

We are now in a position to prove Theorems A and B. First for Theorem A.

Let a_j be an integral solution of (1) through (4). Then we have shown that $p \sum_{j=1}^{r-1} a_j \zeta_r^j = g(\chi^t)^r$ for some integer t relatively prime to r . By Theorem 1, the above states that $\chi(r) = 1$ if and only if $p \sum_j a_j \zeta_r^j \equiv -1 \pmod{(1 - \zeta_r)^{r+1}}$.

Define $b_s, s = 0, 1, \dots, r - 2$, by $b_0 = -pa_{r-1}, b_s = p(a_s - a_{r-1}), s = 1, 2, \dots, r - 2$. Then

$$p \sum_{j=1}^{r-1} a_j \zeta_r^j = \sum_{s=0}^{r-2} b_s \zeta_r^s .$$

Further let

$$C_i = (-1)^i \sum_{s=i}^{r-2} \binom{s}{i} b_s ,$$

where $\binom{s}{i}$ is the binomial coefficient. Then

$$\begin{aligned} p \sum_{j=1}^{r-1} a_j \zeta_r^j &= \sum_{s=0}^{r-2} b_s \zeta_r^s = \sum_s b_s (1 - (1 - \zeta_r))^s \\ &= \sum_s b_s \sum_{i=0}^s (-1)^i \binom{s}{i} (1 - \zeta_r)^i \\ &= \sum_{i=0}^{r-2} C_i (1 - \zeta_r)^i . \end{aligned}$$

The first statement in Theorem 1 states that $g(\chi^t)^r + 1 \equiv 0 \pmod{(1 - \zeta_r)^r}$. Hence,

$$\begin{aligned} \sum_{i=0}^{r-2} C_i (1 - \zeta_r)^i + 1 &\equiv (C_0 + 1) + \sum_{i=1}^{r-2} C_i (1 - \zeta_r)^i \\ &\equiv 0 \pmod{(1 - \zeta_r)^r} \end{aligned}$$

This implies that $C_0 + 1 \equiv 0 \pmod{r^2}$. Hence,

$$\sum_{i=0}^{r-2} C_i(1 - \zeta_r)^i \equiv C_1(1 - \zeta_r) \pmod{(1 - \zeta_r)^{r+1}}$$

or that $\chi(r) = 1$ if and only if

$$(18) \quad C_1 \equiv 0 \pmod{r^2} .$$

Now

$$\begin{aligned} (19) \quad C_1 &= (-1) \sum_{s=1}^{r-2} \binom{s}{1} b_s = - \sum_{s=1}^{r-2} s b_s \\ &= -p \sum_{s=1}^{r-2} s(a_s - a_{r-1}) \\ &= -p \sum_{s=1}^{r-2} s a_s + \frac{1}{2} p(r-2)(r-1) a_{r-1} \\ &\equiv -p \left(\sum_{s=1}^{r-1} s a_s + \frac{1}{2} r a_{r-1} \right) \pmod{r^2} . \end{aligned}$$

Equations (18) and (19) complete the proof of Theorem A.

Theorem B is also derived immediately from Theorem 2. If $q \equiv 1 \pmod{r}$, q a positive rational prime, then in $Q(\zeta_r)$, $(q) = \mathfrak{A}_1 \mathfrak{A}_2 \cdots \mathfrak{A}_{r-1}$, where \mathfrak{A}_j are prime ideals and $\text{Norm}_{Q(\zeta_r), Q} \mathfrak{A}_j = q$.

We may take $0, 1, 2, \dots, q-1$ as a set of residues $\pmod{\mathfrak{A}_1}$. Hence, as $1 - \zeta_r^t \not\equiv 0 \pmod{\mathfrak{A}_1}$, unless $\zeta_r^t = 1$, $\zeta_r \equiv h \pmod{\mathfrak{A}_1}$, where h is a rational integer such that $h^r \equiv 1 \pmod{q}$.

Thus by Theorem 2, $\chi(q) = 1$ if and only if there is a $\beta \in Q(\zeta_r)$ such that $g(\chi^t)^r = p \sum_j a_j \zeta_r^{jt} \equiv p \sum_j a_j h^j \equiv \beta^r \pmod{\mathfrak{A}_1}$.

We may take $\beta = b \in Q$ by the above remarks.

Hence, $\chi_p(q) = 1$ if and only if $\chi_q(p \sum_j a_j h^j) = 1$ where χ_q is a primitive r th power character \pmod{q} .

If we had chosen another h_1 whose order was $r \pmod{q}$, then $h_1 \equiv h^t \pmod{\mathfrak{A}_1}$, and

$$p \sum_j a_j h_1^j \equiv p \sum_j a_j \zeta_r^{jt} \equiv g(\chi^t)^r \pmod{\mathfrak{A}_1} .$$

Thus, any h whose order \pmod{q} is r works equally well in Theorem B.

There are several special cases one can derive when $q \not\equiv 1 \pmod{r}$, in particular, when $q = 2$, and $r = 5, 7$.

If $q = 2, r = 5$, then in $Q(\zeta_5)$, 2 remains a prime because 2^4 is the least power of 2 congruent to 1 $\pmod{5}$. One can easily compute that the only elements in $Q(\zeta_5)$ which are fifth powers $\pmod{2}$ are $1 = -\sum_{j=1}^4 \zeta_5^j, \zeta_5 + \zeta_5^{-1}$, and $\zeta_5^2 + \zeta_5^{-2} \pmod{2}$. Hence, for $r = 5, \chi_p(2) = 1$ if and only if $a_j \equiv a_{5-j} \pmod{2}$.

For $q = 2, r = 7$, then $2^3 \equiv 1 \pmod{7}$. Hence, in $Q(\zeta_7)$, $(2) = \mathfrak{A}_1 \mathfrak{A}_2$ where $\text{Norm} \mathfrak{A}_i = 8$. For $\alpha \equiv \beta^7 \pmod{\mathfrak{A}_1}, \beta \not\equiv 0 \pmod{\mathfrak{A}_1}$, and $\beta \in Q(\zeta_7)$

implies $\alpha \equiv 1 \pmod{\mathfrak{A}_1}$. Hence, for $r = 7$, $\chi_p(2) = 1$ if and only if $a_j \equiv 1 \pmod{2}$ for $j = 1, \dots, 6$.

One could easily generalize this to the case when $r = 2^s - 1$. Then $\chi_p(2) = 1$ if and only if $a_j \equiv 1 \pmod{2}$ for $j = 1, \dots, r - 1$.

BIBLIOGRAPHY

1. H. Davenport and H. Hasse, *Die Nullstellen der Kongruenzzetafunktionen in gewissen zyklischen Fällen*, Journal für die reine und angewandte Mathematik, Band CLXXII, (1935), 151-182.
2. E. Lehmer, *Criteria for cubic and quartic residuacity*, Mathematika, **5**, Part 1, (1958), 20-29.
3. H. Weyl, *Algebraic Theory of Numbers*, Annals of Mathematical Studies, Princeton University Press, 1940.

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