SOME CLASSES OF EQUIVALENT GAUSSIAN PROCESSES ON AN INTERVAL

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1. Introduction. Let $T$ be an index set, $R, S$ real-valued nonnegative definite functions of two variables in $T$, and $m, n$ real-valued functions on $T$. Let $\Omega$ be the set of all real-valued functions on $T$, and $\mathcal{F}$ the Borel field of cylinder sets. There are then unique measures $\mu, \nu$ on $\mathcal{F}$ such that the functions $x_t$ on $\Omega$ defined by $x_t(\omega) = \omega(t)$ form Gaussian stochastic processes, with means respectively $m$ and $n$, and covariances respectively $R$ and $S$. It is shown in [2] that $\mu$ and $\nu$ are either mutually absolutely continuous or totally singular, and a necessary and sufficient condition for equivalence is given.

Suppose now that $T$ is a subset of the real line, and $R(s, t) = \iota(s - t)$, $S(s, t) = \varsigma(s - t)$, where $\iota$ and $\varsigma$ are continuous nonnegative-definite functions, and hence can be written as inverse Fourier transforms of finite measures $d\rho, d\sigma$. Thus, using respectively the measures $\mu$ and $\nu$ on $\Omega$, $x_t - m(t)$ and $x_t - n(t)$ are the restrictions to $T$ of stationary Gaussian processes on the real line. For simplicity, only the case $m = n = 0$ will be considered.

When $T$ is the entire real line, then it is easy to see, by looking at $d\rho$ and $d\sigma$, exactly when $\mu \sim \nu$, as is essentially known (see [3]). The precise conditions are:

a. $\rho$ and $\sigma$ must have identical non-atomic parts.

b. Their points of positive mass be the same, and if the masses are $a_i$ and $b_i$ at $x_i$, then $\sum [(a_i/b_i) - 1]^2$ must be finite.

Now suppose $T$ is a finite interval. The problem of determining from knowledge of $\rho$ and $\sigma$ whether $\mu$ and $\nu$ are equivalent becomes much more difficult. We here discuss only a certain class of cases. Because of stationarity, one need only consider an interval symmetric about 0. Continuity of $\iota$ and $\varsigma$ implies that the Gaussian process is continuous with probability one at any given point, so that it makes no difference whether the interval is open or closed. There is no essential loss of generality, then, in considering only the closed interval $[-\pi, \pi]$. The following facts will then be proven:

**Theorem.** Let $d\rho(x) = \{dx\}(1 + x^2)^u$, where $u$ is an integer $\geq 1$, and let $d\sigma$ be some other finite nonnegative measure on the real line. Write $\tau = \sigma - \rho$. The following conditions are necessary and sufficient that the Gaussian processes induced on $[-\pi, \pi]$ by the Fourier trans-

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forms of $\rho$ and $\sigma$ have equivalent measures on path space:

(a) if $k_n$ is a sequence of $C_\alpha$ functions with support in $]-\pi, \pi[$ and $K_n$ is the Fourier transform of $k_n$, then $\int |K_n|^2 d\sigma \to 0$ implies $\int |K_n|^2 d\rho \to 0$.

(b) The Fourier transform (in the sense of Schwartz distributions) of $(1 + x^2)\sigma(x)$ agrees on $]-2\pi, 2\pi[$ with a function $\psi$ such that

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\psi(s-t)|^2 ds dt < \infty.$$

Remark 1. It will be seen that sufficiency still holds if (a) is weakened to:

(a') $\int |K_n|^2 d\sigma \to 0$ and $K_n \to K$ in $\mathcal{L}_2(\rho)$ implies that $K = 0$ on some set of positive $\rho$-measure.

Remark 2. As a consequence of Remark 1, it is clear that if $\sigma$ has a component which is absolutely continuous with respect to $\rho$, then Condition (a) automatically satisfied.

Retaining the notation of the theorem:

Corollary 1. If $d\sigma = \Phi d\rho$, where $\Phi$ is a function such that $\Phi - 1$ is a finite linear combination of functions in various $L_a(-\infty, \infty)$ classes, $1 \leq a \leq 2$, then the Gaussian processes induced by $\rho$ and $\sigma$ have equivalent measures on path space.

One direction of the following corollary was proven by D. Slepian in [5], using techniques of G. Baxter in [1]:

Corollary 2. If $A_j$ and $B_j$ are polynomials, with degrees respectively $a_j$ and $b_j$, $j = 1, 2$, and $b_j > a_j$, then the Gaussian processes whose spectral measures are $|A_j(x)|^2 B_j(x)^2 dx$ have equivalent measures on path space if and only if

(a) $b_1 - a_1 = b_2 - a_2$

(b) the ratio of the leading coefficients of $A_1$ and $B_1$ has the same absolute value as the ratio of the leading coefficients of $A_2$ and $B_2$.

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2. Some preliminaries on functions of exponential type. First, some notation. Functions will be complex-valued functions of a real variable, unless otherwise stated. $\hat{F}$ will mean the Fourier transform of $F$ (in various degrees of generalization, depending on context), and $\hat{F}^*$ the conjugate Fourier transform. $\text{sup}(f)$ will mean the points where $f \neq 0$. $\mathcal{E}_a = \{F| F$ extends to an entire function of exponential type $\leq a\pi\}$. $\mathcal{H}_a = \mathcal{E}_a \cap \mathcal{L}_2(-\infty, \infty)$, or, by the Payley-Wiener theorem,
u will be a fixed integer $\geq 1$, and $p(x) = (i + x)^u$. $\rho$ is the measure $d\rho(x) = \{1/|p(x)|^u\}dx$. $\mathcal{H}$ will denote the completion of $\mathcal{D}$ in the inner product $\langle F, G \rangle = \int F \bar{G} d\rho$.

Naturally, $\mathcal{H}$ really consists of equivalence classes of functions; but it will turn out that there is a continuous, in fact entire, member in each class. $H_t$ will denote a fixed function of $\mathcal{D}$ such that $h_t = \hat{H}_t$ is nonnegative and has integral 1. For $a > 0$, $h_a(s)$ will be $(1/a)h_t(s/a)$, $H_a(x) = H_t(ax)$, so that $h_a = \hat{H}_a$, and $H_a \in \mathcal{D}_a$. Then $H_a$ vanishes faster than any polynomial, $|H_a(x)| \leq 1$ for all $x$, and $\lim_{a \to 0} H_a(x) = 1$ uniformly on any finite interval.

**Lemma 1.** If $F \in \mathcal{D}_1$ and $\int |F|^2 d\rho < \infty$, then $F \in \mathcal{H}$.

**Proof.** If $(1/2) < c < 1$, then

$$\left( \int |F(cx) - F(x)|^2 d\rho(x) \right)^{1/2} \leq \left( \int_{-b}^b |F(cx) - F(x)|^2 d\rho(x) \right)^{1/2} + \left( \int_{|x| > b} |F(cx)|^2 d\rho(x) \right)^{1/2} + \left( \int_{|x| > b} |F(x)|^2 d\rho(x) \right)^{1/2}.$$

Now,

$$\int_{|x| > b} |F(cx)|^2 d\rho(x) = \frac{1}{c} \int_{|x| > bc} |F(x)|^2 \frac{1}{|p(x)|^2} dx \leq 2 \int_{|x| < (1/2)} |F(x)|^2 \frac{1}{|p(x)|^2} dx.$$

Choosing $b$ large, and then choosing $c$ close enough to 1 to make $|F(cx) - F(x)|$ small on $[-b, b]$, we see that it suffices to show that the function $G: x \to F(cx)$ is in $\mathcal{H}$. Notice that $G \in \mathcal{D}_1$, as $c < 1$.

$H_aG$ is square-integrable, since $H_a$ vanishes faster than $(1/|p|^2)$. So $H_aG$ is in $\mathcal{H}_{a+c}$, its Fourier transform being some $g'$ in $\mathcal{L}_2(-\infty, \infty)$ with support in $[-(a+c)\pi, (a+c)\pi]$. Thus $h_a g' \in \mathcal{D}_{a+c}$, and $H_a^2 G \in \mathcal{D}_{2a+c}$. Choosing $a$ small causes $H_a^2 G$ to be in $\mathcal{D}_1$, and simultaneously causes $\int |H_a^2 G - G|^2 d\rho$ to get small. This proves the lemma.

Let $\mathcal{H} = \{pF | F \in \mathcal{H}_c \}$, and $\mathcal{D} = \{pF | F \in \mathcal{D}_1 \}$. Lemma 1 tells us $\mathcal{H} \subset \mathcal{H}$.

**Lemma 2.** $\mathcal{H}$ is precisely the closure of $\mathcal{D}$ in $\mathcal{H}$.
Proof. First, we see that $\mathcal{H}$ is closed. If $F_n \in \mathcal{H}_1$ and
\[
\left| \int p F_n - G \right|^2 d\rho \to 0,
\]
then $\left| \int F_n(x) - F_m(x) \right|^2 dx \to 0$.

Since $\mathcal{H}_1$ is complete, there is some $F \in \mathcal{H}_1$ with
\[
\int \left| F_n(x) - F(x) \right|^2 dx \to 0.
\]
So some subsequence of the $p F_n$ converges almost everywhere to $p F$. Thus $p F = G$ almost everywhere.

To approximate elements $p F$ in $\mathcal{H}$ by elements in $\mathcal{D}$, just approximate $F$ in $\mathcal{L}_2(-\infty, \infty)$ by elements in $\mathcal{D}_1$, using the technique of Lemma 1.

**Lemma 3.** $\mathcal{H} \subseteq \mathcal{H}$ is precisely the finite-dimensional space $\mathcal{L}$ of functions of the form $x \rightarrow e^{i\pi q(i - x)}$, where $q$ is a polynomial of degree $\leq u - 1$.

**Proof.** Suppose $F \in \mathcal{H} \subseteq \mathcal{H}$. Then $\int \{F(x)/p(x)\} G(x) dx = 0$ for all $G$ in $\mathcal{D}_1$. Now, $(F/p)$ is in $\mathcal{L}_1(-\infty, \infty)$, so it has a Fourier transform $k$ which is likewise square-integrable, and, by Plancherel's theorem, $\int k(s)\overline{g(s)} ds = 0$ for all $g$ in $\mathcal{D}_1$. So $k$ vanishes in $]-\pi, \pi[$.

Since $F \in \mathcal{H}$, $F$ can be approximated in $\mathcal{H}$ by functions $F_n$ in $\mathcal{D}_1$. Each $F_n$ is in $\mathcal{D}_a$ for some $a_n < 1$, since $\text{sup}(F_n) \subset [-\pi, \pi[$, and hence $\subset [-a_n \pi, a_n \pi[$ for some $a_n < 1$. Let $k_n$ be the Fourier transform of $F_n/p$. Then $k_n \rightarrow k$ in $\mathcal{L}_2(-\infty, \infty)$, and $k_n$ is in the domain of the $\mathcal{L}_2$-differential operator $p(-iD) = i^u(I-D)^u$. So $p(-iD)k_n = f_n$, where $f_n$ is the Fourier transform of $F_n$. Since $f_n$ vanishes outside some $[-a_n \pi, a_n \pi[$, $k_n$ must be of the form $\sum_j a_j s^j e^s$ in $]-\infty, -\pi[$ and $\sum_j b_j s^j e^s$ in $[\pi, \infty[$, where $j$ ranges between 0 and $u - 1$. Since $k_n$ is in $\mathcal{L}_2(-\infty, \infty)$, the $b_j$ are zero, and, letting $\varphi$ be the indicator of $]-\infty, -\pi[$, we get $\varphi k_n = \varphi \sum a_j s^j e^s$. This converges in $\mathcal{L}_1(-\infty, \infty)$, so the limit is of the form $\varphi \sum a_j s^j e^s$. Then $k_n \rightarrow 0$ in $\{\pi, \infty[, 0$ in $[-\pi, \pi[,$ and $\sum a_j s^j e^s$ in $]-\infty, -\pi[$, so $k = \varphi \sum a_j s^j e^s$. $F/p$ is then a linear combination of terms like $\int_0^\infty e^{-ix^2} s^j e^s ds$, $0 \leq j \leq u - 1$, which is a linear combination of terms like $e^{ix(i + x)^{-1}}$, $1 \leq j \leq u$. Multiplying by $p$ gives the result.

Combining information from lemmas 1, 2, 3 we get a description of $\mathcal{H}$:

**Proposition.** $\mathcal{H}$ is the orthogonal direct sum of $\mathcal{H}$ and $\mathcal{L}$.

**Lemma 4.** $\mathcal{D} = \mathcal{H} \cap \mathcal{D}_1$.

**Proof.** $\mathcal{D} \subseteq \mathcal{H}$, by definition, since $\mathcal{D}_1 \subset \mathcal{H}_1$. Also $\mathcal{D} \subset \mathcal{D}_1$, since $\mathcal{D}_1$ is closed under multiplication by polynomials (because $\mathcal{D}_1$ is...
closed under differentiation). So \( \mathcal{D} \subset \mathcal{H} \cap \mathcal{D}_1 \), and it remains to show \( \mathcal{D} \supset \mathcal{H} \cap \mathcal{D}_1 \).

Suppose \( G \in \mathcal{H} \). Then \( G \) is a \( \langle \, , \rangle \) limit of elements \( G_n \) in \( \mathcal{D} \), by Lemma 2. \( G_n \) then has the form \( pF_n, F_n \) in \( \mathcal{D}_1 \). Thus \( F_n \) is an \( \mathcal{L}_2(-\infty, \infty) \) Cauchy sequence, hence has a limit \( F \). Then \( pF = G \).

Suppose \( G \) is also in \( \mathcal{D}_1 \). Then \( G \) is infinitely differentiable. Since \( \hat{G} = p\hat{F} = p(-iD)\hat{F} \), we conclude that \( \hat{F} \) is infinitely differentiable. Now it must be shown that \( \hat{F} \) vanishes outside some interval \([ -a\pi, a\pi ] \), \( 0 < a < 1 \). But \( \hat{G} = p(-iD)\hat{F} \) vanishes outside such an interval, so \( \hat{F} \) is analytic outside \([ -a\pi, a\pi ] \). Also, \( \hat{F} \) vanishes outside \([ -\pi, \pi ] \), since each \( \hat{F}_n \) has support in \([ -\pi, \pi ] \). Therefore, \( \hat{F} \) vanishes outside \([ -a\pi, a\pi ] \). So \( \hat{F} \) is in \( \mathcal{D}_1 \), and \( F \) is in \( \mathcal{D}_1 \).

Lemma 5. \( \mathcal{D}_1 / \mathcal{D} \) is finite dimensional.

Proof. \( \mathcal{D}_1 / \mathcal{D} = \mathcal{D}_1 / \mathcal{D} \cap \mathcal{H} \approx (\mathcal{D} + \mathcal{H}) / \mathcal{H} \subset \mathcal{H} / \mathcal{H} \approx \mathcal{D} \).

3. Proof of theorem. In [2] it is shown that a necessary and sufficient condition for equivalence of \( \mu \) and \( \nu \) is that there be an equivalence operator from the closed linear span of \( \{ x_t \mid t \in T \} \) in \( \mathcal{L}_2(\mu) \) to their closed linear span in \( \mathcal{L}_2(\nu) \), sending the \( \mu \)-equivalence class of \( x_t \) to the \( \nu \)-equivalence class of \( x_t \) for each \( t \in T \). (An equivalence operator, as defined in [2], is a linear homeomorphism \( H \) between two Hilbert spaces such that \( I - H^*H \) is Hilbert Schmidt). Actually, we shall want the condition in complex \( \mathcal{L}_2 \), while the proof in [2] is for real \( \mathcal{L}_2 \); however, the transition from the one to the other is immediate.

Under this condition, \( H \) would map \( \int_{-\pi}^{\pi} f(x_t)dt \) as an \( \mathcal{L}_2(\mu) \)-valued integral to \( \int_{-\pi}^{\pi} f(t)x_t dt \) as an \( \mathcal{L}_2(\nu) \)-valued integral, for each \( f \in \mathcal{D}_2 \); and conversely, if \( H \) had this effect on all such \( \int_{-\pi}^{\pi} f(t)x_t dt \), then by choosing a sequence of \( f \) approximating a delta function, one could verify that \( H \) sent the equivalence class of \( x_t \) in \( \mathcal{L}_2(\mu) \) to the equivalence class of \( x_t \) in \( \mathcal{L}_2(\nu) \). Therefore, putting inner products \( ( \, , \) \) and \( ( \, , \gamma \) on \( \mathcal{D}_1 \), by the rules

\[
(f, g) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (s - t)f(s)g(t)dsdt,
\]

\[
(f, g)\gamma = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (s - t)f(s)g(t)dsdt,
\]

and noting that \( (f, g) = \int (\int_{-\pi}^{\pi} f(s)x_s ds)(\int_{-\pi}^{\pi} g(t)x_t dt) d\mu \) and

\[
(f, g)\gamma = \int (\int_{-\pi}^{\pi} f(s)x_s ds)(\int_{-\pi}^{\pi} g(t)x_t dt) dv,
\]

it follows that a necessary and sufficient condition for the equivalence of \( \mu \) and \( \nu \) is the existence of an equivalence operator from the \( ( \, , \) com-
pletion of $\mathcal{D}_1$ to its $(\cdot, \cdot)^*$ completion, and sending the $(\cdot, \cdot)$-equivalence class of $f$ to its $(\cdot, \cdot)^*$-equivalence class.

Now let $\langle F, G' \rangle = \int \overline{F} \overline{G} d\sigma$, where $F$ and $G$ are in $\mathcal{D}_1$ (and hence continuous and bounded, so that the integral exists). Let $\mathcal{K}$ be the closure of $\mathcal{D}_1$ in $L_2(\sigma)$. Let $J$ be the map assigning to $F$ in $\mathcal{K}$ its equivalence class in $\mathcal{D}_1$. Since $\langle F, G' \rangle = (\overline{F}, \overline{G})$, and $\langle F, G' \rangle = (\overline{F}, \overline{G})^*$, the necessary and sufficient condition for the equivalence of $\mu$ and $\nu$ in the theorem is that $J$ be the restriction to $\mathcal{D}_1$ of an equivalence map from $\mathcal{K}$ to $\mathcal{K}$.

To prove sufficiency of the conditions in the theorem, suppose first that $p(x)^2 d\tau(x)$ has a generalized Fourier transform (see [4]) which agrees on $[-2\pi, 2\pi]$ with a function $\psi$ such that $\int \int |\psi(s-t)|^2 dsdt = a^2 < \infty$. We extend $\psi$ by making it 0 outside $[-2\pi, 2\pi]$.

**Lemma 6.** If $F \in \mathcal{D}$, then $\langle F, F' \rangle \leq (1 + a) \langle F, F \rangle$.

**Proof.** Write $F = pG, G \in \mathcal{D}_1$. Then $\int |F|^2 d\sigma = \int |F|^2 d\rho + \int |G|^2 |p|^2 d\tau$. Now, $\tilde{G}$ is in $\mathcal{D}_1$, so $\tilde{G} \ast \tilde{G}$ is infinitely differentiable with support in $[-2\pi, 2\pi]$. Then, by Schwartz's definition of generalized Fourier transform, we get $\int |G|^2 |p|^2 d\tau = \int |\tilde{G} \ast \tilde{G}(s)\psi(s)| ds = \int \int \tilde{G}(s-t)\tilde{G}(1-t)|\psi(s-t)|^2 dsdt$, where $a(s) = \max(-\pi, s - \pi)$ and $b(s) = \min(\pi, s + \pi)$. Letting $s - t = s'$ and $t = - t'$ gives $\int \int \tilde{G}(s')\tilde{G}(t')|\psi(s'-t')| ds'dt'$, whose absolute value, by the Schwartz inequality, is

$$\leq \left[ \int \int \tilde{G}(s)|\psi(s)|^2 ds \right]^{1/2} \left[ \int \int |\psi(s-t)|^2 dsdt \right]^{1/2} = \left( \int |F|^2 d\rho \right) a = \left( \int |F|^2 d\tau \right) a.$$

Pick a complete orthonormal set (c.o.n.s.) $f_1, f_2, \cdots$ for $L_2(-\pi, \pi)$ out of the dense subset $\mathcal{D}_1$. Let $F_n = \hat{f}_n$, and $G_n = pF_n$. Then the $G_n$ form a c.o.n.s. for $\mathcal{K}$ (in $\langle \cdot, \cdot \rangle$) consisting of elements of $\mathcal{D}$, because the $F_n$ are a c.o.n.s. for $\mathcal{K}$ consisting of elements of $\mathcal{D}$.

**Lemma 7.** $\sum_{n=1}^\infty |\langle G_n, G_m \rangle - \langle G_n, G_m \rangle^*|^2 = a^2$.

**Proof.** $\int G_n(x)\overline{G}_m(x) d\tau(x) = \int f_n(s)\overline{f}_m(t)\psi(s-t) dsdt$. By using a change of variable as in the previous lemma, this equals $\int f_n(s)\overline{f}_m(t)\psi(s-t) dsdt$. But the functions $(s, t) \rightarrow (f_n, f_m)$ form a c.o.n.s. in $L_2([\pi, \pi] \times [-\pi, \pi])$, so that $\sum_{n=1}^\infty \sum_{m=1}^\infty |f_n(s)\overline{f}_m(t)\psi(s-t)| dsdt = a^2$.

Now consider the map $J$ from $\mathcal{D}_1$ to $\mathcal{K}$. Lemma 6 implies that its restriction to $\mathcal{D}$ is bounded, and, since $\mathcal{D}/\mathcal{D}$ is finite-dimensional (Lemma 5), $J$ is bounded as an operator from $\mathcal{D}_1$ to $\mathcal{K}$ (a finite-dimensional...
extension of a bounded operator is bounded, as is readily seen). So $J$ extends uniquely to a bounded operator $A$ from $H$ to $K$.

**Lemma 8.** $I - A^* A$ is a Hilbert-Schmidt operator.

**Proof.** Complete the o.n.s. $G_1, G_2, \ldots$ by adding to it a c.o.n.s. $G_0, G_{-1}, \ldots, G_{-u}$ in $L$. Then, letting $k = u - 1$,

$$
\sum_{n, m = 1}^{\infty} \left| \langle (I - A^* A)G_n, G_m \rangle \right|^2
$$

using Parseval’s equality. But $\langle AG_n, AG_m \rangle = \langle G_n, G_m \rangle$ for $n, m > 0$, since such $G_n$ are in $D_1$, so that the sum is exactly

$$
a^2 + 2 \sum_{n = 1}^{\infty} \langle (I - A^* A)G_n, (I - A^* A)G_n \rangle.
$$

In order to complete the proof, it must be shown that $A$ is a homeomorphism from $H$ onto $K$. Since $I - A^* A$ is completely continuous, it will suffice to show

1. that the range of $A$ is dense in $K$.
2. that $A$ sends no nonzero element to zero.

1 is clear, since the range of $A$ contains the range of $J$, which is dense by the very definition of $H$.

We now make use of (a), or rather of the weaker (a'), to prove (2). Suppose, in fact, that $A(K)$ is zero in $K$ for some $K$ in $H$. Let $K_n$ be a sequence of members of $D_1$ converging to $K$ in $K$. Then $K_n$ converges to zero in $K$, since $A(K_n) = J(K_n)$. Then, by (a'), $K = 0$ on a set of positive $\rho$ measure. But the Proposition of the previous section tells us that $K$ is analytic. Thus $K = 0$.

To show the necessity of condition (a), suppose $J$ has an extension to an equivalence operator from $H$ to $K$, which we call $A$. Then (a) is immediate from the fact that $A$ is continuously invertible.

Since $I - A^* A$ is an equivalence operator, $\sum_{n, m = 1}^{\infty} \langle G_n, G_m \rangle - \langle G_n G_m \rangle \leq \langle G_n G_m \rangle < \infty$, where $G_1, G_2, \ldots$ is the c.o.n.s. in $D$ for $H$ previously constructed. Define an operator $Z$ on $L_2([0, \pi] \times [0, \pi])$ as follows: let $f_n(s, t) = f_n(s) f_m(t)$, where $G_n = p f_n$. For $Q = \sum_{n, m} a_{n, m} f_{n, m}$, Let $Z(Q) = \sum_{n, m} a_{n, m} (\langle G_n, G_m \rangle - \langle G_n G_m \rangle)$. Then

$$
| Z(Q) |^2 \leq \sum_{n, m} |a_{n, m}|^2 \sum_{n, m} |\langle G_n, G_m \rangle - \langle G_n G_m \rangle|\.
$$
So $Z(Q)$ has the form $\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} Q(s, t) \mathcal{F}(s, t) \, ds \, dt$ for some $\mathcal{F}$ such that $\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\mathcal{F}(s, t)|^2 \, ds \, dt < \infty$. In particular, consider $f, g \in \mathcal{D}_1$, and let $f = \sum_n a_n f_n, g = \sum_m b_m f_m$. Let $Q(s, t) = f(s)g(t)$. Then $Z(Q) = \sum_n a_n b_m \langle \langle G_n, G_m \rangle \rangle - \langle \langle G_n, G_m \rangle \rangle = \sum_n a_n b_m \left( \mathcal{F}(s) \mathcal{F}(t) \right) dt = \int \tilde{f} \tilde{g} \, |p|^3 \, dt.$

Let $0 < r < 2\pi$, and let $f, g$ have the closure of their supports in $]-\pi, \pi[$. Let $f'(s) = f(s + r), g'(s) = g(s + r)$. Then $f', g'$ are in $\mathcal{D}_1$, and their inverse Fourier transforms satisfy $\hat{f}'(x) = e^{irx} \hat{f}(x) = e^{irx} \hat{g}(x)$. Then

$$\int_{-\pi}^{\pi} f'(s)g'(t)\mathcal{F}(s, t) \, ds \, dt = \int \tilde{f}' \tilde{g}' \, |p|^3 \, dt \quad \text{But} \quad \int_{-\pi}^{\pi} f(s + r)g(t + r)\mathcal{F}(s, t) \, ds \, dt = \int f(s)g(t)\mathcal{F}(s, t) \, ds \, dt.$$
Φ is as in the statement, then \((\Phi - 1)dx\) has a generalized Fourier transform which is a function \(\varphi\) square-summable in any finite interval, so that

\[
\left[ \int_{-\pi}^{\pi} \left| \varphi(s - t) \right|^2 ds dt \right] \leq \left[ \int_{-\pi}^{\pi} \left| \varphi(r) \right|^2 dr \right] \leq 2\pi \left[ \int_{-\pi}^{\pi} \left| \varphi(r) \right|^2 dr \right].
\]

To prove corollary 2: let \(c_j\) be the absolute value of the ratio of the leading terms of \(A_j\) and \(B_j\), and let \(u_j = b_j - a_j = \deg(B_j) - \deg(A_j)\). It is clear in general that equivalence of the Gaussian processes induced by given covariances is unaffected if both covariances are multiplied by the same constant. Thus, we find that the process whose spectral measure is

\[
\left| \frac{A_j(x)}{B_j(x)} \right|^2 dx
\]

has measure on path space equivalent to that whose spectral measure is

\[
\frac{c_j}{(1 + x^u_j)} dx,
\]

because the quotient of

\[
\left| \frac{A_j(x)}{B_j(x)} \right|^2 \text{ by } \frac{c_j}{(1 + x^u_j)}
\]

is of the form: 1 plus a function in \(\mathcal{L}^2(-\infty, \infty)\). So the problem is reduced to whether or not the processes with spectral measures

\[
\frac{1}{(1 + x^u_j)} dx \quad \text{and} \quad \frac{c_jc_i^{-1}}{(1 + x^u_2)} dx
\]

are equivalent. The criterion is that

\[
\left(1 - \frac{c_jc_i^{-1}}{(1 + x^u_2 - u_1)}\right) dx
\]

have a generalized Fourier transform which agrees with a function on \([-2\pi, 2\pi]\) having certain properties. But this generalized Fourier transform is explicitly calculated (see [4]), and is of the required form when and only when \(c_2 = c_1\) and \(u_2 = u_1\).

**Bibliography**


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