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MINIMAL SUPERADDITIVE EXTENSIONS OF SUPERADDITIVE FUNCTIONS

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Introduction. A real valued function f is said to be superadditive on an interval $I = [0, a]$ if it satisfies the inequality $f(x + y) \geq f(x) + f(y)$ whenever x, y and $x + y$ are in I . Such functions have been studied in detail by E. Hille and R. Phillips [1] and R. A. Rosenbaum [2]. In this paper we show that any superadditive function f on I has a minimal superadditive extension F to the non-negative real line E , and then proceed to show that F inherits much of its behavior from the behavior of f . We deal primarily with superadditive functions which are continuous and non-negative.

A simple example of a superadditive function on $[0, a]$ is furnished by a convex function f with $f(0) \leq 0$. Also, if f is convex and $f(0) = 0$, then it is easy to verify that its minimal superadditive extension F is given by

$$F(x) = nf(a) + f(x - na)$$

for $na \leq x < (n + 1)a$. In general, the minimal superadditive extension F is not easily computed. In the sequel we shall discuss two methods for obtaining F . For brevity we shall use the notation f^*F to mean “ F is the minimal superadditive extension of f ”.

1. The decomposition method. DEFINITION. Let $x \in E$. The numbers x^1, \dots, x^n are said to form an a -partition for x if $x^1 + \dots + x^n = x$ and for each $i = 1, \dots, n$ we have $0 \leq x^i \leq a$.

THEOREM 1. *Let f be a superadditive function on $I = [0, a]$. Then the function F defined on E by the equation*

$$F(x) = \sup \Sigma f(u^i),$$

the supremum being taken over all a -partitions of x , is the minimal superadditive extension of f .

Proof. We will show that F is superadditive. The minimality of F will then follow from the fact that any superadditive extension \hat{f} of f must satisfy $\hat{f}(x) \geq \Sigma f(x^i)$ for all $x \in E$ and all a -partitions x^1, \dots, x^n of x . Let $x, y \in E, \varepsilon > 0$. Choose a -partitions x^1, \dots, x^m and y^1, \dots, y^n for

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x and y respectively such that $f(x^1) + \cdots + f(x^m) \geq F(x) - \varepsilon/2$ and $f(y^1) + \cdots + f(y^n) \geq F(y) - \varepsilon/2$. Then the numbers $x^1, \dots, x^m, y^1, \dots, y^n$ form an a -partition for $x + y$ and we have

$$\begin{aligned} F(x + y) &\geq f(x^1) + \cdots + f(x^m) + f(y^1) + \cdots + f(y^n) \\ &\geq F(x) + F(y) - \varepsilon. \end{aligned}$$

Suppose we have an approximation for $F(x)$: that is, a number $\varepsilon > 0$ and an a -partition x^1, \dots, x^n for x such that $F(x) - \sum f(x^i) < \varepsilon$. If among the members of this a -partition there are two, say x^j and x^k such that $u = x^j + x^k \leq a$, then since $f(u) \geq f(x^j) + f(x^k)$, we have

$$F(x) - [f(u) + \sum_{i \neq j, k} f(x^i)] \leq F(x) - \sum_1^n f(x^i) < \varepsilon.$$

In other words, replacing two numbers used in the approximation by their sum $u \leq a$ yields an approximation at least as good as the original. It follows that if x satisfies the inequality $(M - 2)a/2 \leq x \leq (M - 1)a/2$, where M is a positive integer, then there exist arbitrarily good approximations for $F(x)$ using only M terms in the a -partition. If f is continuous, then a simple compactness argument results in the following theorem:

THEOREM 2. *Let f be a continuous superadditive function on $[0, a]$, and let F be its minimal superadditive extension. Let x satisfy the inequality $(M - 2)a/2 \leq x \leq (M - 1)a/2$. Then \exists an a -partition x^1, \dots, x^M for x such that*

$$\sum f(x^i) = F(x).$$

Such an a -partition for x will be called a *decomposition* of x , for which we shall use the notation $\langle x \rangle$ whenever convenient. We will denote by $N(x)$ a number so large that for any continuous superadditive function on $[0, a]$, \exists a decomposition $\langle x \rangle$ of x with at most $N(x)$ members. It follows from the above that we can always let $N(x) = 2x/a + 2$, for example.

Henceforth we shall be concerned primarily with continuous non-negative superadditive functions for which we shall use the abbreviation *csa*. It is readily verified that such functions are non-decreasing and vanish at the origin.

2. Combinations of extensions. One might expect that if the members of a family f of *csa* functions are combined in a linear fashion to give another *csa* function h , then combining the members of the family \tilde{f} of minimal superadditive extensions of functions in f in the same way would give rise to a function H which is the minimal superadditive

extension of h . However this is not always the case. Consider, for example, the functions f and g defined on $[0, 3]$ as follows: $f(0) = 0, f(1) = 0, f(2) = 0, f(3) = 1$ and $g(0) = 0, g(1) = 0, g(2) = 2, g(3) = 3, f$ and g linear on $[n, n+1], n = 0, 1, 2$. Simple computations show that whereas $(F + G)(4) = 5$ and $FG(4) = 4$, the minimal superadditive extensions of $f + g$ and fg take on the values 4 and 3 respectively at $x = 4$. The minimal superadditive extension of a sum (product) of superadditive functions is thus not necessarily the sum (product) of the minimal superadditive extensions. However, some processes do commute with taking minimal superadditive extensions.

THEOREM 3. *Let $\{f_n\}$ be a sequence of csa functions converging to the continuous function f on $I = [0, a]$. Let $f_n^*F_n$. Then f is csa and $f^* \lim_{n \rightarrow \infty} F_n$.*

Proof. That f is superadditive and non-negative is clear. Since for each positive integer n the function f_n is non-decreasing, the convergence of $\{f_n\}$ to f is uniform on I . Given $\varepsilon > 0$ and $x \in E$, let M be such that $n \geq M \Rightarrow \max_{t \in I} |f_n(t) - f(t)| < \varepsilon/N(x)$ where $N(x)$ is a number chosen as in § 1. Let $k > M$ and let $\langle x^k \rangle \equiv x_k^1, \dots, x_k^{N(x)}$ and $\langle x \rangle \equiv x^1, \dots, x^{N(x)}$ be decompositions for x relative to F_k and F respectively. We have

$$F(x) = \sum_{i=1}^{N(x)} f(x^i) \geq \sum_{i=1}^{N(x)} f(x_k^i)$$

and

$$F_k(x) = \sum_{i=1}^{N(x)} f_k(x_k^i) \geq \sum_{i=1}^{N(x)} f_k(x^i).$$

It follows from these two inequalities that

$$|F(x) - F_k(x)| < \varepsilon,$$

for $n \geq M$.

3. Behavior of the minimal superadditive extension. It seems reasonable to expect that the minimal superadditive extension F of a csa function f will enjoy many of the properties of f . To a certain extent this is true and we are able to predict much about the behavior of F by examining the behavior of f .

THEOREM 4. *Let f be csa on $[0, a]$. If f^*F , then F is csa on E .*

Proof. Clearly F is non-negative. To prove that F is continuous let $\varepsilon > 0$ and choose $\delta < a/2\vartheta$ if $u, v \leq a$ and $|u - v| < \delta$ then $|f(u) - f(v)| < \varepsilon$. Now let x and y be points of E for which $|y - x| < \delta$,

say $y = x + h$. Let $\langle y \rangle = y^1, \dots, y^N$ be a decomposition for y with, say, $y^1 \geq a/2$. We have

$$F(y) = \sum_1^N f(y_i) \text{ and } F(x) \geq \sum_2^N f(y^i) + f(y^1 - h).$$

Hence $0 \leq F(y) - F(x) \leq f(y^1) - f(y^1 - h) < \varepsilon$.

In a similar manner one can establish the following theorem, which is stated without proof.

THEOREM 5. *Let f be csa on $[0, a]$. If f^*F , then the following statements hold:*

(a) *If f satisfies a Lipschitz condition with coefficient M , then so does F ;*

(b) *If $\langle y \rangle = y^1, \dots, y^M$ is a decomposition for y and f is differentiable at y^i and y^j , then $f'(y^i) = f'(y^j)$. If, in addition, F is differentiable at y , then $F'(y) = f'(y^i)$.*

One might expect that the differentiability of f on $[0, a]$ would imply the differentiability of F , except possibly at integral multiples of a . Although this turns out not to be the case, we do have the following theorem:

THEOREM 6. *Let f be a csa function on the interval $[0, a]$, with f' continuous on $(0, a)$. For x not an integral multiple of a , let X be the set of points of $[0, a]$ which can be used in a decomposition for x . Then F has a right hand derivative $F_+(x)$ and a left hand derivative $F_-(x)$ at x with*

$$F_+(x) = \sup_{t \in X} f'(t) \equiv S$$

and

$$F_-(x) = \inf_{t \in X} f'(t) \equiv I.$$

Proof. We will prove only the upper equality. The lower can be proved in a similar manner. It suffices to show $D^+F(x) = D_+F(x) = S$ where D^+F and D_+F are the upper and lower right hand derivatives of F . Suppose $\exists \varepsilon > 0 \ni D^+F(x) > S + 2\varepsilon$. Then a sequence $\{h_i\}$ of numbers approaching 0 such that

$$(1) \quad F(x) < F(x + h_i) - (S + \varepsilon)h_i$$

for $i = 1, 2, \dots$. For each positive integer i , let (u^i, v^i, \dots, w^i) be a decomposition for $x + h_i$. Without loss of generality, we assume that the sequence (u^i, v^i, \dots, w^i) converges to, say, (u, v, \dots, w) ; otherwise we consider a convergent subsequence. Since x is not an integral multiple of a , one of the numbers u, v, \dots, w is not equal to 0 or a . Denote such a one by u . From (1) we have

$$(2) \quad F(x) < f(u^i) + f(v^i) + \dots + f(w^i) - (S + \varepsilon)h_i .$$

Choose $N_1 \ni i > N_1$ implies that

$$(3) \quad f(u^i) < f(u^i - h_i) + [f'(u^i - h_i) + \varepsilon/2]h_i .$$

That N_1 can be so chosen follows from the continuity of f' . In fact, let δ be such that $|u - v| < \delta \Rightarrow |f'(u) - f'(v)| < \varepsilon/4$. Now choose N_1 such that $i > N_1 \Rightarrow u - \delta < u^i - h_i < u^i < u + \delta$. If $y \in [u^i - h_i, u^i]$, with $i > N_1$, then $f'(u^i - h_i) + \varepsilon/2 > f'(y)$. Hence (3) is a valid inequality. For $i > N_1$ we have from (2) and (3),

$$(4) \quad F(x) < f(u^i - h_i) + f(v^i) + \dots + f(w^i) + [f'(u^i - h_i) - (S + \varepsilon/2)]h_i .$$

Now the sequence $(u^i - h_i, v^i, \dots, w^i)$ converges to (u, v, \dots, w) and $u + v + \dots + w = x$. Thus, since

$$f(u^i) + f(v^i) + \dots + f(w^i) = F(x + h_i) \geq F(x) ,$$

and F is a superadditive function, we have

$$f(u) + f(v) + \dots + f(w) = F(x)$$

and $u \in X$. Therefore $f'(u) \leq S$. By the continuity of f' , $\lim_{i \rightarrow \infty} f'(u^i - h_i) = f'(u)$. Hence \exists a positive number N_2 such that $i > N_2 \Rightarrow f'(u^i - h_i) < S + \varepsilon/2$. Let $i = \max(N_1, N_2)$. For this i we have from (4),

$$F(x) < f(u^i - h_i) + f(v^i) + \dots + f(w^i) .$$

This is impossible, for $u^i - h_i + v^i + \dots + w^i = x$ for each $i = 1, 2, \dots$ and F is superadditive. We have shown $D_+F(x) \leq S$.

It remains to show $D_+F(x) \geq S$. Let $\varepsilon > 0$, and let (u, v, \dots, w) be a decomposition for x such that $u \neq a$, and $f'(u) > S - \varepsilon/4$. Choose $\delta > 0 \ni h < \delta \Rightarrow f(u + h) > f(u) + (S - \varepsilon/2)h_i$. For $h < \delta$,

$$F(x + h) \geq f(u + h) + f(v) + \dots + f(w) > F(x) + (S - \varepsilon/2)h .$$

The first and third members of this inequality give

$$\frac{F(x + h) - F(x)}{h} > S + \varepsilon/2 .$$

Since ε was arbitrary, $D_+F(x) \geq S$, and the proof of the theorem is complete.

We now proceed to obtain a linear upper bound for F . If f is *csa* on $[0, a]$, then the function g defined by $g(x) = f(x)/x$ is continuous on $[0, a]$ and satisfies $g(nx) \geq g(x)$, $n = 1, 2, \dots$, whenever $nx \leq a$. It follows that g attains a maximum at some point of $(0, a]$.

THEOREM 7. *Let f be *csa* on $[0, a]$, f^*F , and let g be defined as*

above. Let t be a point of $(0, a]$ at which g attains its maximum M . Then

- (a) $F(x)/x \leq M$ for all $x > 0$,
- (b) $F(x)/x = M$ if x is an integral multiple of t ,
- (c) $\lim_{x \rightarrow \infty} F(x)/x = M$,
- (d) $\max_{x \in [0, a]} [Mx - f(x)] = \max_{x \in E} [Mx - F(x)]$,
- (e) $\lim_{x \rightarrow \infty} [F(x) - Mx] = 0$ if f is differentiable at t .

Proof. The proofs of (a), (b), (c) and (d) are straightforward and will be omitted. Let us then turn to (e). For each $x \in E$, write x in the form $x = nt + y$, where n is an integer and $0 \leq y < t$. Define a function F^* by $F^*(nt + y) = nf(t + y/n)$, $n = 1, 2, \dots$. Clearly $F^*(x) \leq F(x) \leq Mx$ for all $x \in E$. We will show that $\lim_{x \rightarrow \infty} [Mx - F^*(x)] = 0$. By the definition of F^* we have

$$Mx - F^*(x) = M(nt + y) - nf(t + y/n).$$

Noting that $f(t) = Mt$, we see that the right hand member of this last equation can be written in the form

$$(1) \quad y \left[M - \frac{f(t + y/n) - f(t)}{y/n} \right]$$

Now let $x \rightarrow \infty$. Then y is bounded between 0 and t and $n \rightarrow \infty$. The expression (1) approaches 0, since $f'(t) = M$.

We observe that the function F^* of the preceding theorem is asymptotic to F with $F^* \leq F$. Hence $F(x)$ is bounded between $F^*(x)$ and Mx , two functions which are easy to calculate, and whose difference is small when x is large.

4. The polygonal method. The minimal superadditive extension of a *csa* function may also be obtained as the limit of a sequence of polygonal functions. A function p is said to be *polygonal* if p is continuous and piecewise linear. The point $x \in [0, a]$ is called a *vertex* of p if $(x, p(x))$ is a vertex of the polygon forming the graph of p .

THEOREM 8. *Let p be polygonal on $[0, a]$ with equally spaced vertices. Then p is superadditive if and only if p is superadditive on its vertices.*

Proof. If p is superadditive, then p is clearly superadditive on its vertices. To prove the converse consider the function g defined on the set

$$D \equiv \{(x, y): 0 \leq x, y \leq a \text{ and } x + y \leq a\}$$

by the equation $g(x, y) = p(x + y) - p(x) - p(y)$. It is easy to verify that g is planar on any triangle T of the form

$$T = \{(x, y): u_1 \leq x \leq u_2; v_1 \leq y \leq v_2, x + y \leq (\text{or } \geq) u_2 + v_2\},$$

where (u_1, v_1) and (u_2, v_2) are pairs of successive vertices of p . Hence g attains its minimum on T at one of the points (u_i, v_i) and therefore its minimum on D at a point (u, v) where both u and v are vertices of p . Thus, if g is anywhere negative then g is negative at a point whose two coordinates are vertices of p . This proves the theorem.

Now let p be a polygonal function on $[0, a]$ with vertices at $0, v, 2v, \dots, mv = a$. We define a function P on E as follows:

$$P(x) = p(x) \quad \text{for } x \leq a$$

$$P(Mv) = \max_{K=1, \dots, M-1} [P(Kv) + P(Mv - Kv)] \quad M \text{ an integer } \geq m + 1$$

and

$$P \text{ linear on } [Mv, (M + 1)v], \quad M = m, m + 1, \dots.$$

P will be called the function associated with p . It is easy to see that if p is *csa*, then P is *csa*.

DEFINITION. A sequence $\{p_n\}$ of functions defined on $[0, a]$ is called a p -sequence if

- (i) each p_n is a polygonal function
- (ii) the vertices of p_n are $Ka/2^n$, $K = 0, 1, \dots, 2^n$
- (iii) $P_n(Ka/2^m) = p_m(Ka/2^m)$ if $m \leq n$.

In terms of this concept we have

THEOREM 9. Let $\{p_n\}$ be a p -sequence covering to the *csa* function f on $[0, a]$. For each positive integer n let P_n be the function associated with p_n . Then, if f^*F , $\{P_n\}$ converges to F on E .

Proof. It suffices to show that P_n approaches F on $[0, 2a]$. Let $F^*(x) = \overline{\lim}_{n \rightarrow \infty} P_n(x)$. It is easy to check that F^* is superadditive. Let V_k be the set of vertices of P_k in $[a, 2a]$, and let $V = \bigcup_1^\infty V_k$. If $v \in V$, then $\lim_{n \rightarrow \infty} P_n(v)$ exists since the sequence $\{P_n(v)\}$ is ultimately non-decreasing and $P_n(v) \leq F(v)$ for all n . We have $\lim_{n \rightarrow \infty} P_n(v) \leq F(v)$. But since F^* is superadditive, we have $F^* \geq F$. Hence $F^* = F$ on V . By standard arguments involving the continuity of F , the density of V in $[a, 2a]$, and the monotonicity of each P_n and F^* , it follows that $F \equiv F^*$ and that $F^* = \lim_{n \rightarrow \infty} P_n(x)$.

5. Superadditive functions in n -dimensions. It turns out that many of the results obtained in one dimension have their analogues in n -di-

mensions. The interval $I \equiv [0, a]$ is replaced by a fundamental region R defined by the inequalities $0 \leq x_i \leq a_i, i = 1, \dots, n$, where the a_i are arbitrary positive numbers. The decomposition method works, just as it does on the line, and we can prove with little difficulty that to any superadditive function f on R there corresponds a minimal superadditive extension F to $E_n^+ \equiv \{(x_1, \dots, x_n): 0 \leq x_i, i = 1, \dots, n\}$. We can also prove a theorem corresponding to Theorem 5, the derivatives here being directional derivatives. In Theorem 7 a certain line $l(x) = Mx$ played an important role. In n -dimensions, for each direction θ we have a plane P_θ which plays the role of l in some direction, and when the function P , defined on the fundamental region R by the equation

$$P(z) = \inf_{\theta} P_{\theta}(z) ,$$

is extended to E_n^+ by homogeneity it is the minimal concave superadditive function which bounds F from above. It can be proved, at least in E_2^+ , that

$$n \max_{z \in R} [P(z) - f(z)] \geq \max_{z \in E_n^+} [P(z) - F(z)] .$$

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