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NORMAL EXTENSIONS OF FORMALLY NORMAL OPERATORS

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1. Introduction. Let \mathfrak{H} be a Hilbert space. If T is any operator in \mathfrak{H} its domain will be denoted by $\mathfrak{D}(T)$, its null space by $\mathfrak{N}(T)$. A *formally normal* operator N in \mathfrak{H} is a densely defined closed operator such that $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$, and $\|Nf\| = \|N^*f\|$ for all $f \in \mathfrak{D}(N)$. Intimately associated with such an N is the operator \bar{N} which is the restriction of N^* to $\mathfrak{D}(N)$. The operator N is formally normal if and only if \bar{N} is. A *normal operator* N in \mathfrak{H} is a formally normal operator for which $\mathfrak{D}(N) = \mathfrak{D}(N^*)$; in this case $\bar{N} = N^*$. A densely defined closed operator N is normal if and only if $N^*N = NN^*$.¹

Let N be formally normal in \mathfrak{H} . Since $\bar{N} \subset N^*$ we have $N \subset \bar{N}^*$, where $\bar{N}^* = (\bar{N})^*$. Thus we see that a closed symmetric operator is a formally normal operator such that $N = \bar{N}$, and a self-adjoint operator is a normal operator such that $N = \bar{N}$ ($= N^*$). If a closed symmetric operator has a normal extension in \mathfrak{H} , this extension is self-adjoint. It is known that a closed symmetric operator may not have a self-adjoint extension in \mathfrak{H} . Necessary and sufficient conditions for such extensions were given by von Neumann.² However, until recently, conditions under which a formally normal operator N can be extended to a normal one in \mathfrak{H} were known only for certain special cases.^{3,4} Kilpi⁵ considered the problem in terms of the real and imaginary parts of N . It is the purpose of this note to characterize the normal extensions of N in a manner similar to the von Neumann solution for the symmetric case.

If N_1 is a normal extension of a formally normal operator N in \mathfrak{H} , then it is easy to see that $N \subset N_1 \subset \bar{N}^*$, and $\bar{N} \subset N_1^* \subset N^*$. In Theorem 1 we describe $\mathfrak{D}(\bar{N}^*)$ and $\mathfrak{D}(N^*)$ for any two operators N, \bar{N} satisfying $N \subset \bar{N}^*$, $\bar{N} \subset N^*$. With the aid of this result a characterization of the normal extensions N_1 of a formally normal N in \mathfrak{H} is given in Theorem 2. It is indicated in Theorem 3 how the domains of normal extensions

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¹ See, e.g., B. v. Sz. Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, *Ergeb. Math.*, **5** (1942), 33.

² *Ibid*; p. 39.

³ Y. Kilpi, "Über lineare normale Transformationen im Hilbertschen Raum", *Annales Academiae Scientiarum Fennicae*, Series A-I, No. **154** (1953).

⁴ R. H. Davis, "Singular normal differential operators", Technical Report No. 10, Department of Mathematics, University of California, Berkeley, Calif., (1955).

⁵ Y. Kilpi, "Über das komplexe Momentenproblem", *Annales Academiae Scientiarum Fennicae*, Series A-I, No. **236** (1957).

can be described by abstract boundary conditions.

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2. Domains.

THEOREM 1. *Let N, \bar{N} be two closed densely defined operators in a Hilbert space \mathfrak{H} such that $N \subset \bar{N}^*$, $\bar{N} \subset N^*$. Then*

$$\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N) + \mathfrak{M}, \quad \mathfrak{D}(N^*) = \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}},$$

where $\mathfrak{M} = \mathfrak{R}(I + N^*\bar{N}^*)$, $\bar{\mathfrak{M}} = \mathfrak{R}(I + \bar{N}^*N^*)$. Here I is the identity operator, and the sums are direct sums.

Proof. Let N, \bar{N} be any two closed densely defined operators in \mathfrak{H} such that $N \subset \bar{N}^*$, $\bar{N} \subset N^*$. Then $(Nf, g) = (f, \bar{N}g)$ for all $f \in \mathfrak{D}(N)$, $g \in \mathfrak{D}(\bar{N})$. Define an operator \mathcal{N} in the Hilbert space $\mathfrak{H}_2 = \mathfrak{H} \oplus \mathfrak{H}$ with domain $\mathfrak{D}(\mathcal{N})$ the set of all $\hat{f} = \{f_1, f_2\}$ with $f_1 \in \mathfrak{D}(N)$, $f_2 \in \mathfrak{D}(\bar{N})$, and such that $\mathcal{N}\hat{f} = \{\bar{N}f_2, Nf_1\}$. Then \mathcal{N} is closed symmetric. Indeed $\mathfrak{D}(\mathcal{N})$ is dense in $\mathfrak{H} \oplus \mathfrak{H}$, and, if $\hat{f} = \{f_1, f_2\}$, $\hat{g} = \{g_1, g_2\}$ are in $\mathfrak{D}(\mathcal{N})$, we have

$$(\mathcal{N}\hat{f}, \hat{g}) = (\bar{N}f_2, g_1) + (Nf_1, g_2) = (f_1, \bar{N}g_2) + (f_2, Ng_1) = (\hat{f}, \mathcal{N}\hat{g}).$$

Since N and \bar{N} are closed, so is \mathcal{N} . The adjoint \mathcal{N}^* of \mathcal{N} has domain $\mathfrak{D}(\mathcal{N}^*)$ the set of all $\hat{g} = \{g_1, g_2\}$ such that $g_1 \in \mathfrak{D}(\bar{N}^*)$, $g_2 \in \mathfrak{D}(N^*)$; and $\mathcal{N}^*\hat{g} = \{N^*g_2, \bar{N}^*g_1\}$.

We now show that the defect spaces of \mathcal{N} , namely,

$$\mathfrak{G}(+i) = \{\hat{\phi} \in \mathfrak{D}(\mathcal{N}^*) : \mathcal{N}^*\hat{\phi} = i\hat{\phi}\},$$

$$\mathfrak{G}(-i) = \{\hat{\psi} \in \mathfrak{D}(\mathcal{N}^*) : \mathcal{N}^*\hat{\psi} = -i\hat{\psi}\},$$

have the same dimension. We have $\hat{\phi} = \{\phi_1, \phi_2\} \in \mathfrak{G}(+i)$ if and only if $\phi_1 \in \mathfrak{D}(\bar{N}^*)$, $\phi_2 \in \mathfrak{D}(N^*)$, $N^*\phi_2 = i\phi_1$, $\bar{N}^*\phi_1 = i\phi_2$. The latter is true if and only if $N^*(-\phi_2) = -i\phi_1$, $\bar{N}^*\phi_1 = -i(-\phi_2)$. Thus we see that the unitary map \mathcal{U} of \mathfrak{H}_2 onto itself given by $\mathcal{U}\{f_1, f_2\} = \{f_1, -f_2\}$ carries $\mathfrak{G}(-i)$ onto $\mathfrak{G}(+i)$ in an isometric way. This proves $\dim \mathfrak{G}(+i) = \dim \mathfrak{G}(-i)$.

We note that $\{\phi_1, \phi_2\} \in \mathfrak{G}(+i)$ if and only if $\phi_1 \in \mathfrak{D}(N^*\bar{N}^*)$, $(I + N^*\bar{N}^*)\phi_1 = 0$, and $\phi_2 = -i\bar{N}^*\phi_1$. Alternatively $\{\phi_1, \phi_2\} \in \mathfrak{G}(+i)$ if and only if $\phi_2 \in \mathfrak{D}(\bar{N}^*N^*)$, $(I + \bar{N}^*N^*)\phi_2 = 0$, and $\phi_1 = -iN^*\phi_2$. Thus we see that the algebraic dimensions of the spaces $\mathfrak{M} = \mathfrak{R}(I + N^*\bar{N}^*)$, $\bar{\mathfrak{M}} = \mathfrak{R}(I + \bar{N}^*N^*)$, $\mathfrak{G}(+i)$, and $\mathfrak{G}(-i)$ are all the same. Further it is easy to see that \bar{N}^* maps \mathfrak{M} one-to-one onto $\bar{\mathfrak{M}}$, the inverse mapping being $-N^*$ restricted to $\bar{\mathfrak{M}}$.

Since $\dim \mathfrak{G}(+i) = \dim \mathfrak{G}(-i)$ the operator \mathcal{N} has self-adjoint

extensions in \mathfrak{S}_2 . They are in a one-to-one correspondence with the isometries of $\mathfrak{G}(-i)$ onto $\mathfrak{G}(+i)$. If \mathcal{S} is a self-adjoint extension of \mathcal{N} there is a unique isometry \mathcal{V} of $\mathfrak{G}(-i)$ onto $\mathfrak{G}(+i)$ such that $\mathfrak{D}(\mathcal{S}) = \mathfrak{D}(\mathcal{N}) + (\mathcal{S} - \mathcal{V})\mathfrak{G}(-i)$, where \mathcal{S} is the identity operator on \mathfrak{S}_2 . Let us consider that self-adjoint extension \mathcal{S} of \mathcal{N} determined in this way by the isometry $-\mathcal{U}$ restricted to $\mathfrak{G}(-i)$. Then we have $\hat{h} \in \mathfrak{D}(\mathcal{S})$ if and only if $\hat{h} = \hat{f} + \hat{\psi} + \mathcal{U}\hat{\psi}$, for some $\hat{f} \in \mathfrak{D}(\mathcal{N})$, $\hat{\psi} \in \mathfrak{G}(-i)$. If $\hat{h} = \{h_1, h_2\}$, $\hat{f} = \{f_1, f_2\}$, $\hat{\psi} = \{\psi_1, \psi_2\}$, this means $h_1 = f_1 + 2\psi_1$, $h_2 = f_2$, where $f_1 \in \mathfrak{D}(N)$, $\psi_1 \in \mathfrak{M}$, $f_2 \in \mathfrak{D}(\bar{N})$. Thus $\mathfrak{D}(\mathcal{S})$ is the set of all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(N) + \mathfrak{M}$, $h_2 \in \mathfrak{D}(\bar{N})$. Now the operator \mathcal{S}_1 with domain all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(\bar{N}^*)$, $h_2 \in \mathfrak{D}(\bar{N})$, and such that $\mathcal{S}_1\{h_1, h_2\} = \{\bar{N}h_2, \bar{N}^*h_1\}$, is readily seen to be a self-adjoint operator in \mathfrak{S}_2 satisfying $\mathcal{N} \subset \mathcal{S} \subset \mathcal{S}_1 \subset N^*$. Hence $\mathcal{S} = \mathcal{S}_1$, and we see that $\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N) + \mathfrak{M}$. The sum is a direct one, for if $f \in \mathfrak{D}(N) \cap \mathfrak{M}$, $0 = (I + N^*\bar{N}^*)f = f + N^*Nf$ implying $0 = (f + N^*Nf, f) = \|f\|^2 + \|Nf\|^2$, or $f = 0$.

A similar argument shows that the self-adjoint extension \mathcal{S} of \mathcal{N} determined by the isometry \mathcal{V} equal to \mathcal{U} restricted to $\mathfrak{G}(-i)$ has domain the set of all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(N)$, $h_2 \in \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$. This operator is equal to the self-adjoint extension of \mathcal{N} having domain the set of all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(N)$, $h_2 \in \mathfrak{D}(N^*)$, implying that $\mathfrak{D}(N^*) = \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$, a direct sum. This completes the proof of Theorem 1.

Note added in proof. The results of Theorem 1 can be obtained more directly, although some of the discussion given in the proof above is required for our proof of Theorem 2. Let $\mathfrak{G}(T)$ denote the graph of an operator T . If A, B are any two closed operators with dense domain, and $A \subset B$, then it is easy to see that $\mathfrak{G}(B) \ominus \mathfrak{G}(A)$ is the set of all $\{u, Bu\} \in \mathfrak{G}(B)$ such that $u \in \mathfrak{R}(I + A^*B)$. Since

$$\mathfrak{G}(B) = \mathfrak{G}(A) \oplus [\mathfrak{G}(B) \ominus \mathfrak{G}(A)],$$

we have $\mathfrak{D}(B) = \mathfrak{D}(A) + \mathfrak{R}(I + A^*B)$, a direct sum. This implies Theorem 1.

3. Normal extensions.

THEOREM 2. *If N_1 is a normal extension of a formally normal operator N in a Hilbert space \mathfrak{S} , then there exists a unique linear map W of \mathfrak{M} onto itself satisfying*

- (i) $W^2 = I$,
- (ii) $\|\phi\|^2 + \|\bar{N}^*\phi\|^2 = \|W\phi\|^2 + \|\bar{N}^*W\phi\|^2$, ($\phi \in \mathfrak{M}$),
- (iii) $(I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M}$,
- (iv) $\|\bar{N}^*(I - W)\phi\| = \|N^*(I - W)\phi\|$, ($\phi \in \mathfrak{M}$).

In terms of W we have

$$(1) \quad \mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}, \quad N_1f = \bar{N}^*f, \quad (f \in \mathfrak{D}(N_1)).$$

Conversely, if W is any linear map of \mathfrak{M} onto \mathfrak{M} satisfying (i)—(iv) above, then the operator N_1 defined by (1) is a normal extension of N in \mathfrak{S} .

REMARKS. Condition (i) implies that $P_1 = (1/2)(I + W)$ and $P_2 = (1/2)(I - W)$ are projections (not necessarily orthogonal) in \mathfrak{M} , and \mathfrak{M} is the direct sum of $\mathfrak{M}_1 = P_1\mathfrak{M}$ and $\mathfrak{M}_2 = P_2\mathfrak{M}$. If $\phi \in \mathfrak{M}$, then $\phi \in \mathfrak{M}_1$ if and only if $W\phi = \phi$, and $\phi \in \mathfrak{M}_2$ if and only if $W\phi = -\phi$.

Condition (ii) implies that if $\phi, \phi' \in \mathfrak{M}$ then

$$(\phi, \phi') + (\bar{N}^*\phi, \bar{N}^*\phi') = (W\phi, W\phi') + (\bar{N}^*W\phi, \bar{N}^*W\phi').$$

If $\phi \in \mathfrak{M}_1, \phi' \in \mathfrak{M}_2$ we see that $(\phi, \phi') + (\bar{N}^*\phi, \bar{N}^*\phi') = 0$, which means that the graph of \bar{N}^* restricted to \mathfrak{M}_1 is orthogonal to the graph of \bar{N}^* restricted to \mathfrak{M}_2 .

Since \bar{N}^* is one-to-one from \mathfrak{M} onto $\bar{\mathfrak{M}}$, condition (iii) implies that $\mathfrak{M}_2 = \bar{N}^*\mathfrak{M}_1 \subset \mathfrak{M} \cap \bar{\mathfrak{M}}$, and \mathfrak{M}_2 has the same algebraic dimension as \mathfrak{M}_1 . In particular the dimension of \mathfrak{M} must be even.

Proof of Theorem 2. Let N_1 be a normal extension of the formally normal operator N in \mathfrak{S} . Then we have $N \subset N_1 \subset \bar{N}^*, \bar{N} \subset N_1^* \subset N^*$. Let the operator \mathcal{N}_1 in \mathfrak{S}_2 be defined with domain all $\{h_1, h_2\}$ such that $h_1 \in \mathfrak{D}(N_1), h_2 \in \mathfrak{D}(N_1^*)$, and so that $\mathcal{N}_1\{h_1, h_2\} = \{N_1^*h_2, N_1h_1\}$. Then it is easily seen that \mathcal{N}_1 is a self-adjoint extension of the operator \mathcal{N} defined in the proof of Theorem 1.

Let \mathcal{N}_1 be any self-adjoint extension of \mathcal{N} , and let \mathcal{V} be the unique isometry of $\mathfrak{E}(-i)$ onto $\mathfrak{E}(+i)$ such that $\mathfrak{D}(\mathcal{N}_1) = \mathfrak{D}(\mathcal{N}) + (\mathcal{I} - \mathcal{V})\mathfrak{E}(-i)$. Then we may write $\mathcal{V} = \mathcal{W}\mathcal{U}$, where \mathcal{U} is the isometry defined on $\mathfrak{E}(-i)$ to $\mathfrak{E}(+i)$ by $\mathcal{U}\{\psi_1, \psi_2\} = \{\psi_1, -\psi_2\}$, and \mathcal{W} is a unitary map of $\mathfrak{E}(+i)$ onto itself. For $\{\phi_1, \phi_2\} \in \mathfrak{E}(+i)$ let $\mathcal{W}\{\phi_1, \phi_2\} = \{\chi_1, \chi_2\}$. Then $\phi_1, \chi_1 \in \mathfrak{M}$ and $\phi_2 = -i\bar{N}^*\phi_1, \chi_2 = -i\bar{N}^*\chi_1$. Define the map W of \mathfrak{M} into \mathfrak{M} by $W\phi_1 = \chi_1$. Then W is linear, and since \mathcal{W} is unitary, W is onto, and

$$\| \{\phi, -i\bar{N}^*\phi\} \|^2 = \| \{W\phi, -i\bar{N}^*W\phi\} \|^2, \quad (\phi \in \mathfrak{M}),$$

or

$$(2) \quad \| \phi \|^2 + \| \bar{N}^*\phi \|^2 = \| W\phi \|^2 + \| \bar{N}^*W\phi \|^2, \quad (\phi \in \mathfrak{M}).$$

Conversely, suppose W is a linear map of \mathfrak{M} onto \mathfrak{M} satisfying (2). Then for $\hat{\phi} = \{\phi, -i\bar{N}^*\phi\} \in \mathfrak{E}(+i)$ define $\mathcal{W}\hat{\phi} = \{W\phi, -i\bar{N}^*W\phi\}$. Then \mathcal{W} maps $\mathfrak{E}(+i)$ onto $\mathfrak{E}(+i)$ and (2) implies that \mathcal{W} is unitary. Thus we see that the self-adjoint extensions \mathcal{N}_1 of \mathcal{N} are in a one-to-one correspondence with the linear maps W of \mathfrak{M} onto \mathfrak{M} satisfying (2). We have $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$ if and only if \hat{h} can be represented in

the form $\hat{h} = \hat{f} + (\mathcal{I} - \mathcal{W}\mathcal{U})\hat{\psi}$, where $\hat{f} = \{f_1, f_2\} \in \mathfrak{D}(\mathcal{N})$, $\hat{\psi} = \{\phi, i\bar{N}^*\phi\} \in \mathfrak{C}(-i)$. This means $h_1 = f_1 + (I - W)\phi$, $h_2 = f_2 + i\bar{N}^*(I + W)\phi$, where $f_1 \in \mathfrak{D}(\mathcal{N})$, $f_2 \in \mathfrak{D}(\bar{N})$, $\phi \in \mathfrak{M}$.

The self-adjoint extension \mathcal{N}_1 arising from the normal extension N_1 of N has the property that if $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$ then so does $\mathcal{P}_1\hat{h} = \{h_1, 0\}$. It will now be shown that a self-adjoint extension \mathcal{N}_1 of \mathcal{N} has this property if and only if the W corresponding to \mathcal{N}_1 satisfies $W^2 = I$. First suppose $\mathcal{P}_1\hat{h} \in \mathfrak{D}(\mathcal{N}_1)$ for all $\hat{h} \in \mathfrak{D}(\mathcal{N}_1)$. Letting $h_1 = f_1 + (I - W)\phi$, $h_2 = f_2 + i\bar{N}^*(I + W)\phi$ as above, we see that this implies that there exist elements $f'_1 \in \mathfrak{D}(N)$, $f'_2 \in \mathfrak{D}(\bar{N})$, $\phi' \in \mathfrak{M}$, such that

$$\begin{aligned} f_1 + (I - W)\phi &= f'_1 + (I - W)\phi' , \\ 0 &= f'_2 + i\bar{N}^*(I + W)\phi' . \end{aligned}$$

Since $\mathfrak{D}(N) + \mathfrak{M}$ and $\mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$ are direct sums these equations imply that $f_1 = f'_1$, $(I - W)\phi = (I - W)\phi'$, $f'_2 = 0$, and $\bar{N}^*(I + W)\phi' = 0$. The last equation implies $(I + W)\phi' = 0$ since \bar{N}^* is one-to-one from \mathfrak{M} to $\bar{\mathfrak{M}}$. Thus we have

$$(3) \quad \begin{aligned} \phi' + W\phi' &= 0 , \\ \phi' - W\phi' &= \phi - W\phi , \end{aligned}$$

from which results $2\phi' = (I - W)\phi$. Returning to the first equation in (3) we obtain $(I + W)(I - W)\phi = (I - W^2)\phi = 0$ for all $\phi \in \mathfrak{M}$, showing that $W^2 = I$. Conversely, suppose $W^2 = I$ on \mathfrak{M} . Then if $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$, $h_1 = f_1 + (I - W)\phi$, $h_2 = f_2 + i\bar{N}^*(I + W)\phi$, define $\phi' = (1/2)(I - W)\phi$. Then equations (3) will be valid, implying that

$$\begin{aligned} f_1 + (I - W)\phi &= f_1 + (I - W)\phi' , \\ 0 &= 0 + i\bar{N}^*(I + W)\phi' , \end{aligned}$$

which shows that $\mathcal{P}_1\hat{h} = \{h_1, 0\} \in \mathfrak{D}(\mathcal{N}_1)$.

If \mathcal{N}_1 is any self-adjoint extension of \mathcal{N} for which $W^2 = I$, then $\mathfrak{D}(\mathcal{N}_1)$ consists of those $\{h_1, h_2\}$ such that $h_1 = f_1 + (I - W)\phi$, $h_2 = f_2 + i\bar{N}^*(I + W)\phi'$, for some $f_1 \in \mathfrak{D}(N)$, $f_2 \in \mathfrak{D}(\bar{N})$, and $\phi, \phi' \in \mathfrak{M}$. The point is that ϕ and ϕ' need not now be the same element. Indeed, if h_1, h_2 have such representations let $\phi'' = (1/2)(I - W)\phi + (1/2)(I + W)\phi'$. Then $(I - W)\phi = (I - W)\phi''$, and $(I + W)\phi' = (I + W)\phi''$, which implies that $\{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$. For such an \mathcal{N}_1 define N_1 to be the operator in \mathfrak{G} with $\mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}$, and $N_1h_1 = \bar{N}^*h_1$ for $h_1 \in \mathfrak{D}(N_1)$. Similarly define N_2 on $\mathfrak{D}(N_2) = \mathfrak{D}(\bar{N}) + \bar{N}^*(I + W)\mathfrak{M}$ by $N_2h_2 = N^*h_2$ for $h_2 \in \mathfrak{D}(N_2)$. In terms of N_1 and N_2 we have $\{h_1, h_2\} \in \mathfrak{D}(\mathcal{N}_1)$ if and only if $h_1 \in \mathfrak{D}(N_1)$, $h_2 \in \mathfrak{D}(N_2)$, and $\mathcal{N}_1\{h_1, h_2\} = \{N_2h_2, N_1h_1\}$. A short computation shows that $\mathfrak{D}(\mathcal{N}_1^*)$ is the set of all $\{g_1, g_2\}$ such that $g_1 \in \mathfrak{D}(N_2^*)$,

$g_2 \in \mathfrak{D}(N_1^*)$, and $\mathcal{N}_1^*\{g_1, g_2\} = \{N_1^*g_2, N_2^*g_1\}$. But since $\mathcal{N}_1 = \mathcal{N}_1^*$ we obtain $N_2 = N_1^*$. Hence $\mathfrak{D}(\mathcal{N}_1)$ consists of all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(N_1)$, $h_2 \in \mathfrak{D}(N_1^*)$, and $\mathcal{N}_1\{h_1, h_2\} = \{N_1^*h_2, N_1h_1\}$. Here

$$(4) \quad \begin{aligned} \mathfrak{D}(N_1) &= \mathfrak{D}(N) + (I - W)\mathfrak{M}, \\ \mathfrak{D}(N_1^*) &= \mathfrak{D}(\bar{N}) + \bar{N}^*(I + W)\mathfrak{M}, \end{aligned}$$

and $N \subset N_1 \subset \bar{N}^*$, $\bar{N} \subset N_1^* \subset N^*$. Thus any self-adjoint extension \mathcal{N}_1 of \mathcal{N} having the property that $W^2 = I$ determines a unique operator N_1 in \mathfrak{H} as above, which is easily seen to be closed. In particular, if N_1 is a normal extension of N , then the equalities (4) hold.

It remains to characterize those \mathcal{N}_1 such that $W^2 = I$ for which N_1 is normal, that is $\mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$ and $\|N_1h\| = \|N_1^*h\|$, $h \in \mathfrak{D}(N_1)$. We claim that this is true if and only if

$$(5) \quad (I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M},$$

and

$$(6) \quad \|\bar{N}^*(I - W)\phi\| = \|N^*(I - W)\phi\|, \quad (\phi \in \mathfrak{M}).$$

If (5) is valid then (4) implies that $\mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$, since $\mathfrak{D}(N) = \mathfrak{D}(\bar{N})$. Let $h \in \mathfrak{D}(N_1)$, $h = f + (I - W)\phi$, $f \in \mathfrak{D}(N)$, $\phi \in \mathfrak{M}$. Then $(I - W)\phi \in \mathfrak{M} \cap \bar{\mathfrak{M}}$, and we have $N_1h = Nf + \bar{N}^*(I - W)\phi$, $N_1^*h = \bar{N}f + N^*(I - W)\phi$. Thus

$$\begin{aligned} \|N_1h\|^2 &= \|Nf\|^2 + (Nf, \bar{N}^*(I - W)\phi) + (\bar{N}^*(I - W)\phi, Nf) \\ &\quad + \|\bar{N}^*(I - W)\phi\|^2, \end{aligned}$$

and

$$\begin{aligned} \|N_1^*h\|^2 &= \|\bar{N}f\|^2 + (\bar{N}f, N^*(I - W)\phi) + (N^*(I - W)\phi, \bar{N}f) \\ &\quad + \|N^*(I - W)\phi\|^2. \end{aligned}$$

Since N is formally normal $\|Nf\| = \|\bar{N}f\|$. Moreover $\bar{N}^*(I - W)\phi \in \bar{\mathfrak{M}}$ implies that $(Nf, \bar{N}^*(I - W)\phi) = (f, N^*\bar{N}^*(I - W)\phi) = -(f, (I - W)\phi)$, and similarly $(\bar{N}f, N^*(I - W)\phi) = -(f, (I - W)\phi)$. Using (6) we see that $\|N_1h\| = \|N_1^*h\|$ for all $h \in \mathfrak{D}(N_1)$, proving that N_1 is normal.

Conversely, suppose N_1 is normal. Then (6) is clearly valid, for $(I - W)\phi \in \mathfrak{D}(N_1)$ by (4). Suppose $h \in \mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$ and $h = f + (I - W)\phi = f' + \bar{N}^*(I + W)\phi'$ with $f, f' \in \mathfrak{D}(N)$, $\phi, \phi' \in \mathfrak{M}$. We show that $f = f'$ and $(I - W)\phi = \bar{N}^*(I + W)\phi'$. Applying this to $f = 0$ we obtain $(I - W)\mathfrak{M} \subset \bar{N}^*(I + W)\mathfrak{M}$, and with $f' = 0$ we get $\bar{N}^*(I + W)\mathfrak{M} \subset (I - W)\mathfrak{M}$, proving (5). Now for any $g \in \mathfrak{D}(N)$ we have $(N_1h, N_1g) = (N_1^*h, N_1^*g)$, or

$$(Nf, Ng) + (\bar{N}^*(I - W)\phi, Ng) = (\bar{N}f', \bar{N}g) - ((I + W)\phi', \bar{N}g).$$

Since $(\bar{N}f', \bar{N}g) = (Nf', Ng)$ and $(\bar{N}^*(I - W)\phi, Ng) = -((I - W)\phi, g)$, this yields

$$(Nf, Ng) - ((I - W)\phi, g) = (Nf', Ng) - (\bar{N}^*(I + W)\phi', g) ,$$

or

$$(N(f - f'), Ng) + (\bar{N}^*(I + W)\phi' - (I - W)\phi, g) = 0 .$$

But $\bar{N}^*(I + W)\phi' - (I - W)\phi = f - f'$, and hence

$$(N(f - f'), Ng) + (f - f', g) = 0$$

for all $g \in \mathfrak{D}(N)$. Letting $g = f - f'$ we obtain $f = f'$ as desired. This completes the proof of Theorem 2.

4. Abstract boundary conditions. For $u \in \mathfrak{D}(\bar{N}^*)$, $v \in \mathfrak{D}(N^*)$ define $\langle uv \rangle = (\bar{N}^*u, v) - (u, N^*v)$.

THEOREM 3. *If N_1 is a normal extension of the formally normal operator N such that $\mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}$, then $\mathfrak{D}(N_1)$ may be described as the set of all $u \in \mathfrak{D}(\bar{N}^*)$ satisfying $\langle u\alpha \rangle = 0$ for all $\alpha \in (I - W)\mathfrak{M}$.⁶*

REMARK. For differential operators the conditions $\langle u\alpha \rangle = 0$ become boundary conditions. They are self-adjoint ones, that is, $\langle \alpha\alpha' \rangle = 0$ for all $\alpha, \alpha' \in (I - W)\mathfrak{M}$. Indeed $\alpha, \alpha' \in \mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$ and for any $\alpha \in \mathfrak{D}(N_1)$, $\alpha' \in \mathfrak{D}(N_1^*)$ we have $(\bar{N}^*\alpha, \alpha') = (N_1\alpha, \alpha') = (\alpha, N_1^*\alpha') = (\alpha, N^*\alpha')$.

Proof of Theorem 3. If $u \in \mathfrak{D}(N_1)$, $\alpha \in (I - W)\mathfrak{M} \subset \mathfrak{D}(N_1^*)$, the above argument shows that $\langle u\alpha \rangle = 0$. Conversely suppose $u \in \mathfrak{D}(\bar{N}^*)$ and $\langle u\alpha \rangle = 0$ for all $\alpha \in (I - W)\mathfrak{M}$. Let $u = f + (I - W)\phi + (I + W)\phi$, where $f \in \mathfrak{D}(N)$, $\phi \in \mathfrak{M}$. We note that $\langle \ \rangle$ is linear in the first spot, and $f + (I - W)\phi \in \mathfrak{D}(N_1)$. Thus $\langle (I + W)\phi\alpha \rangle = 0$ for all $\alpha \in (I - W)\mathfrak{M}$. Let $\alpha = \bar{N}^*(I + W)\phi \in (I - W)\mathfrak{M}$, since $(I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M}$. Then

$$0 = \langle (I + W)\phi \bar{N}^*(I + W)\phi \rangle = (\bar{N}^*(I + W)\phi, \bar{N}^*(I + W)\phi) + ((I + W)\phi, (I + W)\phi) ,$$

which proves that $(I + W)\phi = 0$, and hence $u \in \mathfrak{D}(N_1)$ as desired.

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⁶ A result similar to Theorem 3 appears in the report by Davis (4) for the case when $\dim (\mathfrak{D}(\bar{N}^*)/\mathfrak{D}(N)) < \infty$.

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