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SUMMABILITY OF DERIVED CONJUGATE SERIES

BILLY JOE BOYER

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SUMMABILITY OF DERIVED CONJUGATE SERIES

B. J. BOYER

1. Introduction. In a recent paper ([3] it was shown that the summability of the successively derived Fourier series of a *CP* integrable function could be characterized by that of the Fourier series of another *CP* integrable function. It is the purpose of the present paper to give analogous theorems for the successively derived conjugate series of a Fourier series.

2. Definitions. The terminology used in [3] will be continued in this paper. In addition let us define:

$$\psi(t) = \psi(t, r, x) = \frac{1}{2}[f(x + t) + (-1)^{r-1}f(x - t)] \tag{1}$$

$$Q(t) = \sum_{i=0}^{[\frac{r-1}{2}]} \frac{\bar{a}_{r-1-2i}}{(r - 1 - 2i)!} t^{r-1-2i} \tag{2}$$

$$g(t) = r!t^{-r}[\psi(t) - Q(t)] \tag{3}$$

The rth derived conjugate series of the Fourier series of $f(t)$ at $t = x$ will be denoted by $D_rCFSf(x)$, and the nth mean of order (α, β) of $D_rCFSf(x)$ by $\bar{S}^r_{\alpha,\beta}(f, x, n)$.

3. Lemmas.

LEMMA 1. *For $\alpha = 0, \beta > 1$ or $\alpha > 0, \beta \geqq 0$, and $r \geqq 0$,*

$$\bar{\lambda}^{(r)}_{1+\alpha,\beta}(x) = -\pi^{-1}r!(-x)^{r+1} + 0(|x|^{-1-\alpha}\log^{-\beta}|x|) + 0(|x|^{-r-2}) \text{ as } |x| \to \infty .$$

This is a result due to Bosanquet and Linfoot [2].

LEMMA 2. *For $\alpha > 0, \beta \geqq 0$ or $\alpha = 0, \beta > 0$ and*

$$r \geqq 0, x^r\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(x) = \sum_{i,j=0}^{r} B^r_{ij}(\alpha, \beta)\bar{\lambda}_{1+\alpha+r-i,\beta+j}(x) ,$$

where the B^r_{ij} are independent from x and have the properties:

(i) $B^r_{ij}(\alpha, 0) = 0$ *for* $j \geqq 1$;

(ii) $B^r_{r0}(\alpha, \beta) \neq 0$;

(iii) $\sum_{i,j=0}^{r} B^r_{ij}(\alpha, \beta) = (-1)^r r!$.

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The proofs of (i) and (ii) will be found in [3], Lemma 2, taking the imaginary parts of the equations there. Part (iii) follows immediately from the first part of the lemma and Lemma 1.

LEMMA 3. *For $n > 0$, $\alpha = 0$, $\beta > 1$ or $\alpha > 0$, $\beta \geqq 0$, and $r \geqq 0$,*

$$\left(\frac{d}{dt}\right)^r \left\{2B\pi^{-1}\sum_{\nu \leqq n}\left(1 - \frac{\nu}{n}\right)^{\alpha} \log^{-\beta}\left(\frac{C}{1 - \frac{\nu}{n}}\right) \sin \nu t\right\}$$

$$= 2n^{r+1} \sum_{k=-\infty}^{\infty} \bar{\lambda}^{(r)}_{1+\alpha,\beta}[n(t + 2k\pi)] .$$

Proof. Smith ([6], Lemma 6) has shown that for every odd, Lebesgue integrable function, $Z(t)$, of period 2π,

$$\bar{S}_{\alpha,\beta}(Z, 0, n) = -2n\int_0^{\infty} Z(t)\bar{\lambda}_{1+\alpha,\beta}(nt)dt .$$

Since the right side of this equation can be written

$$-2n\int_0^{\pi} Z(t) \sum_{k=-\infty}^{\infty} \bar{\lambda}_{1+\alpha,\beta}[n(t + 2k\pi)]dt$$

for every such $Z(t)$, the lemma is true for $r = 0$. For $r \geqq 1$ the interchange of $(d/dt)^r$ and $\sum_{-\infty}^{\infty}$ is justified by uniform convergence.

The following lemma is a direct consequence of Lemma 3:

LEMMA 4. *Let $f(x) \in CP[-\pi, \pi]$ and be of period 2π. For $n > 0$ and $\alpha = 0$, $\beta > 1$ or $\alpha > 0$, $\beta \geqq 0$,*

$$\bar{S}^r_{\alpha,\beta}(f, x, n) = 2(-n)^{r+1}\int_0^{\pi} \psi(t) \sum_{k=-\infty}^{\infty} \bar{\lambda}^{(r)}_{1+\alpha,\beta}[n(t + 2k\pi)]dt .$$

LEMMA 5. *For $\alpha \geqq 0$, $\beta \geqq 0$, $n > 0$ and $r \geqq 0$,*

$$n^{r+1}\int_0^{\infty} Q(t)\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(nt)dt = 0 ,$$

where $Q(t)$ is defined by (2).

Proof. If $r = 0$, then $Q(t) = 0$. For $r \geqq 1$ and $i = 0, 1, \cdots [r-1/2]$, the truth of the lemma follows from the equation:

$$\int_0^{\infty} x^{r-1-2i}\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(x)dx = 0 ,$$

which is easily verified by means of $r - 1 - 2i$ integrations by parts.

The final two lemmas of this section give the appropriate representation of the nth mean of $D_rCFSf(x)$.

LEMMA 6. *Let* $f(x) \in C_\lambda P[-\pi, \pi]$ *and be of period* 2π. *Let* m, $0 \leqq m \leqq \lambda + 1$, *be an integer for which* $\Psi_m(t) \in L[0, \pi]$. *Then, for* $\alpha = m$, $\beta > 1$ *or* $\alpha > m$, $\beta \geqq 0$ *and* $r \geqq 0$,

$$\bar{S}^r_{\alpha+r,\beta}(f, x, n) = 2(-n)^{r+1}\int_0^\pi [\psi(t) - Q(t)]\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(nt)dt + C_r + o(1)$$

as $n \to \infty$, *where*

$$C_r = 2\pi^{-1}(-1)^{r+1}\int_0^\pi \psi(t)\left(\frac{d}{dt}\right)^r\left[\frac{1}{2}ctn\frac{1}{2}t - t^{-1}\right]dt + 2r!\pi^{-1}\int_\pi^\infty t^{-r-1}Q(t)dt\,. \tag{4}$$

Proof. It follows from Lemmas 4 and 5 that

$$\begin{aligned}\bar{S}^r_{\alpha+r,\beta}(f, x, n) &= 2(-n)^{r+1}\int_0^\pi [\psi(t) - Q(t)]\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(nt)dt \\ &\quad + 2(-n)^{r+1}\int_0^\pi \psi(t)\sum_{-\infty}^{\infty}{}'\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}[n(t + 2k\pi)]\,dt \\ &\quad + -2(-n)^{r+1}\int_\pi^\infty Q(t)\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}(nt)dt \\ &= I_1 + I_2 + I_3\,.\end{aligned} \tag{5}$$

Since the degree of $Q(t)$ is $r - 1$, Lemma 1 shows that

$$I_3 = 2r!\pi^{-1}\int_\pi^\infty t^{-r-1}Q(t)dt + o(1)\,. \tag{6}$$

Let us define:

$$J(n, t) = 2(-n)^{r+1}\sum_{-\infty}^{\infty}{}'\{\bar{\lambda}^{(r)}_{1+\alpha+r,\beta}[n(t + 2k\pi)] - (-1)^r r!\pi^{-1}[n(t + 2k\pi)]^{-r-1}\}\,.$$

Again appealing to Lemma 1, we see that $\lim_{n\to\infty}(\partial/\partial t)^j J(n, t) = 0$ uniformly for $t \in [0, \pi]$ and $j = 0, 1, \cdots, m$.

With the aid of the well-known cotangent expansion I_2 may be written:

$$I_2 = \int_0^\pi \psi(t)J(n, t)dt + (-1)^{r+1}2\pi^{-1}\int_0^\pi \psi(t)\left(\frac{d}{dt}\right)^r\left[\frac{1}{2}ctn\frac{1}{2}t - t^{-1}\right]dt\,. \tag{7}$$

But after m integrations by parts, it is seen that

$$\int_0^\pi \psi(t)J(n, t)dt = o(1)\,. \tag{8}$$

The lemma now follows from equations (5), (6), (7), and (8).

A particular, but useful, case of Lemma 6 is

LEMMA 7. *Let* $f(x) \in C_\lambda P[-\pi, \pi]$ *and be of period* 2π. *If* $g(t) \in C_\mu P[0, \pi]$, *where* $g(t)$ *is defined by* (3), *then*

$$\bar{S}_{\alpha,\beta}(g, 0, n) = -2n\int_0^\pi g(t)\bar{\lambda}_{1+\alpha,\beta}(nt)dt - 2\pi^{-1}\int_0^\pi g(t)\left(\frac{1}{2}ctn\frac{1}{2}t - t^{-1}\right)dt + o(1)$$

for $\alpha = 1 + \xi, \beta > 1$ *or* $\alpha > 1 + \xi, \beta \geqq 0$, *where* $\xi = \min[\mu, \max(r, \lambda)]$.

The hypothese of Lemma 6 are fulfilled, because $t^r g(t) \in C_\lambda P[0, \pi]$ implies $G_{1+\xi}(t) \in L[0, \pi]$ by Lemma 6 of [3].

4. Theorems.

THEOREM 1. *Let* $f(x) \in C_\lambda P[-\pi, \pi]$ *and be of period* 2π. *If there exist constants* $\bar{a}_{r-1-2i}$, $i = 0, 1, \cdots [r - 1/2]$, *such that*

(i) $g(t) \in C_\mu P[0, \pi]$ *for some integer* μ;

(ii) $CFSg(0) = s(\alpha, \beta)$ *for* $\alpha = 1 + \xi, \beta > 1$ *or* $\alpha > 1 + \xi, \beta \geqq 0$, *where* $\xi = \min[\mu, \max(r, \lambda)]$;

then $D_r CFSf(x) = S(\alpha + r, \beta)$, $s = \pi^{-1}\int_0^\pi g(t)ctn(1/2)tdt$ *and*

$$S = -2\pi^{-1}\int_0^\pi t^{-1}g(t)dt + C_r,$$

where C_r *is defined by equation* (4).

THEOREM 2. *Let* $f(x) \in C_\lambda P[-\pi, \pi]$ *and be of period* 2π. *If* $D_r CFSf(x) = S(\alpha + r, \beta)$ *for* $\alpha = 1 + \lambda, \beta > 1$ *or* $\alpha > 1 + \lambda, \beta \geqq 0$, *then there exist constants* $\bar{a}_{r-1-2i}$, $i = 0, 1, \cdots [r - 1/2]$, *such that*

(i) $g(t) \in C_\mu P[0, \pi]$ *for some integer* μ:

(ii) $CFSg(0) = s(\alpha', \beta')$, *where*

$\alpha' = 1 + \xi, \beta' > 1$ *if* $1 + \lambda \leqq \alpha < 1 + \xi$ *or* $\alpha = 1 + \xi, \beta \leqq 1$ $\alpha' = \alpha$, $\beta' = \beta$ *if* $\alpha = 1 + \xi, \beta > 1$ *or* $\alpha > 1 + \xi, \beta \geqq 0$, *and* ξ, s *and* S *have the values given in Theorem* 1.

Before passing to the proofs of these theorems, let us observe that the existence of the constants $\bar{a}_{r-1-2i}$ implies their uniqueness from the definition of $g(t)$. In fact, it can be shown that the $\bar{a}_{r-1-2i}$ are given by

$$D_{r-1-2i}FSf(x) = \bar{a}_{r-1-2i}(C), \qquad i = 0, 1, \cdots \left[\frac{r-1}{2}\right].$$

[1] Bosanquet ([1], Theorem 1) has shown this for $f(x)$ Lebesgue integrable and (C) replaced by Abel summability.

In addition it can be shown that when $f(x) \in L$, the sum, S, of $D_r CFSf(x)$ may be written

$$S = -2\pi^{-1}\int_{\to o(C)}^{\infty} t^{-1}g(t)dt \ .^{2}$$

Proof of Theorem 1. That $s = -\pi^{-1}\int_0^{\pi} g(t)ctn(1/2)tdt$ follows from the consistency of (α, β) summability and a result due to Sargent ([4], Theorem 3). Therefore, both $g(t)ctn(1/2)t$ and $t^{-1}g(t)$ are CP integrable over $[0, \pi]$.

From Lemma 7 we have

$$(9) \qquad \bar{S}_{\alpha,\beta}(g, 0, n) - s = -2n\int_0^{\pi} g(t)[\bar{\lambda}_{1+\alpha,\beta}(nt) - (\pi nt)^{-1}]dt + o(1) \ .$$

The left side of (9) is $o(1)$ by hypothesis. By consistency equation (9) remains valid if α is replaced by $\alpha + r - i$ and β by $\beta + j$, $i, j = 0, 1, \cdots r$. Therefore,

$$-2n\int_0^{\pi} g(t) \sum_{i,j=0}^{r} B_{ij}^{r}(\alpha, \beta)[\bar{\lambda}_{1+\alpha+r-i,\beta+j}(nt) - (\pi nt)^{-1}]dt = o(1) \ .$$

With the aid of Lemmas 2 and 6, the last equation becomes

$$\bar{S}_{\alpha+r,\beta}^{r}(f, x, n) = -2\pi^{-1}\int_0^{\pi} t^{-1}g(t)dt + C_r + o(1) \ .$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Due to the length of this proof and its similarity to the proof of Theorem 2, ([3]), only a brief outline of the proof will be given.

Putting $Q(t) = 0$, $\beta = 0$ and $p > \alpha + r$ in Lemma 6 and integrating the right-hand side of the resulting equation $\lambda + 1$ times, one can show that

$$D_{r+\lambda+1}CFS(\Psi_{\lambda+1}, 0, n) \text{ is summable } (C, p) \ .$$

A result due to Bosanquet ([1], Theorem 1) and the stepwise procedure employed in the proof of Theorem 2 ([3], equations 18 through 22) lead to the conclusion: $t^{-r-1}[\psi(t) - Q(t)] \in CP[0, \pi]$ for an appropriate polynomial $Q(t)$, i.e., $t^{-1}g(t) \in CP[0, \pi]$. From this statement and a results due to Sargent ([4], Theorem 3), $g(t) \in C_\mu P[0, \pi]$ for some integer μ and $CFSg(0) = s(C)$, where $s = \pi^{-1}\int_0^{\pi} g(t)ctn(1/2)tdt$.[3]

[2] Ibid. The difference in sign is due to the distinction between allied and conjugate series.

[3] The CP integrability of $g(t)ctn(1/2)t$ is equivalent to that of $t^{-1}g(t)$.

That S, the $(\alpha + r, \beta)$ sum of $D_rCFSf(x)$, has the value

$$-2\pi^{-1}\int_0^\pi t^{-1}g(t)dt + C_r$$

follows immediately from Theorem 1 and the consistency of the summability scale.

Thus, it remains to prove only the order relations (α', β') in (ii) of the theorem. A straightforward calculation using the representations in Lemmas 6 and 7, the properties of the $B_{ij}^r(\alpha, \beta)$ in Lemma 2, and the consistency of the summability scale applied to $D_rCFSf(x)$, leads to the following equations:

$$\sum_{i,j=0}^{r} B_{ij}^r(\alpha' + k, \beta')\Big[\bar{S}_{\alpha'+k+r-i,\beta'+j}(g, 0, n) - \pi^{-1}\int_0^\pi g(t)ctn\frac{1}{2}tdt\Big] = o(1) ,$$

for $k = 0, 1, 2, \cdots$.

The expression in brackets may be considered the nth mean of order $(\alpha' + k + r - i, \beta' + j)$ of a series formed from $CFSg(0)$ by altering the first term. Since this series is summable (C) to 0, then Lemma 8 [3] shows that $CFSg(0) = s(\alpha', \beta')$.

The following theorem gives a sufficient condition for the (α, β) summability of $CFSg(0)$ for $\beta \neq 0$. Since the proof follows the usual lines for Riesz summability, it is omitted.

THEOREM 3. *Let $g(t)$ be an odd function of period 2π. If $t^{-1}g(t) \in C_kP[0, \pi]$, where k is a non-negative integer, then*

$$CFSg(0) = -\pi^{-1}\int_0^\pi g(t)ctn\frac{1}{2}tdt(1 + k, \beta), \beta > 1 .$$

As an application of these theorems it can be shown that

$$D_rCFSf(0, m) = S(1 + m + 2r, \beta), \beta > 1 ,$$

where $f(x; m)$ is either $x^{-m}\sin x^{-1}$ or $x^{-m}\cos x^{-1}$, $m = 0, 1, 2, \cdots$.

The following results may be deduced from Theorems 1 and 2. It is assumed that $f(x) \in C_\lambda P[-\pi, \pi]$ and is of period 2π. The values of S and s, when either exists, and ξ are given in Theorem 1.

(A). If $g(t) \in C_\mu P[0, \pi]$, then for $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi, \beta \geqq 0$, $D_rCFSf(x) = S(\alpha + r, \beta)$ if and only if $CFSg(0) = s(\alpha, \beta)$.

(B). For $\alpha = 1 + \max(r, \lambda), \beta > 1$ or $\alpha > 1 + \max(r, \lambda), \beta \geqq 0$, $D_rCFSf(x) = S(\alpha + r, \beta)$ if and only if $g(t) \in CP[0, \pi]$ and $CFSg(0) = s(\alpha, \beta)$.

These results generalize, to various degrees, results obtained by Takahashi and Wang [7] and Bosanquet [1].

A weak, but none the less interesting, form of these results is

(C). If $f(x) \in CP[-\pi, \pi]$ and is of period 2π, then in order that $D_r CFSf(x)$ be summable (C), it is necessary and sufficient that $g(t) \in CP[0, \pi]$ and $CFSg(0)$ be summable (C).

REFERENCES

1. L. S. Bosanquet, *On the derived allied series of a Fourier series*, Proc. London Math. Soc. (2), **49** (1947), 63-76.

2. L. S. Bosanquet and E. H. Linfoot, *Generalized means and the summability of Fourier series*, Quart. Journal of Math. (Oxford), **2** (1931), 207-229.

3. B. J. Boyer, *On the summability of derived Fourier series*, Pacific J. Math. (To be published.)

4. W. L. C. Sargent, *On the summability (C) of allied series*, Proc. London Math. Soc., **50** (1949), 330-348.

5. A. H. Smith, *On the summability of derived conjugate series of the Fourier-Lebesgue type*, Bull. Amer. Math. Soc., **40** (1934), 406-412.

6. ———, *Summability of conjugate Fourier series*, Duke Math. Journal. **4** (1938), 270-276.

7. T. Takahashi and F. T. Wang, *Some notes on trigonometric series*, Tohoku Math. Journal, **41** (1935), 169-187.

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