

Pacific Journal of Mathematics

MINIMAL SUPERADDITIVE EXTENSIONS OF SUPERADDITIVE FUNCTIONS

ANDREW MICHAEL BRUCKNER

MINIMAL SUPERADDITIVE EXTENSIONS OF SUPERADDITIVE FUNCTIONS

ANDREW BRUCKNER

Introduction. A real valued function f is said to be superadditive on an interval $I = [0, a]$ if it satisfies the inequality $f(x + y) \geq f(x) + f(y)$ whenever x, y and $x + y$ are in I . Such functions have been studied in detail by E. Hille and R. Phillips [1] and R. A. Rosenbaum [2]. In this paper we show that any superadditive function f on I has a minimal superadditive extension F to the non-negative real line E , and then proceed to show that F inherits much of its behavior from the behavior of f . We deal primarily with superadditive functions which are continuous and non-negative.

A simple example of a superadditive function on $[0, a]$ is furnished by a convex function f with $f(0) \leq 0$. Also, if f is convex and $f(0) = 0$, then it is easy to verify that its minimal superadditive extension F is given by

$$F(x) = nf(a) + f(x - na)$$

for $na \leq x < (n + 1)a$. In general, the minimal superadditive extension F is not easily computed. In the sequel we shall discuss two methods for obtaining F . For brevity we shall use the notation f^*F to mean " F is the minimal superadditive extension of f ".

1. The decomposition method. DEFINITION. Let $x \in E$. The numbers x^1, \dots, x^n are said to form an α -partition for x if $x^1 + \dots + x^n = x$ and for each $i = 1, \dots, n$ we have $0 \leq x^i \leq \alpha$.

THEOREM 1. Let f be a superadditive function on $I = [0, a]$. Then the function F defined on E by the equation

$$F(x) = \sup \Sigma f(u^i),$$

the supremum being taken over all α -partitions of x , is the minimal superadditive extension of f .

Proof. We will show that F is superadditive. The minimality of F will then follow from the fact that any superadditive extension \hat{f} of f must satisfy $\hat{f}(x) \geq \Sigma f(x^i)$ for all $x \in E$ and all α -partitions x^1, \dots, x^n of x . Let $x, y \in E, \varepsilon > 0$. Choose α -partitions x^1, \dots, x^m and y^1, \dots, y^n for

Received November 6, 1959. This paper is part of the author's doctoral thesis, and the author is indebted to Professor John Green for his guidance in its preparation. Thanks are also due the National Science Foundation for their support.

x and y respectively such that $f(x^1) + \cdots + f(x^m) \geq F(x) - \varepsilon/2$ and $f(y^1) + \cdots + f(y^n) \geq F(y) - \varepsilon/2$. Then the numbers $x^1, \dots, x^m, y^1, \dots, y^n$ form an a -partition for $x + y$ and we have

$$\begin{aligned} F(x + y) &\geq f(x^1) + \cdots + f(x^m) + f(y^1) + \cdots + f(y^n) \\ &\geq F(x) + F(y) - \varepsilon. \end{aligned}$$

Suppose we have an approximation for $F(x)$: that is, a number $\varepsilon > 0$ and an a -partition x^1, \dots, x^n for x such that $F(x) - \sum f(x^i) < \varepsilon$. If among the members of this a -partition there are two, say x^j and x^k such that $u = x^j + x^k \leq a$, then since $f(u) \geq f(x^j) + f(x^k)$, we have

$$F(x) - [f(u) + \sum_{i \neq j, k} f(x^i)] \leq F(x) - \sum_1^n f(x^i) < \varepsilon.$$

In other words, replacing two numbers used in the approximation by their sum $u \leq a$ yields an approximation at least as good as the original. It follows that if x satisfies the inequality $(M - 2)a/2 \leq x \leq (M - 1)a/2$, where M is a positive integer, then there exist arbitrarily good approximations for $F(x)$ using only M terms in the a -partition. If f is continuous, then a simple compactness argument results in the following theorem:

THEOREM 2. *Let f be a continuous superadditive function on $[0, a]$, and let F be its minimal superadditive extension. Let x satisfy the inequality $(M - 2)a/2 \leq x \leq (M - 1)a/2$. Then \exists an a -partition x^1, \dots, x^M for x such that*

$$\sum f(x^i) = F(x).$$

Such an a -partition for x will be called a *decomposition* of x , for which we shall use the notation $\langle x \rangle$ whenever convenient. We will denote by $N(x)$ a number so large that for any continuous superadditive function on $[0, a]$, \exists a decomposition $\langle x \rangle$ of x with at most $N(x)$ members. It follows from the above that we can always let $N(x) = 2x/a + 2$, for example.

Henceforth we shall be concerned primarily with continuous non-negative superadditive functions for which we shall use the abbreviation *csa*. It is readily verified that such functions are non-decreasing and vanish at the origin.

2. Combinations of extensions. One might expect that if the members of a family f of *csa* functions are combined in a linear fashion to give another *csa* function h , then combining the members of the family \tilde{f} of minimal superadditive extensions of functions in f in the same way would give rise to a function H which is the minimal superadditive

extension of h . However this is not always the case. Consider, for example, the functions f and g defined on $[0, 3]$ as follows: $f(0) = 0, f(1) = 0, f(2) = 0, f(3) = 1$ and $g(0) = 0, g(1) = 0, g(2) = 2, g(3) = 3, f$ and g linear on $[n, n + 1], n = 0, 1, 2$. Simple computations show that whereas $(F + G)(4) = 5$ and $FG(4) = 4$, the minimal superadditive extensions of $f + g$ and fg take on the values 4 and 3 respectively at $x = 4$. The minimal superadditive extension of a sum (product) of superadditive functions is thus not necessarily the sum (product) of the minimal superadditive extensions. However, some processes do commute with taking minimal superadditive extensions.

THEOREM 3. *Let $\{f_n\}$ be a sequence of csa functions converging to the continuous function f on $I = [0, a]$. Let $f_n^*F_n$. Then f is csa and $f^*\lim_{n \rightarrow \infty} F_n$.*

Proof. That f is superadditive and non-negative is clear. Since for each positive integer n the function f_n is non-decreasing, the convergence of $\{f_n\}$ to f is uniform on I . Given $\varepsilon > 0$ and $x \in E$, let M be such that $n \geq M \Rightarrow \max_{t \in I} |f_n(t) - f(t)| < \varepsilon/N(x)$ where $N(x)$ is a number chosen as in § 1. Let $k > M$ and let $\langle x^k \rangle \equiv x_k^1, \dots, x_k^{N(x)}$ and $\langle x \rangle \equiv x^1, \dots, x^{N(x)}$ be decompositions for x relative to F_k and F respectively. We have

$$F(x) = \sum_{i=1}^{N(x)} f(x^i) \geq \sum_{i=1}^{N(x)} f(x_k^i)$$

and

$$F_k(x) = \sum_{i=1}^{N(x)} f_k(x_k^i) \geq \sum_{i=1}^{N(x)} f_k(x^i).$$

It follows from these two inequalities that

$$F(x) - F_k(x) < \varepsilon,$$

for $n \geq M$.

3. Behavior of the minimal superadditive extension. It seems reasonable to expect that the minimal superadditive extension F of a csa function f will enjoy many of the properties of f . To a certain extent this is true and we are able to predict much about the behavior of F by examining the behavior of f .

THEOREM 4. *Let f be csa on $[0, a]$. If f^*F , then F is csa on E .*

Proof. Clearly F is non-negative. To prove that F is continuous let $\varepsilon > 0$ and choose $\delta < a/2\vartheta$ if $u, v \leq a$ and $|u - v| < \delta$ then $|f(u) - f(v)| < \varepsilon$. Now let x and y be points of E for which $|y - x| < \delta$,

say $y = x + h$. Let $\langle y \rangle = y^1, \dots, y^N$ be a decomposition for y with, say, $y^1, \geq a/2$. We have

$$F(y) = \sum_1^N f(y_i) \text{ and } F(x) \geq \sum_2^N f(y^i) + f(y^1 - h).$$

Hence $0 \leq F(y) - F(x) \leq f(y^1) - f(y^1 - h) < \epsilon$.

In a similar manner one can establish the following theorem, which is stated without proof.

THEOREM 5. *Let f be csa on $[0, a]$. If f^*F , then the following statements hold:*

(a) *If f satisfies a Lipschitz condition with coefficient M , then so does F ;*

(b) *If $\langle y \rangle = y^1, \dots, y^M$ is a decomposition for y and f is differentiable at y^i and y^j , then $f'(y^i) = f'(y^j)$. If, in addition, F is differentiable at y , then $F'(y) = f'(y^i)$.*

One might expect that the differentiability of f on $[0, a]$ would imply the differentiability of F , except possibly at integral multiples of a . Although this turns out not to be the case, we do have the following theorem:

THEOREM 6. *Let f be a csa function on the interval $[0, a]$, with f' continuous on $(0, a)$. For x not an integral multiple of a , let X be the set of points of $[0, a]$ which can be used in a decomposition for x . Then F has a right hand derivative $F_+(x)$ and a left hand derivative $F_-(x)$ at x with*

$$F_+(x) = \sup_{t \in X} f'(t) \equiv S$$

and

$$F_-(x) = \inf_{t \in X} f'(t) \equiv I.$$

Proof. We will prove only the upper equality. The lower can be proved in a similar manner. It suffices to show $D^+F(x) = D_+F(x) = S$ where D^+F and D_+F are the upper and lower right hand derivatives of F . Suppose $\exists \epsilon > 0 \ni D^+F(x) > S + 2\epsilon$. Then a sequence $\{h_i\}$ of numbers approaching 0 such that

$$(1) \quad F(x) < F(x + h_i) - (S + \epsilon)h_i$$

for $i = 1, 2, \dots$. For each positive integer i , let (u^i, v^i, \dots, w^i) be a decomposition for $x + h_i$. Without loss of generality, we assume that the sequence (u^i, v^i, \dots, w^i) converges to, say, (u, v, \dots, w) ; otherwise we consider a convergent subsequence. Since x is not an integral multiple of a , one of the numbers u, v, \dots, w is not equal to 0 or a . Denote such a one by u . From (1) we have

$$(2) \quad F(x) < f(u^i) + f(v^i) + \dots + f(w^i) - (S + \varepsilon)h_i .$$

Choose $N_1 \ni i > N_1$ implies that

$$(3) \quad f(u^i) < f(u^i - h_i) + [f'(u^i - h_i) + \varepsilon/2]h_i .$$

That N_1 can be so chosen follows from the continuity of f' . In fact, let δ be such that $|u - v| < \delta \Rightarrow |f'(u) - f'(v)| < \varepsilon/4$. Now choose N_1 such that $i > N_1 \Rightarrow u - \delta < u^i - h_i < u^i < u + \delta$. If $y \in [u^i - h_i, u^i]$, with $i > N_1$, then $f'(u^i - h_i) + \varepsilon/2 > f'(y)$. Hence (3) is a valid inequality. For $i > N_1$ we have from (2) and (3),

$$(4) \quad F(x) < f(u^i - h_i) + f(v^i) + \dots + f(w^i) + [f'(u^i - h_i) - (S + \varepsilon/2)]h_i .$$

Now the sequence $(u^i - h_i, v^i, \dots, w^i)$ converges to (u, v, \dots, w) and $u + v + \dots + w = x$. Thus, since

$$f(u^i) + f(v^i) + \dots + f(w^i) = F(x + h_i) \geq F(x) ,$$

and F is a superadditive function, we have

$$f(u) + f(v) + \dots + f(w) = F(x)$$

and $u \in X$. Therefore $f'(u) \leq S$. By the continuity of f' , $\lim_{i \rightarrow \infty} f'(u^i - h_i) = f'(u)$. Hence \exists a positive number N_2 such that $i > N_2 \Rightarrow f'(u^i - h_i) < S + \varepsilon/2$. Let $i = \max(N_1, N_2)$. For this i we have from (4),

$$F(x) < f(u^i - h_i) + f(v^i) + \dots + f(w^i) .$$

This is impossible, for $u^i - h_i + v^i + \dots + w^i = x$ for each $i = 1, 2, \dots$ and F is superadditive. We have shown $D^+F(x) \leq S$.

It remains to show $D_+F(x) \geq S$. Let $\varepsilon > 0$, and let (u, v, \dots, w) be a decomposition for x such that $u \neq a$, and $f'(u) > S - \varepsilon/4$. Choose $\delta > 0 \ni h < \delta \Rightarrow f(u + h) > f(u) + (S - \varepsilon/2)h_i$. For $h < \delta$,

$$F(x + h) \geq f(u + h) + f(v) + \dots + f(w) > F(x) + (S - \varepsilon/2)h .$$

The first and third members of this inequality give

$$\frac{F(x + h) - F(x)}{h} > S + \varepsilon/2 .$$

Since ε was arbitrary, $D_+F(x) \geq S$, and the proof of the theorem is complete.

We now proceed to obtain a linear upper bound for F . If f is *csa* on $[0, a]$, then the function g defined by $g(x) = f(x)/x$ is continuous on $[0, a]$ and satisfies $g(nx) \geq g(x)$, $n = 1, 2, \dots$, whenever $nx \leq a$. It follows that g attains a maximum at some point of $(0, a]$.

THEOREM 7. *Let f be csa on $[0, a]$, f^*F , and let g be defined as*

above. Let t be a point of $(0, a]$ at which g attains its maximum M . Then

- (a) $F(x)/x \leq M$ for all $x > 0$,
- (b) $F(x)/x = M$ if x is an integral multiple of t ,
- (c) $\lim_{x \rightarrow \infty} F(x)/x = M$,
- (d) $\max_{x \in [0, a]} [Mx - f(x)] = \max_{x \in E} [Mx - F(x)]$,
- (e) $\lim_{x \rightarrow \infty} [F(x) - Mx] = 0$ if f is differentiable at t .

Proof. The proofs of (a), (b), (c) and (d) are straightforward and will be omitted. Let us then turn to (e). For each $x \in E$, write x in the form $x = nt + y$, where n is an integer and $0 \leq y < t$. Define a function F^* by $F^*(nt + y) = nf(t + y/n)$, $n = 1, 2, \dots$. Clearly $F^*(x) \leq F(x) \leq Mx$ for all $x \in E$. We will show that $\lim_{x \rightarrow \infty} [Mx - F^*(x)] = 0$. By the definition of F^* we have

$$Mx - F^*(x) = M(nt + y) - nf(t + y/n).$$

Noting that $f(t) = Mt$, we see that the right hand member of this last equation can be written in the form

$$(1) \quad y \left[M - \frac{f(t + y/n) - f(t)}{y/n} \right]$$

Now let $x \rightarrow \infty$. Then y is bounded between 0 and t and $n \rightarrow \infty$. The expression (1) approaches 0, since $f'(t) = M$.

We observe that the function F^* of the preceding theorem is asymptotic to F with $F^* \leq F$. Hence $F(x)$ is bounded between $F^*(x)$ and Mx , two functions which are easy to calculate, and whose difference is small when x is large.

4. The polygonal method. The minimal superadditive extension of a *csa* function may also be obtained as the limit of a sequence of polygonal functions. A function p is said to be *polygonal* if p is continuous and piecewise linear. The point $x \in [0, a]$ is called a *vertex* of p if $(x, p(x))$ is a vertex of the polygon forming the graph of p .

THEOREM 8. *Let p be polygonal on $[0, a]$ with equally spaced vertices. Then p is superadditive if and only if p is superadditive on its vertices.*

Proof. If p is superadditive, then p is clearly superadditive on its vertices. To prove the converse consider the function g defined on the set

$$D \equiv \{(x, y): 0 \leq x, y \leq a \text{ and } x + y \leq a\}$$

by the equation $g(x, y) = p(x + y) - p(x) - p(y)$. It is easy to verify that g is planar on any triangle T of the form

$$T = \{(x, y): u_1 \leq x \leq u_2; v_1 \leq y \leq v_2, x + y \leq (\text{or } \geq) u_2 + v_2\},$$

where (u_1, v_1) and (u_2, v_2) are pairs of successive vertices of p . Hence g attains its minimum on T at one of the points (u_i, v_i) and therefore its minimum on D at a point (u, v) where both u and v are vertices of p . Thus, if g is anywhere negative then g is negative at a point whose two coordinates are vertices of p . This proves the theorem.

Now let p be a polygonal function on $[0, a]$ with vertices at $0, v, 2v, \dots, mv = a$. We define a function P on E as follows:

$$P(x) = p(x) \quad \text{for } x \leq a$$

$$P(Mv) = \max_{K=1, \dots, M-1} [P(Kv) + P(Mv - Kv)] \quad M \text{ an integer } \geq m + 1$$

and

$$P \text{ linear on } [Mv, (M + 1)v], \quad M = m, m + 1, \dots$$

P will be called the function associated with p . It is easy to see that if p is *csa*, then P is *csa*.

DEFINITION. A sequence $\{p_n\}$ of functions defined on $[0, a]$ is called a *p*-sequence if

- (i) each p_n is a polygonal function
- (ii) the vertices of p_n are $Ka/2^n$, $K = 0, 1, \dots, 2^n$
- (iii) $P_n(Ka/2^m) = p_m(Ka/2^m)$ if $m \leq n$.

In terms of this concept we have

THEOREM 9. *Let $\{p_n\}$ be a p-sequence covering to the csa function f on $[0, a]$. For each positive integer n let P_n be the function associated with p_n . Then, if f^*F , $\{P_n\}$ converges to F on E .*

Proof. It suffices to show that P_n approaches F on $[0, 2a]$. Let $F^*(x) = \overline{\lim}_{n \rightarrow \infty} P_n(x)$. It is easy to check that F^* is superadditive. Let V_k be the set of vertices of P_k in $[a, 2a]$, and let $V = \bigcup_1^\infty V_k$. If $v \in V$, then $\lim_{n \rightarrow \infty} P_n(v)$ exists since the sequence $\{P_n(v)\}$ is ultimately non-decreasing and $P_n(v) \leq F(v)$ for all n . We have $\lim_{n \rightarrow \infty} P_n(v) \leq F(v)$. But since F^* is superadditive, we have $F^* \geq F$. Hence $F^* = F$ on V . By standard arguments involving the continuity of F , the density of V in $[a, 2a]$, and the monotonicity of each P_n and F^* , it follows that $F \equiv F^*$ and that $F^* = \lim_{n \rightarrow \infty} P_n(x)$.

5. **Superadditive functions in n -dimensions.** It turns out that many of the results obtained in one dimension have their analogues in n -di-

mensions. The interval $I \equiv [0, a]$ is replaced by a fundamental region R defined by the inequalities $0 \leq x_i \leq a_i, i = 1, \dots, n$, where the a_i are arbitrary positive numbers. The decomposition method works, just as it does on the line, and we can prove with little difficulty that to any superadditive function f on R there corresponds a minimal superadditive extension F to $E_n^+ \equiv \{(x_1, \dots, x_n): 0 \leq x_i, i = 1, \dots, n\}$. We can also prove a theorem corresponding to Theorem 5, the derivatives here being directional derivatives. In Theorem 7 a certain line $l(x) = Mx$ played an important role. In n -dimensions, for each direction θ we have a plane P_θ which plays the role of l in some direction, and when the function P , defined on the fundamental region R by the equation

$$P(z) = \inf_{\theta} P_{\theta}(z) ,$$

is extended to E_n^+ by homogeneity it is the minimal concave superadditive function which bounds F from above. It can be proved, at least in E_2^+ , that

$$n \max_{z \in R} [P(z) - f(z)] \geq \max_{z \in E_h^+} [P(z) - F(z)] .$$

BIBLIOGRAPHY

1. E. Hille and R. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications vol. **XXXI** ch. 7 pp. 237-255.
2. R. A. Rosenbaum, *Subadditive functions*, Duke Math J., **17** (1950), p 227-247.

UNIVERSITY OF CALIFORNIA,
SANTA BARBARA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG
Stanford University
Stanford, California

F. H. BROWNELL
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

L. J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY
D. DERRY

E. HEWITT
A. HORN
L. NACHBIN

M. OHTSUKA
H. L. ROYDEN
M. M. SCHIFFER

E. SPANIER
E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

M. Altman, <i>An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space</i>	1107
Nesmith Cornett Ankeny, <i>Criterion for rth power residuacity</i>	1115
Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures I</i>	1125
Frank Featherstone Bonsall, <i>Positive operators compact in an auxiliary topology</i>	1131
Billy Joe Boyer, <i>Summability of derived conjugate series</i>	1139
Delmar L. Boyer, <i>A note on a problem of Fuchs</i>	1147
Hans-Joachim Bremermann, <i>The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces</i>	1149
Andrew Michael Bruckner, <i>Minimal superadditive extensions of superadditive functions</i>	1155
Billy Finney Bryant, <i>On expansive homeomorphisms</i>	1163
Jean W. Butler, <i>On complete and independent sets of operations in finite algebras</i>	1169
Lucien Le Cam, <i>An approximation theorem for the Poisson binomial distribution</i>	1181
Paul Civin, <i>Involutions on locally compact rings</i>	1199
Earl A. Coddington, <i>Normal extensions of formally normal operators</i>	1203
Jacob Feldman, <i>Some classes of equivalent Gaussian processes on an interval</i>	1211
Shaul Foguel, <i>Weak and strong convergence for Markov processes</i>	1221
Martin Fox, <i>Some zero sum two-person games with moves in the unit interval</i>	1235
Robert Pertsch Gilbert, <i>Singularities of three-dimensional harmonic functions</i>	1243
Branko Grünbaum, <i>Partitions of mass-distributions and of convex bodies by hyperplanes</i>	1257
Sidney Morris Harmon, <i>Regular covering surfaces of Riemann surfaces</i>	1263
Edwin Hewitt and Herbert S. Zuckerman, <i>The multiplicative semigroup of integers modulo m</i>	1291
Paul Daniel Hill, <i>Relation of a direct limit group to associated vector groups</i>	1309
Calvin Virgil Holmes, <i>Commutator groups of monomial groups</i>	1313
James Fredrik Jakobsen and W. R. Utz, <i>The non-existence of expansive homeomorphisms on a closed 2-cell</i>	1319
John William Jewett, <i>Multiplication on classes of pseudo-analytic functions</i>	1323
Helmut Klingen, <i>Analytic automorphisms of bounded symmetric complex domains</i>	1327
Robert Jacob Koch, <i>Ordered semigroups in partially ordered semigroups</i>	1333
Marvin David Marcus and N. A. Khan, <i>On a commutator result of Taussky and Zassenhaus</i>	1337
John Glen Marica and Steve Jerome Bryant, <i>Unary algebras</i>	1347
Edward Peter Merkes and W. T. Scott, <i>On univalence of a continued fraction</i>	1361
Shu-Teh Chen Moy, <i>Asymptotic properties of derivatives of stationary measures</i>	1371
John William Neuberger, <i>Concerning boundary value problems</i>	1385
Edward C. Posner, <i>Integral closure of differential rings</i>	1393
Marian Reichaw-Reichbach, <i>Some theorems on mappings onto</i>	1397
Marvin Rosenblum and Harold Widom, <i>Two extremal problems</i>	1409
Morton Lincoln Slater and Herbert S. Wilf, <i>A class of linear differential-difference equations</i>	1419
Charles Robson Storey, Jr., <i>The structure of threads</i>	1429
J. François Treves, <i>An estimate for differential polynomials in $\partial/\partial z_1, \dots, \partial/\partial z_n$</i>	1447
J. D. Weston, <i>On the representation of operators by convolutions integrals</i>	1453
James Victor Whittaker, <i>Normal subgroups of some homeomorphism groups</i>	1469