

Pacific Journal of Mathematics

SUMMABILITY OF DERIVED CONJUGATE SERIES

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SUMMABILITY OF DERIVED CONJUGATE SERIES

B. J. BOYER

1. Introduction. In a recent paper ([3] it was shown that the summability of the successively derived Fourier series of a *CP* integrable function could be characterized by that of the Fourier series of another *CP* integrable function. It is the purpose of the present paper to give analogous theorems for the successively derived conjugate series of a Fourier series.

2. Definitions. The terminology used in [3] will be continued in this paper. In addition let us define:

$$(1) \quad \psi(t) = \psi(t, r, x) = \frac{1}{2}[f(x + t) + (-1)^{r-1}f(x - t)]$$

$$(2) \quad Q(t) = \sum_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} \frac{\bar{a}_{r-1-2i}}{(r-1-2i)!} t^{r-1-2i}$$

$$(3) \quad g(t) = r!t^{-r}[\psi(t) - Q(t)]$$

The *r*th derived conjugate series of the Fourier series of *f*(*t*) at *t* = *x* will be denoted by *D,CFSf*(*x*), and the *n*th mean of order (*α*, *β*) of *D,CFSf*(*x*) by $\bar{S}_{\alpha,\beta}^r(f, x, n)$.

3. Lemmas.

LEMMA 1. For $\alpha = 0, \beta > 1$ or $\alpha > 0, \beta \geq 0$, and $r \geq 0$,

$$\bar{\lambda}_{1+\alpha,\beta}^{(r)}(x) = -\pi^{-1}r!(-x)^{r+1} + 0(|x|^{-1-\alpha} \log^{-\beta} |x|) + 0(|x|^{-r-2}) \text{ as } |x| \rightarrow \infty .$$

This is a result due to Bosanquet and Linfoot [2].

LEMMA 2. For $\alpha > 0, \beta \geq 0$ or $\alpha = 0, \beta > 0$ and

$$r \geq 0, x^r \bar{\lambda}_{1+\alpha+r,\beta}^{(r)}(x) = \sum_{i,j=0}^r B_{ij}^r(\alpha, \beta) \bar{\lambda}_{1+\alpha+r-i,\beta+j}(x) ,$$

where the B_{ij}^r are independent from *x* and have the properties:

- (i) $B_{ij}^r(\alpha, 0) = 0$ for $j \geq 1$;
- (ii) $B_{r0}^r(\alpha, \beta) \neq 0$;
- (iii) $\sum_{i,j=0}^r B_{ij}^r(\alpha, \beta) = (-1)^r r!$.

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The proofs of (i) and (ii) will be found in [3], Lemma 2, taking the imaginary parts of the equations there. Part (iii) follows immediately from the first part of the lemma and Lemma 1.

LEMMA 3. For $n > 0, \alpha = 0, \beta > 1$ or $\alpha > 0, \beta \geq 0$, and $r \geq 0$,

$$\left(\frac{d}{dt}\right)^r \left\{ 2B\pi^{-1} \sum_{\nu \leq n} \left(1 - \frac{\nu}{n}\right)^\alpha \log^{-\beta} \left(\frac{C}{1 - \frac{\nu}{n}}\right) \sin \nu t \right\} \\ = 2n^{r+1} \sum_{k=-\infty}^{\infty} \bar{\lambda}_{1+\alpha, \beta}^{(r)} [n(t + 2k\pi)] .$$

Proof. Smith ([6], Lemma 6) has shown that for every odd, Lebesgue integrable function, $Z(t)$, of period 2π ,

$$\bar{S}_{\alpha, \beta}(Z, 0, n) = -2n \int_0^\infty Z(t) \bar{\lambda}_{1+\alpha, \beta}(nt) dt .$$

Since the right side of this equation can be written

$$-2n \int_0^\pi Z(t) \sum_{k=-\infty}^{\infty} \bar{\lambda}_{1+\alpha, \beta} [n(t + 2k\pi)] dt$$

for every such $Z(t)$, the lemma is true for $r = 0$. For $r \geq 1$ the interchange of $(d/dt)^r$ and $\sum_{-\infty}^{\infty}$ is justified by uniform convergence.

The following lemma is a direct consequence of Lemma 3:

LEMMA 4. Let $f(x) \in CP[-\pi, \pi]$ and be of period 2π . For $n > 0$ and $\alpha = 0, \beta > 1$ or $\alpha > 0, \beta \geq 0$,

$$\bar{S}_{\alpha, \beta}^r(f, x, n) = 2(-n)^{r+1} \int_0^\pi \psi_r(t) \sum_{k=-\infty}^{\infty} \bar{\lambda}_{1+\alpha, \beta}^{(r)} [n(t + 2k\pi)] dt .$$

LEMMA 5. For $\alpha \geq 0, \beta \geq 0, n > 0$ and $r \geq 0$,

$$n^{r+1} \int_0^\infty Q(t) \bar{\lambda}_{1+\alpha+r, \beta}^{(r)}(nt) dt = 0 ,$$

where $Q(t)$ is defined by (2).

Proof. If $r = 0$, then $Q(t) = 0$. For $r \geq 1$ and $i = 0, 1, \dots [r-1/2]$, the truth of the lemma follows from the equation:

$$\int_0^\infty x^{r-1-2i} \bar{\lambda}_{1+\alpha+r, \beta}^{(r)}(x) dx = 0 ,$$

which is easily verified by means of $r - 1 - 2i$ integrations by parts.

The final two lemmas of this section give the appropriate representation of the n th mean of $D_r CFSf(x)$.

LEMMA 6. Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π . Let $m, 0 \leq m \leq \lambda + 1$, be an integer for which $\Psi_m(t) \in L[0, \pi]$. Then, for $\alpha = m, \beta > 1$ or $\alpha > m, \beta \geq 0$ and $r \geq 0$,

$$\bar{S}_{\alpha+r, \beta}^r(f, x, n) = 2(-n)^{r+1} \int_0^\pi [\psi(t) - Q(t)] \bar{\lambda}_{1+\alpha+r, \beta}^{(r)}(nt) dt + C_r + o(1)$$

as $n \rightarrow \infty$, where

$$(4) \quad C_r = 2\pi^{-1}(-1)^{r+1} \int_0^\pi \psi(t) \left(\frac{d}{dt}\right)^r \left[\frac{1}{2} ctn \frac{1}{2} t - t^{-1} \right] dt + 2r! \pi^{-1} \int_\pi^\infty t^{-r-1} Q(t) dt .$$

Proof. It follows from Lemmas 4 and 5 that

$$(5) \quad \begin{aligned} \bar{S}_{\alpha+r, \beta}^r(f, x, n) &= 2(-n)^{r+1} \int_0^\pi [\psi(t) - Q(t)] \bar{\lambda}_{1+\alpha+r, \beta}^{(r)}(nt) dt \\ &+ 2(-n)^{r+1} \int_0^\pi \psi(t) \sum_{-\infty}^\infty \bar{\lambda}_{1+\alpha+r, \beta}^{(r)}[n(t + 2k\pi)] dt \\ &+ -2(-n)^{r+1} \int_\pi^\infty Q(t) \bar{\lambda}_{1+\alpha+r, \beta}^{(r)}(nt) dt \\ &= I_1 + I_2 + I_3 . \end{aligned}$$

Since the degree of $Q(t)$ is $r - 1$, Lemma 1 shows that

$$(6) \quad I_3 = 2r! \pi^{-1} \int_\pi^\infty t^{-r-1} Q(t) dt + o(1) .$$

Let us define:

$$J(n, t) = 2(-n)^{r+1} \sum_{-\infty}^\infty \{ \bar{\lambda}_{1+\alpha+r, \beta}^{(r)}[n(t + 2k\pi)] - (-1)^r r! \pi^{-1} [n(t + 2k\pi)]^{-r-1} \} .$$

Again appealing to Lemma 1, we see that $\lim_{n \rightarrow \infty} (\partial/\partial t)^j J(n, t) = 0$ uniformly for $t \in [0, \pi]$ and $j = 0, 1, \dots, m$.

With the aid of the well-known cotangent expansion I_2 may be written:

$$(7) \quad I_2 = \int_0^\pi \psi(t) J(n, t) dt + (-1)^{r+1} 2\pi^{-1} \int_0^\pi \psi(t) \left(\frac{d}{dt}\right)^r \left[\frac{1}{2} ctn \frac{1}{2} t - t^{-1} \right] dt .$$

But after m integrations by parts, it is seen that

$$(8) \quad \int_0^\pi \psi(t) J(n, t) dt = o(1) .$$

The lemma now follows from equations (5), (6), (7), and (8).

A particular, but useful, case of Lemma 6 is

LEMMA 7. *Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π . If $g(t) \in C_\mu P[0, \pi]$, where $g(t)$ is defined by (3), then*

$$\begin{aligned} \bar{S}_{\alpha, \beta}(g, 0, n) &= -2n \int_0^\pi g(t) \bar{\lambda}_{1+\alpha, \beta}(nt) dt \\ &\quad - 2\pi^{-1} \int_0^\pi g(t) \left(\frac{1}{2} ctn \frac{1}{2} t - t^{-1} \right) dt + o(1) \end{aligned}$$

for $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi, \beta \geq 0$, where $\xi = \min [\mu, \max (r, \lambda)]$.

The hypotheses of Lemma 6 are fulfilled, because $t^r g(t) \in C_\lambda P[0, \pi]$ implies $G_{1+\xi}(t) \in L[0, \pi]$ by Lemma 6 of [3].

4. Theorems.

THEOREM 1. *Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π . If there exist constants $\bar{a}_{r-1-2i}, i = 0, 1, \dots [r - 1/2]$, such that*

- (i) $g(t) \in C_\mu P[0, \pi]$ for some integer μ ;
- (ii) $CFSg(0) = s(\alpha, \beta)$ for $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi, \beta \geq 0$, where $\xi = \min [\mu, \max (r, \lambda)]$;

then $D_r CFSf(x) = S(\alpha + r, \beta), s = \pi^{-1} \int_0^\pi g(t) ctn(1/2)t dt$ and

$$S = -2\pi^{-1} \int_0^\pi t^{-1} g(t) dt + C_r,$$

where C_r is defined by equation (4).

THEOREM 2. *Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π . If $D_r CFSf(x) = S(\alpha + r, \beta)$ for $\alpha = 1 + \lambda, \beta > 1$ or $\alpha > 1 + \lambda, \beta \geq 0$, then there exist constants $\bar{a}_{r-1-2i}, i = 0, 1, \dots [r - 1/2]$, such that*

- (i) $g(t) \in C_\mu P[0, \pi]$ for some integer μ ;
- (ii) $CFSg(0) = s(\alpha', \beta')$, where

$\alpha' = 1 + \xi, \beta' > 1$ if $1 + \lambda \leq \alpha < 1 + \xi$ or $\alpha = 1 + \xi, \beta \leq 1, \alpha' = \alpha, \beta' = \beta$ if $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi, \beta \geq 0$, and ξ, s and S have the values given in Theorem 1.

Before passing to the proofs of these theorems, let us observe that the existence of the constants \bar{a}_{r-1-2i} implies their uniqueness from the definition of $g(t)$. In fact, it can be shown that the \bar{a}_{r-1-2i} are given by

$$D_{r-1-2i} F S f(x) = \bar{a}_{r-1-2i}(C), \quad i = 0, 1, \dots \left[\frac{r-1}{2} \right].$$

¹ Bosanquet ([1], Theorem 1) has shown this for $f(x)$ Lebesgue integrable and (C) replaced by Abel summability.

In addition it can be shown that when $f(x) \in L$, the sum, S , of $D_r CFS f(x)$ may be written

$$S = -2\pi^{-1} \int_{-o(C)}^{\infty} t^{-1}g(t)dt .^2$$

Proof of Theorem 1. That $s = -\pi^{-1} \int_0^{\pi} g(t)ctn(1/2)t dt$ follows from the consistency of (α, β) summability and a result due to Sargent ([4], Theorem 3). Therefore, both $g(t)ctn(1/2)t$ and $t^{-1}g(t)$ are CP integrable over $[0, \pi]$.

From Lemma 7 we have

$$(9) \quad \bar{S}_{\alpha, \beta}(g, 0, n) - s = -2n \int_0^{\pi} g(t)[\bar{\lambda}_{1+\alpha, \beta}(nt) - (\pi nt)^{-1}]dt + o(1) .$$

The left side of (9) is $o(1)$ by hypothesis. By consistency equation (9) remains valid if α is replaced by $\alpha + r - i$ and β by $\beta + j, i, j = 0, 1, \dots, r$. Therefore,

$$-2n \int_0^{\pi} g(t) \sum_{i, j=0}^r B_{ij}^r(\alpha, \beta)[\bar{\lambda}_{1+\alpha+r-i, \beta+j}(nt) - (\pi nt)^{-1}]dt = o(1) .$$

With the aid of Lemmas 2 and 6, the last equation becomes

$$\bar{S}_{\alpha+r, \beta}^r(f, x, n) = -2\pi^{-1} \int_0^{\pi} t^{-1}g(t)dt + C_r + o(1) .$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Due to the length of this proof and its similarity to the proof of Theorem 2, ([3]), only a brief outline of the proof will be given.

Putting $Q(t) = 0, \beta = 0$ and $p > \alpha + r$ in Lemma 6 and integrating the right-hand side of the resulting equation $\lambda + 1$ times, one can show that

$$D_{r+\lambda+1} CFS(\Psi_{\lambda+1}, 0, n) \text{ is summable } (C, p) .$$

A result due to Bosanquet ([1], Theorem 1) and the stepwise procedure employed in the proof of Theorem 2 ([3], equations 18 through 22) lead to the conclusion: $t^{-r-1}[\psi(t) - Q(t)] \in CP[0, \pi]$ for an appropriate polynomial $Q(t)$, i.e., $t^{-1}g(t) \in CP[0, \pi]$. From this statement and a results due to Sargent ([4], Theorem 3), $g(t) \in C_{\mu}P[0, \pi]$ for some integer μ and $CFSg(0) = s(C)$, where $s = \pi^{-1} \int_0^{\pi} g(t)ctn(1/2)t dt .^3$

² Ibid. The difference in sign is due to the distinction between allied and conjugate series.

³ The CP integrability of $g(t)ctn(1/2)t$ is equivalent to that of $t^{-1}g(t)$.

That S , the $(\alpha + r, \beta)$ sum of $D_r CFSf(x)$, has the value

$$-2\pi^{-1} \int_0^\pi t^{-1} g(t) dt + C_r$$

follows immediately from Theorem 1 and the consistency of the summability scale.

Thus, it remains to prove only the order relations (α', β') in (ii) of the theorem. A straightforward calculation using the representations in Lemmas 6 and 7, the properties of the $B_{ij}^r(\alpha, \beta)$ in Lemma 2, and the consistency of the summability scale applied to $D_r CFSf(x)$, leads to the following equations:

$$\sum_{i,j=0}^r B_{ij}^r(\alpha' + k, \beta') \left[\bar{S}_{\alpha'+k+r-i, \beta'+j}(g, 0, n) - \pi^{-1} \int_0^\pi g(t) ctn \frac{1}{2} t dt \right] = o(1),$$

for $k = 0, 1, 2, \dots$

The expression in brackets may be considered the n th mean of order $(\alpha' + k + r - i, \beta' + j)$ of a series formed from $CFSg(0)$ by altering the first term. Since this series is summable (C) to 0, then Lemma 8 [3] shows that $CFSg(0) = s(\alpha', \beta')$.

The following theorem gives a sufficient condition for the (α, β) summability of $CFSg(0)$ for $\beta \neq 0$. Since the proof follows the usual lines for Riesz summability, it is omitted.

THEOREM 3. *Let $g(t)$ be an odd function of period 2π . If $t^{-1}g(t) \in C_k P[0, \pi]$, where k is a non-negative integer, then*

$$CFSg(0) = -\pi^{-1} \int_0^\pi g(t) ctn \frac{1}{2} t dt (1 + k, \beta), \beta > 1.$$

As an application of these theorems it can be shown that

$$D_r CFSf(0, m) = S(1 + m + 2r, \beta), \beta > 1,$$

where $f(x; m)$ is either $x^{-m} \sin x^{-1}$ or $x^{-m} \cos x^{-1}$, $m = 0, 1, 2, \dots$

The following results may be deduced from Theorems 1 and 2. It is assumed that $f(x) \in C_\lambda P[-\pi, \pi]$ and is of period 2π . The values of S and s , when either exists, and ξ are given in Theorem 1.

(A). If $g(t) \in C_\mu P[0, \pi]$, then for $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi, \beta \geq 0, D_r CFSf(x) = S(\alpha + r, \beta)$ if and only if $CFSg(0) = s(\alpha, \beta)$.

(B). For $\alpha = 1 + \max(r, \lambda), \beta > 1$ or $\alpha > 1 + \max(r, \lambda), \beta \geq 0, D_r CFSf(x) = S(\alpha + r, \beta)$ if and only if $g(t) \in CP[0, \pi]$ and $CFSg(0) = s(\alpha, \beta)$.

These results generalize, to various degrees, results obtained by Takahashi and Wang [7] and Bosanquet [1].

A weak, but none the less interesting, form of these results is

(C). If $f(x) \in CP[-\pi, \pi]$ and is of period 2π , then in order that $D, CFSf(x)$ be summable (C), it is necessary and sufficient that $g(t) \in CP[0, \pi]$ and $CFSg(0)$ be summable (C).

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