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## **NORMAL EXTENSIONS OF FORMALLY NORMAL OPERATORS**

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# NORMAL EXTENSIONS OF FORMALLY NORMAL OPERATORS

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**1. Introduction.** Let  $\mathfrak{H}$  be a Hilbert space. If  $T$  is any operator in  $\mathfrak{H}$  its domain will be denoted by  $\mathfrak{D}(T)$ , its null space by  $\mathfrak{N}(T)$ . A *formally normal* operator  $N$  in  $\mathfrak{H}$  is a densely defined closed operator such that  $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$ , and  $\|Nf\| = \|N^*f\|$  for all  $f \in \mathfrak{D}(N)$ . Intimately associated with such an  $N$  is the operator  $\bar{N}$  which is the restriction of  $N^*$  to  $\mathfrak{D}(N)$ . The operator  $N$  is formally normal if and only if  $\bar{N}$  is. A *normal operator*  $N$  in  $\mathfrak{H}$  is a formally normal operator for which  $\mathfrak{D}(N) = \mathfrak{D}(N^*)$ ; in this case  $\bar{N} = N^*$ . A densely defined closed operator  $N$  is normal if and only if  $N^*N = NN^*$ .<sup>1</sup>

Let  $N$  be formally normal in  $\mathfrak{H}$ . Since  $\bar{N} \subset N^*$  we have  $N \subset \bar{N}^*$ , where  $\bar{N}^* = (\bar{N})^*$ . Thus we see that a closed symmetric operator is a formally normal operator such that  $N = \bar{N}$ , and a self-adjoint operator is a normal operator such that  $N = \bar{N}$  ( $= N^*$ ). If a closed symmetric operator has a normal extension in  $\mathfrak{H}$ , this extension is self-adjoint. It is known that a closed symmetric operator may not have a self-adjoint extension in  $\mathfrak{H}$ . Necessary and sufficient conditions for such extensions were given by von Neumann.<sup>2</sup> However, until recently, conditions under which a formally normal operator  $N$  can be extended to a normal one in  $\mathfrak{H}$  were known only for certain special cases.<sup>3,4</sup> Kilpi<sup>5</sup> considered the problem in terms of the real and imaginary parts of  $N$ . It is the purpose of this note to characterize the normal extensions of  $N$  in a manner similar to the von Neumann solution for the symmetric case.

If  $N_1$  is a normal extension of a formally normal operator  $N$  in  $\mathfrak{H}$ , then it is easy to see that  $N \subset N_1 \subset \bar{N}^*$ , and  $\bar{N} \subset N_1^* \subset N^*$ . In Theorem 1 we describe  $\mathfrak{D}(\bar{N}^*)$  and  $\mathfrak{D}(N^*)$  for any two operators  $N, \bar{N}$  satisfying  $N \subset \bar{N}^*$ ,  $\bar{N} \subset N^*$ . With the aid of this result a characterization of the normal extensions  $N_1$  of a formally normal  $N$  in  $\mathfrak{H}$  is given in Theorem 2. It is indicated in Theorem 3 how the domains of normal extensions

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<sup>1</sup> See, e.g., B. v. Sz. Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, *Ergeb. Math.*, **5** (1942), 33.

<sup>2</sup> *Ibid*; p. 39.

<sup>3</sup> Y. Kilpi, "Über lineare normale Transformationen im Hilbertschen Raum", *Annales Academiae Scientiarum Fennicae, Series A-I*, No. **154** (1953).

<sup>4</sup> R. H. Davis, "Singular normal differential operators", Technical Report No. 10, Department of Mathematics, University of California, Berkeley, Calif., (1955).

<sup>5</sup> Y. Kilpi, "Über das komplexe Momentenproblem", *Annales Academiae Scientiarum Fennicae, Series A-I*, No. **236** (1957).

can be described by abstract boundary conditions.

I would like to thank Ralph Phillips for instructive conversations during this work.

## 2. Domains.

**THEOREM 1.** *Let  $N, \bar{N}$  be two closed densely defined operators in a Hilbert space  $\mathfrak{H}$  such that  $N \subset \bar{N}^*$ ,  $\bar{N} \subset N^*$ . Then*

$$\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N) + \mathfrak{M}, \quad \mathfrak{D}(N^*) = \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}},$$

where  $\mathfrak{M} = \mathfrak{R}(I + N^*\bar{N}^*)$ ,  $\bar{\mathfrak{M}} = \mathfrak{R}(I + \bar{N}^*N^*)$ . Here  $I$  is the identity operator, and the sums are direct sums.

*Proof.* Let  $N, \bar{N}$  be any two closed densely defined operators in  $\mathfrak{H}$  such that  $N \subset \bar{N}^*$ ,  $\bar{N} \subset N^*$ . Then  $(Nf, g) = (f, \bar{N}g)$  for all  $f \in \mathfrak{D}(N)$ ,  $g \in \mathfrak{D}(\bar{N})$ . Define an operator  $\mathcal{N}$  in the Hilbert space  $\mathfrak{H}_2 = \mathfrak{H} \oplus \mathfrak{H}$  with domain  $\mathfrak{D}(\mathcal{N})$  the set of all  $\hat{f} = \{f_1, f_2\}$  with  $f_1 \in \mathfrak{D}(N)$ ,  $f_2 \in \mathfrak{D}(\bar{N})$ , and such that  $\mathcal{N}\hat{f} = \{\bar{N}f_2, Nf_1\}$ . Then  $\mathcal{N}$  is closed symmetric. Indeed  $\mathfrak{D}(\mathcal{N})$  is dense in  $\mathfrak{H} \oplus \mathfrak{H}$ , and, if  $\hat{f} = \{f_1, f_2\}$ ,  $\hat{g} = \{g_1, g_2\}$  are in  $\mathfrak{D}(\mathcal{N})$ , we have

$$(\mathcal{N}\hat{f}, \hat{g}) = (\bar{N}f_2, g_1) + (Nf_1, g_2) = (f_1, \bar{N}g_2) + (f_2, Ng_1) = (\hat{f}, \mathcal{N}\hat{g}).$$

Since  $N$  and  $\bar{N}$  are closed, so is  $\mathcal{N}$ . The adjoint  $\mathcal{N}^*$  of  $\mathcal{N}$  has domain  $\mathfrak{D}(\mathcal{N}^*)$  the set of all  $\hat{g} = \{g_1, g_2\}$  such that  $g_1 \in \mathfrak{D}(\bar{N}^*)$ ,  $g_2 \in \mathfrak{D}(N^*)$ ; and  $\mathcal{N}^*\hat{g} = \{N^*g_2, \bar{N}^*g_1\}$ .

We now show that the defect spaces of  $\mathcal{N}$ , namely,

$$\begin{aligned} \mathfrak{C}(+i) &= \{\hat{\phi} \in \mathfrak{D}(\mathcal{N}^*) : \mathcal{N}^*\hat{\phi} = i\hat{\phi}\}, \\ \mathfrak{C}(-i) &= \{\hat{\psi} \in \mathfrak{D}(\mathcal{N}^*) : \mathcal{N}^*\hat{\psi} = -i\hat{\psi}\}, \end{aligned}$$

have the same dimension. We have  $\hat{\phi} = \{\phi_1, \phi_2\} \in \mathfrak{C}(+i)$  if and only if  $\phi_1 \in \mathfrak{D}(\bar{N}^*)$ ,  $\phi_2 \in \mathfrak{D}(N^*)$ ,  $N^*\phi_2 = i\phi_1$ ,  $\bar{N}^*\phi_1 = i\phi_2$ . The latter is true if and only if  $N^*(-\phi_2) = -i\phi_1$ ,  $\bar{N}^*\phi_1 = -i(-\phi_2)$ . Thus we see that the unitary map  $\mathcal{U}$  of  $\mathfrak{H}_2$  onto itself given by  $\mathcal{U}\{f_1, f_2\} = \{f_1, -f_2\}$  carries  $\mathfrak{C}(-i)$  onto  $\mathfrak{C}(+i)$  in an isometric way. This proves  $\dim \mathfrak{C}(+i) = \dim \mathfrak{C}(-i)$ .

We note that  $\{\phi_1, \phi_2\} \in \mathfrak{C}(+i)$  if and only if  $\phi_1 \in \mathfrak{D}(N^*\bar{N}^*)$ ,  $(I + N^*\bar{N}^*)\phi_1 = 0$ , and  $\phi_2 = -i\bar{N}^*\phi_1$ . Alternatively  $\{\phi_1, \phi_2\} \in \mathfrak{C}(+i)$  if and only if  $\phi_2 \in \mathfrak{D}(\bar{N}^*N^*)$ ,  $(I + \bar{N}^*N^*)\phi_2 = 0$ , and  $\phi_1 = -iN^*\phi_2$ . Thus we see that the algebraic dimensions of the spaces  $\mathfrak{M} = \mathfrak{R}(I + N^*\bar{N}^*)$ ,  $\bar{\mathfrak{M}} = \mathfrak{R}(I + \bar{N}^*N^*)$ ,  $\mathfrak{C}(+i)$ , and  $\mathfrak{C}(-i)$  are all the same. Further it is easy to see that  $\bar{N}^*$  maps  $\mathfrak{M}$  one-to-one onto  $\bar{\mathfrak{M}}$ , the inverse mapping being  $-N^*$  restricted to  $\bar{\mathfrak{M}}$ .

Since  $\dim \mathfrak{C}(+i) = \dim \mathfrak{C}(-i)$  the operator  $\mathcal{N}$  has self-adjoint

extensions in  $\mathfrak{E}_2$ . They are in a one-to-one correspondence with the isometries of  $\mathfrak{E}(-i)$  onto  $\mathfrak{E}(+i)$ . If  $\mathcal{S}$  is a self-adjoint extension of  $\mathcal{N}$  there is a unique isometry  $\mathcal{V}$  of  $\mathfrak{E}(-i)$  onto  $\mathfrak{E}(+i)$  such that  $\mathfrak{D}(\mathcal{S}) = \mathfrak{D}(\mathcal{N}) + (\mathcal{S} - \mathcal{V})\mathfrak{E}(-i)$ , where  $\mathcal{S}$  is the identity operator on  $\mathfrak{E}_2$ . Let us consider that self-adjoint extension  $\mathcal{S}$  of  $\mathcal{N}$  determined in this way by the isometry  $-\mathcal{U}$  restricted to  $\mathfrak{E}(-i)$ . Then we have  $\hat{h} \in \mathfrak{D}(\mathcal{S})$  if and only if  $\hat{h} = \hat{f} + \hat{\psi} + \mathcal{U}\hat{\psi}$ , for some  $\hat{f} \in \mathfrak{D}(\mathcal{N})$ ,  $\hat{\psi} \in \mathfrak{E}(-i)$ . If  $\hat{h} = \{h_1, h_2\}$ ,  $\hat{f} = \{f_1, f_2\}$ ,  $\hat{\psi} = \{\psi_1, \psi_2\}$ , this means  $h_1 = f_1 + 2\psi_1$ ,  $h_2 = f_2$ , where  $f_1 \in \mathfrak{D}(N)$ ,  $\psi_1 \in \mathfrak{M}$ ,  $f_2 \in \mathfrak{D}(\bar{N})$ . Thus  $\mathfrak{D}(\mathcal{S})$  is the set of all  $\{h_1, h_2\}$  with  $h_1 \in \mathfrak{D}(N) + \mathfrak{M}$ ,  $h_2 \in \mathfrak{D}(\bar{N})$ . Now the operator  $\mathcal{S}_1$  with domain all  $\{h_1, h_2\}$  with  $h_1 \in \mathfrak{D}(\bar{N}^*)$ ,  $h_2 \in \mathfrak{D}(\bar{N})$ , and such that  $\mathcal{S}_1\{h_1, h_2\} = \{\bar{N}h_2, \bar{N}^*h_1\}$ , is readily seen to be a self-adjoint operator in  $\mathfrak{E}_2$  satisfying  $\mathcal{N} \subset \mathcal{S} \subset \mathcal{S}_1 \subset N^*$ . Hence  $\mathcal{S} = \mathcal{S}_1$ , and we see that  $\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N) + \mathfrak{M}$ . The sum is a direct one, for if  $f \in \mathfrak{D}(N) \cap \mathfrak{M}$ ,  $0 = (I + N^*\bar{N}^*)f = f + N^*Nf$  implying  $0 = (f + N^*Nf, f) = \|f\|^2 + \|Nf\|^2$ , or  $f = 0$ .

A similar argument shows that the self-adjoint extension  $\mathcal{S}$  of  $\mathcal{N}$  determined by the isometry  $\mathcal{V}$  equal to  $\mathcal{U}$  restricted to  $\mathfrak{E}(-i)$  has domain the set of all  $\{h_1, h_2\}$  with  $h_1 \in \mathfrak{D}(N)$ ,  $h_2 \in \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$ . This operator is equal to the self-adjoint extension of  $\mathcal{N}$  having domain the set of all  $\{h_1, h_2\}$  with  $h_1 \in \mathfrak{D}(N)$ ,  $h_2 \in \mathfrak{D}(N^*)$ , implying that  $\mathfrak{D}(N^*) = \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$ , a direct sum. This completes the proof of Theorem 1.

*Note added in proof.* The results of Theorem 1 can be obtained more directly, although some of the discussion given in the proof above is required for our proof of Theorem 2. Let  $\mathfrak{G}(T)$  denote the graph of an operator  $T$ . If  $A, B$  are any two closed operators with dense domain, and  $A \subset B$ , then it is easy to see that  $\mathfrak{G}(B) \ominus \mathfrak{G}(A)$  is the set of all  $\{u, Bu\} \in \mathfrak{G}(B)$  such that  $u \in \mathfrak{N}(I + A^*B)$ . Since

$$\mathfrak{G}(B) = \mathfrak{G}(A) \oplus [\mathfrak{G}(B) \ominus \mathfrak{G}(A)],$$

we have  $\mathfrak{D}(B) = \mathfrak{D}(A) + \mathfrak{N}(I + A^*B)$ , a direct sum. This implies Theorem 1.

### 3. Normal extensions.

**THEOREM 2.** *If  $N_1$  is a normal extension of a formally normal operator  $N$  in a Hilbert space  $\mathfrak{E}$ , then there exists a unique linear map  $W$  of  $\mathfrak{M}$  onto itself satisfying*

- (i)  $W^2 = I$ ,
- (ii)  $\|\phi\|^2 + \|\bar{N}^*\phi\|^2 = \|W\phi\|^2 + \|\bar{N}^*W\phi\|^2$ , ( $\phi \in \mathfrak{M}$ ),
- (iii)  $(I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M}$ ,
- (iv)  $\|\bar{N}^*(I - W)\phi\| = \|N^*(I - W)\phi\|$ , ( $\phi \in \mathfrak{M}$ ).

*In terms of  $W$  we have*

$$(1) \quad \mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}, \quad N_1f = \bar{N}^*f, \quad (f \in \mathfrak{D}(N_1)).$$

Conversely, if  $W$  is any linear map of  $\mathfrak{M}$  onto  $\mathfrak{M}$  satisfying (i)—(iv) above, then the operator  $N_1$  defined by (1) is a normal extension of  $N$  in  $\mathfrak{G}$ .

REMARKS. Condition (i) implies that  $P_1 = (1/2)(I + W)$  and  $P_2 = (1/2)(I - W)$  are projections (not necessarily orthogonal) in  $\mathfrak{M}$ , and  $\mathfrak{M}$  is the direct sum of  $\mathfrak{M}_1 = P_1\mathfrak{M}$  and  $\mathfrak{M}_2 = P_2\mathfrak{M}$ . If  $\phi \in \mathfrak{M}$ , then  $\phi \in \mathfrak{M}_1$  if and only if  $W\phi = \phi$ , and  $\phi \in \mathfrak{M}_2$  if and only if  $W\phi = -\phi$ .

Condition (ii) implies that if  $\phi, \phi' \in \mathfrak{M}$  then

$$(\phi, \phi') + (\bar{N}^*\phi, \bar{N}^*\phi') = (W\phi, W\phi') + (\bar{N}^*W\phi, \bar{N}^*W\phi').$$

If  $\phi \in \mathfrak{M}_1, \phi' \in \mathfrak{M}_2$  we see that  $(\phi, \phi') + (\bar{N}^*\phi, \bar{N}^*\phi') = 0$ , which means that the graph of  $\bar{N}^*$  restricted to  $\mathfrak{M}_1$  is orthogonal to the graph of  $\bar{N}^*$  restricted to  $\mathfrak{M}_2$ .

Since  $\bar{N}^*$  is one-to-one from  $\mathfrak{M}$  onto  $\bar{\mathfrak{M}}$ , condition (iii) implies that  $\mathfrak{M}_2 = \bar{N}^*\mathfrak{M}_1 \subset \mathfrak{M} \cap \bar{\mathfrak{M}}$ , and  $\mathfrak{M}_2$  has the same algebraic dimension as  $\mathfrak{M}_1$ . In particular the dimension of  $\mathfrak{M}$  must be even.

*Proof of Theorem 2.* Let  $N_1$  be a normal extension of the formally normal operator  $N$  in  $\mathfrak{G}$ . Then we have  $N \subset N_1 \subset \bar{N}^*, \bar{N} \subset N_1^* \subset N^*$ . Let the operator  $\mathcal{N}_1$  in  $\mathfrak{G}_2$  be defined with domain all  $\{h_1, h_2\}$  such that  $h_1 \in \mathfrak{D}(N_1), h_2 \in \mathfrak{D}(N_1^*)$ , and so that  $\mathcal{N}_1\{h_1, h_2\} = \{N_1^*h_2, N_1h_1\}$ . Then it is easily seen that  $\mathcal{N}_1$  is a self-adjoint extension of the operator  $\mathcal{N}$  defined in the proof of Theorem 1.

Let  $\mathcal{N}_1$  be any self-adjoint extension of  $\mathcal{N}$ , and let  $\mathcal{V}$  be the unique isometry of  $\mathfrak{G}(-i)$  onto  $\mathfrak{G}(+i)$  such that  $\mathfrak{D}(\mathcal{N}_1) = \mathfrak{D}(\mathcal{N}) + (\mathcal{I} - \mathcal{V})\mathfrak{G}(-i)$ . Then we may write  $\mathcal{V} = \mathcal{W}\mathcal{U}$ , where  $\mathcal{U}$  is the isometry defined on  $\mathfrak{G}(-i)$  to  $\mathfrak{G}(+i)$  by  $\mathcal{U}\{\psi_1, \psi_2\} = \{\psi_1, -\psi_2\}$ , and  $\mathcal{W}$  is a unitary map of  $\mathfrak{G}(+i)$  onto itself. For  $\{\phi_1, \phi_2\} \in \mathfrak{G}(+i)$  let  $\mathcal{W}\{\phi_1, \phi_2\} = \{\chi_1, \chi_2\}$ . Then  $\phi_1, \chi_1 \in \mathfrak{M}$  and  $\phi_2 = -i\bar{N}^*\phi_1, \chi_2 = -i\bar{N}^*\chi_1$ . Define the map  $W$  of  $\mathfrak{M}$  into  $\mathfrak{M}$  by  $W\phi_1 = \chi_1$ . Then  $W$  is linear, and since  $\mathcal{W}$  is unitary,  $W$  is onto, and

$$\|\{\phi, -i\bar{N}^*\phi\}\|^2 = \|\{W\phi, -i\bar{N}^*W\phi\}\|^2, \quad (\phi \in \mathfrak{M}),$$

or

$$(2) \quad \|\phi\|^2 + \|\bar{N}^*\phi\|^2 = \|W\phi\|^2 + \|\bar{N}^*W\phi\|^2, \quad (\phi \in \mathfrak{M}).$$

Conversely, suppose  $W$  is a linear map of  $\mathfrak{M}$  onto  $\mathfrak{M}$  satisfying (2). Then for  $\hat{\phi} = \{\phi, -i\bar{N}^*\phi\} \in \mathfrak{G}(+i)$  define  $\mathcal{W}\hat{\phi} = \{W\phi, -i\bar{N}^*W\phi\}$ . Then  $\mathcal{W}$  maps  $\mathfrak{G}(+i)$  onto  $\mathfrak{G}(+i)$  and (2) implies that  $\mathcal{W}$  is unitary. Thus we see that the self-adjoint extensions  $\mathcal{N}_1$  of  $\mathcal{N}$  are in a one-to-one correspondence with the linear maps  $W$  of  $\mathfrak{M}$  onto  $\mathfrak{M}$  satisfying (2). We have  $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$  if and only if  $\hat{h}$  can be represented in

the form  $\hat{h} = \hat{f} + (\mathcal{I} - \mathcal{W}\mathcal{U})\hat{\psi}$ , where  $\hat{f} = \{f_1, f_2\} \in \mathfrak{D}(\mathcal{N})$ ,  $\hat{\psi} = \{\phi, i\bar{N}^*\phi\} \in \mathfrak{G}(-i)$ . This means  $h_1 = f_1 + (I - W)\phi$ ,  $h_2 = f_2 + i\bar{N}^*(I + W)\phi$ , where  $f_1 \in \mathfrak{D}(\mathcal{N})$ ,  $f_2 \in \mathfrak{D}(\bar{N})$ ,  $\phi \in \mathfrak{M}$ .

The self-adjoint extension  $\mathcal{N}_1$  arising from the normal extension  $N_1$  of  $N$  has the property that if  $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$  then so does  $\mathcal{P}_1\hat{h} = \{h_1, 0\}$ . It will now be shown that a self-adjoint extension  $\mathcal{N}_1$  of  $\mathcal{N}$  has this property if and only if the  $W$  corresponding to  $\mathcal{N}_1$  satisfies  $W^2 = I$ . First suppose  $\mathcal{P}_1\hat{h} \in \mathfrak{D}(\mathcal{N}_1)$  for all  $\hat{h} \in \mathfrak{D}(\mathcal{N}_1)$ . Letting  $h_1 = f_1 + (I - W)\phi$ ,  $h_2 = f_2 + i\bar{N}^*(I + W)\phi$  as above, we see that this implies that there exist elements  $f'_1 \in \mathfrak{D}(N)$ ,  $f'_2 \in \mathfrak{D}(\bar{N})$ ,  $\phi' \in \mathfrak{M}$ , such that

$$\begin{aligned} f_1 + (I - W)\phi &= f'_1 + (I - W)\phi', \\ 0 &= f'_2 + i\bar{N}^*(I + W)\phi'. \end{aligned}$$

Since  $\mathfrak{D}(N) + \mathfrak{M}$  and  $\mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$  are direct sums these equations imply that  $f_1 = f'_1$ ,  $(I - W)\phi = (I - W)\phi'$ ,  $f'_2 = 0$ , and  $\bar{N}^*(I + W)\phi' = 0$ . The last equation implies  $(I + W)\phi' = 0$  since  $\bar{N}^*$  is one-to-one from  $\bar{\mathfrak{M}}$  to  $\mathfrak{M}$ . Thus we have

$$(3) \quad \begin{aligned} \phi' + W\phi' &= 0, \\ \phi' - W\phi' &= \phi - W\phi, \end{aligned}$$

from which results  $2\phi' = (I - W)\phi$ . Returning to the first equation in (3) we obtain  $(I + W)(I - W)\phi = (I - W^2)\phi = 0$  for all  $\phi \in \mathfrak{M}$ , showing that  $W^2 = I$ . Conversely, suppose  $W^2 = I$  on  $\mathfrak{M}$ . Then if  $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$ ,  $h_1 = f_1 + (I - W)\phi$ ,  $h_2 = f_2 + i\bar{N}^*(I + W)\phi$ , define  $\phi' = (1/2)(I - W)\phi$ . Then equations (3) will be valid, implying that

$$\begin{aligned} f_1 + (I - W)\phi &= f_1 + (I - W)\phi', \\ 0 &= 0 + i\bar{N}^*(I + W)\phi', \end{aligned}$$

which shows that  $\mathcal{P}_1\hat{h} = \{h_1, 0\} \in \mathfrak{D}(\mathcal{N}_1)$ .

If  $\mathcal{N}_1$  is any self-adjoint extension of  $\mathcal{N}$  for which  $W^2 = I$ , then  $\mathfrak{D}(\mathcal{N}_1)$  consists of those  $\{h_1, h_2\}$  such that  $h_1 = f_1 + (I - W)\phi$ ,  $h_2 = f_2 + i\bar{N}^*(I + W)\phi'$ , for some  $f_1 \in \mathfrak{D}(N)$ ,  $f_2 \in \mathfrak{D}(\bar{N})$ , and  $\phi, \phi' \in \mathfrak{M}$ . The point is that  $\phi$  and  $\phi'$  need not now be the same element. Indeed, if  $h_1, h_2$  have such representations let  $\phi'' = (1/2)(I - W)\phi + (1/2)(I + W)\phi'$ . Then  $(I - W)\phi = (I - W)\phi''$ , and  $(I + W)\phi' = (I + W)\phi''$ , which implies that  $\{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$ . For such an  $\mathcal{N}_1$  define  $N_1$  to be the operator in  $\mathfrak{S}$  with  $\mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}$ , and  $N_1h_1 = \bar{N}^*h_1$  for  $h_1 \in \mathfrak{D}(N_1)$ . Similarly define  $N_2$  on  $\mathfrak{D}(N_2) = \mathfrak{D}(\bar{N}) + \bar{N}^*(I + W)\mathfrak{M}$  by  $N_2h_2 = N^*h_2$  for  $h_2 \in \mathfrak{D}(N_2)$ . In terms of  $N_1$  and  $N_2$  we have  $\{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$  if and only if  $h_1 \in \mathfrak{D}(N_1)$ ,  $h_2 \in \mathfrak{D}(N_2)$ , and  $\mathcal{N}_1\{h_1, h_2\} = \{N_2h_2, N_1h_1\}$ . A short computation shows that  $\mathfrak{D}(\mathcal{N}_1^*)$  is the set of all  $\{g_1, g_2\}$  such that  $g_1 \in \mathfrak{D}(N_2^*)$ ,

$g_2 \in \mathfrak{D}(N_1^*)$ , and  $\mathcal{N}_1^* \{g_1, g_2\} = \{N_1^*g_2, N_2^*g_1\}$ . But since  $\mathcal{N}_1 = \mathcal{N}_1^*$  we obtain  $N_2 = N_1^*$ . Hence  $\mathfrak{D}(\mathcal{N}_1)$  consists of all  $\{h_1, h_2\}$  with  $h_1 \in \mathfrak{D}(N_1)$ ,  $h_2 \in \mathfrak{D}(N_1^*)$ , and  $\mathcal{N}_1 \{h_1, h_2\} = \{N_1^*h_2, N_1h_1\}$ . Here

$$(4) \quad \begin{aligned} \mathfrak{D}(N_1) &= \mathfrak{D}(N) + (I - W)\mathfrak{M}, \\ \mathfrak{D}(N_1^*) &= \mathfrak{D}(\bar{N}) + \bar{N}^*(I + W)\mathfrak{M}, \end{aligned}$$

and  $N \subset N_1 \subset \bar{N}^*$ ,  $\bar{N} \subset N_1^* \subset N^*$ . Thus any self-adjoint extension  $\mathcal{N}_1$  of  $\mathcal{N}$  having the property that  $W^2 = I$  determines a unique operator  $N_1$  in  $\mathfrak{H}$  as above, which is easily seen to be closed. In particular, if  $N_1$  is a normal extension of  $N$ , then the equalities (4) hold.

It remains to characterize those  $\mathcal{N}_1$  such that  $W^2 = I$  for which  $N_1$  is normal, that is  $\mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$  and  $\|N_1h\| = \|N_1^*h\|$ ,  $h \in \mathfrak{D}(N_1)$ . We claim that this is true if and only if

$$(5) \quad (I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M},$$

and

$$(6) \quad \|\bar{N}^*(I - W)\phi\| = \|N^*(I - W)\phi\|, \quad (\phi \in \mathfrak{M}).$$

If (5) is valid then (4) implies that  $\mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$ , since  $\mathfrak{D}(N) = \mathfrak{D}(\bar{N})$ . Let  $h \in \mathfrak{D}(N_1)$ ,  $h = f + (I - W)\phi$ ,  $f \in \mathfrak{D}(N)$ ,  $\phi \in \mathfrak{M}$ . Then  $(I - W)\phi \in \mathfrak{M} \cap \bar{\mathfrak{M}}$ , and we have  $N_1h = Nf + \bar{N}^*(I - W)\phi$ ,  $N_1^*h = \bar{N}f + N^*(I - W)\phi$ . Thus

$$\begin{aligned} \|N_1h\|^2 &= \|Nf\|^2 + (Nf, \bar{N}^*(I - W)\phi) + (\bar{N}^*(I - W)\phi, Nf) \\ &\quad + \|\bar{N}^*(I - W)\phi\|^2, \end{aligned}$$

and

$$\begin{aligned} \|N_1^*h\|^2 &= \|\bar{N}f\|^2 + (\bar{N}f, N^*(I - W)\phi) + (N^*(I - W)\phi, \bar{N}f) \\ &\quad + \|N^*(I - W)\phi\|^2. \end{aligned}$$

Since  $N$  is formally normal  $\|Nf\| = \|\bar{N}f\|$ . Moreover  $\bar{N}^*(I - W)\phi \in \bar{\mathfrak{M}}$  implies that  $(Nf, \bar{N}^*(I - W)\phi) = (f, N^*\bar{N}^*(I - W)\phi) = -(f, (I - W)\phi)$ , and similarly  $(\bar{N}f, N^*(I - W)\phi) = -(f, (I - W)\phi)$ . Using (6) we see that  $\|N_1h\| = \|N_1^*h\|$  for all  $h \in \mathfrak{D}(N_1)$ , proving that  $N_1$  is normal.

Conversely, suppose  $N_1$  is normal. Then (6) is clearly valid, for  $(I - W)\phi \in \mathfrak{D}(N_1)$  by (4). Suppose  $h \in \mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$  and  $h = f + (I - W)\phi = f' + \bar{N}^*(I + W)\phi'$  with  $f, f' \in \mathfrak{D}(N)$ ,  $\phi, \phi' \in \mathfrak{M}$ . We show that  $f = f'$  and  $(I - W)\phi = \bar{N}^*(I + W)\phi'$ . Applying this to  $f = 0$  we obtain  $(I - W)\mathfrak{M} \subset \bar{N}^*(I + W)\mathfrak{M}$ , and with  $f' = 0$  we get  $\bar{N}^*(I + W)\mathfrak{M} \subset (I - W)\mathfrak{M}$ , proving (5). Now for any  $g \in \mathfrak{D}(N)$  we have  $(N_1h, N_1g) = (N_1^*h, N_1^*g)$ , or

$$(Nf, Ng) + (\bar{N}^*(I - W)\phi, Ng) = (\bar{N}f', \bar{N}g) - ((I + W)\phi', \bar{N}g).$$

Since  $(\bar{N}f', \bar{N}g) = (Nf', Ng)$  and  $(\bar{N}^*(I - W)\phi, Ng) = -((I - W)\phi, g)$ , this yields

$$(Nf, Ng) - ((I - W)\phi, g) = (Nf', Ng) - (\bar{N}^*(I + W)\phi', g) ,$$

or

$$(N(f - f'), Ng) + (\bar{N}^*(I + W)\phi' - (I - W)\phi, g) = 0 .$$

But  $\bar{N}^*(I + W)\phi' - (I - W)\phi = f - f'$ , and hence

$$(N(f - f'), Ng) + (f - f', g) = 0$$

for all  $g \in \mathfrak{D}(N)$ . Letting  $g = f - f'$  we obtain  $f = f'$  as desired. This completes the proof of Theorem 2.

**4. Abstract boundary conditions.** For  $u \in \mathfrak{D}(\bar{N}^*)$ ,  $v \in \mathfrak{D}(N^*)$  define  $\langle uv \rangle = (\bar{N}^*u, v) - (u, N^*v)$ .

**THEOREM 3.** *If  $N_1$  is a normal extension of the formally normal operator  $N$  such that  $\mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}$ , then  $\mathfrak{D}(N_1)$  may be described as the set of all  $u \in \mathfrak{D}(\bar{N}^*)$  satisfying  $\langle u\alpha \rangle = 0$  for all  $\alpha \in (I - W)\mathfrak{M}$ .*<sup>6</sup>

**REMARK.** For differential operators the conditions  $\langle u\alpha \rangle = 0$  become boundary conditions. They are self-adjoint ones, that is,  $\langle \alpha\alpha' \rangle = 0$  for all  $\alpha, \alpha' \in (I - W)\mathfrak{M}$ . Indeed  $\alpha, \alpha' \in \mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$  and for any  $\alpha \in \mathfrak{D}(N_1)$ ,  $\alpha' \in \mathfrak{D}(N_1^*)$  we have  $(\bar{N}^*\alpha, \alpha') = (N_1\alpha, \alpha') = (\alpha, N_1^*\alpha') = (\alpha, N^*\alpha')$ .

*Proof of Theorem 3.* If  $u \in \mathfrak{D}(N_1)$ ,  $\alpha \in (I - W)\mathfrak{M} \subset \mathfrak{D}(N_1^*)$ , the above argument shows that  $\langle u\alpha \rangle = 0$ . Conversely suppose  $u \in \mathfrak{D}(\bar{N}^*)$  and  $\langle u\alpha \rangle = 0$  for all  $\alpha \in (I - W)\mathfrak{M}$ . Let  $u = f + (I - W)\phi + (I + W)\phi$ , where  $f \in \mathfrak{D}(N)$ ,  $\phi \in \mathfrak{M}$ . We note that  $\langle \cdot \rangle$  is linear in the first spot, and  $f + (I - W)\phi \in \mathfrak{D}(N_1)$ . Thus  $\langle (I + W)\phi \alpha \rangle = 0$  for all  $\alpha \in (I - W)\mathfrak{M}$ . Let  $\alpha = \bar{N}^*(I + W)\phi \in (I - W)\mathfrak{M}$ , since  $(I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M}$ . Then

$$0 = \langle (I + W)\phi \bar{N}^*(I + W)\phi \rangle = (\bar{N}^*(I + W)\phi, \bar{N}^*(I + W)\phi) + ((I + W)\phi, (I + W)\phi) ,$$

which proves that  $(I + W)\phi = 0$ , and hence  $u \in \mathfrak{D}(N_1)$  as desired.

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<sup>6</sup> A result similar to Theorem 3 appears in the report by Davis (4) for the case when  $\dim(\mathfrak{D}(\bar{N}^*)/\mathfrak{D}(N)) < \infty$ .





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