

Pacific Journal of Mathematics

**SOME ZERO SUM TWO-PERSON GAMES WITH MOVES IN
THE UNIT INTERVAL**

MARTIN FOX

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Introduction. Consider the following zero sum two person game. The players alternately choose points $t_i \in [0, 1]$ for $i = 1, 2, \dots, n$, the choice being made by player I if i is odd and by player II if i is even. After the i th move the player who is to make the $(i + 1)$ st move observes the value of $\phi_i(t_1, t_2, \dots, t_i)$ where ϕ_i is some function on the i -dimensional closed unit cube to some set A_i . The payoff is $f(t_1, t_2, \dots, t_n)$ where f is a continuous, real-valued function.

If all the ϕ_i are constant we have the case of no information. Ville [1] showed that in this case such a game has a value. At the other extreme, if the ϕ_i are all one-to-one we have the case of perfect information so the game has a value.

The purpose of the present paper is to show that, in general, games of the form introduced in the first paragraph do not have values and to consider two cases in which they do. The counter-examples to be presented will be compared with Ville's classical example of a game on the unit square which has no value.

It is shown in §2 that the games considered always have values when $n = 2$.

An example of a game with no value is presented in §3. In this example $n = 3$ and the ϕ_i take only a finite number of values.

In §4 it is shown that the additional hypothesis of continuity of the ϕ_i is not sufficient to guarantee existence of a value. In that example $n = 4$. The case $n = 3$ with continuous ϕ_i remains unsolved.

Section 5 deals with a special case for which n is arbitrary and yet the game has a value. In this case the ϕ_i each take only a finite number of values and each is constant on sets which are finite unions of i -dimensional generalized intervals.

1. Preliminary remarks. In this section the notation to be used in this paper will be introduced. This will be facilitated by the introduction of the normal forms of the games under consideration.

A pure strategy for player I is a vector $x = (x_1, x_2, \dots, x_{[(n+1)/2]})$ where $x_i \in [0, 1]$ and the x_i for $i = 2, 3, \dots, [(n + 1)/2]$ are functions on A_{2i-2} to $[0, 1]$. If moves $t_1, t_2, \dots, t_{2i-2}$ have been made, then the i th move made by player I (the $(2i - 1)$ st move in the game) will be $x_i(\phi_{2i-2}(t_1, t_2, \dots, t_{2i-2}))$. His first move will be x_1 .

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A pure strategy for player II is a vector $y = (y_1, y_2, \dots, y_{[n/2]})$ where each y_i is a function on A_{2i-1} to $[0, 1]$. If moves $t_1, t_2, \dots, t_{2i-1}$ have been made, then the i th move made by player II (the $(2i)$ th move in the game) will be $y_i(\phi_{2i-1}(t_1, t_2, \dots, t_{2i-1}))$.

When player I uses the pure strategy x and player II uses the pure strategy y let $t_i(x, y)$ be the i th move made in the game. The t_i are defined recursively as follows:

$$\begin{aligned} t_1(x, y) &= x_1 ; \\ t_{2i}(x, y) &= y_i(\phi_{2i-1}(t_1(x, y), t_2(x, y), \dots, t_{2i-1}(x, y))) \\ &\quad \text{for } i = 1, 2, \dots, [n/2] ; \\ t_{2i-1}(x, y) &= x_i(\phi_{2i-2}(t_1(x, y), t_2(x, y), \dots, t_{2i-2}(x, y))) \\ &\quad \text{for } i = 2, 3, \dots, [(n+1)/2] . \end{aligned}$$

The payoff function is given by $M(x, y) = f(t_1(x, y), t_2(x, y), \dots, t_n(x, y))$. The payoff as a function of mixed strategies will also be denoted by M .

In our case, since the moves are points in $[0, 1]$, the strategy spaces X and Y are products, usually infinite dimensional, each coordinate space being $[0, 1]$. Hence, the choice of a strategy by player I is equivalent to the choice of a distribution function F on X . It will be convenient to let the space P of mixed strategies for player I be the family of all distribution functions on X which assign probability 1 to a finite subset of X . The same will be done for Q , the space of mixed strategies for player II.

If H is a distribution function on the real line and S is any subset of the real line which is Borel measurable, we will let HS be the probability assigned to S by H .

For $F \in P$ we let $F_{i,\alpha}$ denote the marginal distribution function of the coordinate of player I's strategy which corresponds to his i th move when $\phi_{2i-2} = \alpha$. Similar notation will be used for $G \in Q$.

2. The case $n = 2$. In this section it will be shown that any game \mathcal{G} of the type given in the introduction for which $n = 2$ has a value. It is not even necessary to assume that ϕ_1 is a measurable function.

For any $\alpha \in A_1$ let $\mathcal{G}(\alpha) = (\phi_1^{-1}(\alpha), [0, 1], M_\alpha)$ where M_α is f restricted to $\phi_1^{-1}(\alpha) \times [0, 1]$. It follows by the proof used for Ville's minimax theorem that each $\mathcal{G}(\alpha)$ has a value $v(\alpha)$. Let

$$v = \sup_{\alpha \in A_1} v(\alpha) .$$

Fix $\varepsilon > 0$ and let α^* be such that $v(\alpha^*) > v - \varepsilon$. For each $\alpha \in A_1$ let $F^{(\alpha)}$ and $G^{(\alpha)}$ be ε -good strategies for players I and II, respectively, in $\mathcal{G}(\alpha)$. The distribution function $F^{(\alpha)}$ assigns probability 1 to a finite subset of $\phi_1^{-1}(\alpha)$. Since $F^{(\alpha^*)}$ is a distribution function on $[0, 1]$ which

is the strategy space for player I in \mathcal{S} , it can also be used as a strategy in \mathcal{S} . Let y be any pure strategy for player II in \mathcal{S} . Since $y_1(\alpha^*) \in [0, 1]$, it follows that $y_1(\alpha^*)$ is a pure strategy for player II in $\mathcal{S}(\alpha^*)$. Hence,

$$\begin{aligned} M(F^{(\alpha^*)}, y) &= \int_{\phi_1^{-1}(\alpha^*)} f(t, y_1(\alpha^*))F^{(\alpha^*)}(dt) \\ &= \int M_{\alpha^*}(t, y_1(\alpha^*))F^{(\alpha^*)}(dt) \\ &= M_{\alpha^*}(F^{(\alpha^*)}, y_1(\alpha^*)) \\ &> v(\alpha^*) - \varepsilon > v - 2\varepsilon . \end{aligned}$$

Let G be any strategy for player II in \mathcal{S} such that $G_{1,\alpha} = G^{(\alpha)}$ for all $\alpha \in A_1$. Let x be any pure strategy for player I in \mathcal{S} . For some $\alpha \in A_1$ it must be true that $x \in \phi_1^{-1}(\alpha)$ so that x is also a pure strategy for player I in $\mathcal{S}(\alpha)$. Then,

$$\begin{aligned} M(x, G) &= \int f(x, t)G_{1,\alpha}(dt) \\ &= \int M_{\alpha}(x, t)G^{(\alpha)}(dt) \\ &= M_{\alpha}(x, G^{(\alpha)}) \\ &< v(\alpha) + \varepsilon \leq v + \varepsilon . \end{aligned}$$

From the two inequalities obtained above it follows that the value of \mathcal{S} is v .

3. A counter-example for $n = 3$. In this section the counter-example for $n = 3$ will be given. The functions ϕ_i ($i = 1, 2$) each take only a finite number of values. The similarity of this example to Ville's example will be discussed.

For this example let

$$\begin{aligned} \phi_1(t_1) &\equiv 0 ; \\ \phi_2(t_1, t_2) &= \begin{cases} -1 & \text{if } t_1 = 0 \text{ or } 0 < \min(t_2, 1 - t_2) \leq t_1 ; \\ t_2 & \text{if } t_2 = 0 \text{ or } 1 \text{ and } t_1 \neq 0 ; \\ 2 & \text{if } 0 < t_1 < t_2 \leq \frac{1}{2} \\ 3 & \text{if } 0 < t_1 < 1 - t_2 < \frac{1}{2} \end{cases} \end{aligned}$$

$$f(t_1, t_2, t_3) = -|t_3 - t_2|.$$

Let F be any strategy for player I. Fix $\varepsilon > 0$ and let $\delta \in (0, \varepsilon)$ be sufficiently small so that $F_1(0, \delta) < \varepsilon$. Let $G\{\delta\} = G\{1 - \delta\} = 1/2$. Then,

$$\begin{aligned}
M(F, G) &\leq -\frac{1}{2} (F_1[\delta, 1] + F_1\{0\}) \left[\int |t_3 - \delta| F_{2,-1}(dt_3) \right. \\
&\quad \left. + \int |t_3 - (1 - \delta)| F_{2,-1}(dt_3) \right] \\
&< -\frac{1}{2} (1 - \varepsilon) \left[\left(\frac{1}{2} - \delta \right) + \left(1 - \delta - \frac{1}{2} \right) \right] < -\frac{1}{2} + \frac{3}{2} \varepsilon
\end{aligned}$$

so that

$$\sup_F \inf_G M(F, G) \leq -\frac{1}{2}.$$

Let G be any strategy for player II. Fix $\varepsilon > 0$ and let $x_1 \in (0, 1/2)$ be sufficiently small so that $G(0, x_1] + G[1 - x_1, 1) < \varepsilon$.

Let

$$x_2(\alpha) = \begin{cases} \frac{1}{2} & \text{if } \alpha = -1; \\ \alpha & \text{if } \alpha = 0 \text{ or } 1; \\ \frac{1}{4} & \text{if } \alpha = 2; \\ \frac{3}{4} & \text{if } \alpha = 3. \end{cases}$$

Let $x = (x_1, x_2)$ so that x is a pure strategy for player I. Then,

$$\begin{aligned}
M(G, x) &\geq -\int_{(0, x_1]} \left(\frac{1}{2} - t_2 \right) G(dt_2) - \int_{[1 - x_1, 1)} \left(t_2 - \frac{1}{2} \right) G(dt_2) \\
&\quad - \int_{[0, 1/2]} \left| \frac{1}{4} - t_2 \right| G(dt_2) - \int_{(1/2, 1]} \left| \frac{3}{4} - t_2 \right| G(dt_2) \\
&> -\varepsilon - \frac{1}{4}
\end{aligned}$$

so that

$$\inf_G \sup_F M(F, G) \geq -\frac{1}{4}$$

and the game has no value.

In Ville's example the payoff function is such as to force each player to attempt to choose a point closer to 1 than does his opponent without actually choosing 1. It is impossible for either player to guarantee he will achieve this with any preassigned positive probability no matter what pure strategy his opponent may use. In the example just presented a similar situation arises on the first two moves. In Ville's example the competition to choose a point close to the endpoint is

a direct competition over payoff. In the present example this competition is over the information player I will receive, which, of course, helps determine the payoff. If on his first move player I chooses a point closer to 0 (but not 0) than the choice of his opponent is to both 0 and 1, then he will obtain more accurate information about the location of his opponent's choice than would be the case otherwise. Player II is prevented from choosing an endpoint since to do so would be to give his opponent perfect information.

4. A counter-example with continuous ϕ_i . In this section a counter-example will be presented in which the functions ϕ_i are all continuous. In this example $n = 4$. Again a comparison will be made with Ville's example.

Let

$$\begin{aligned} \phi_1(t_1) &\equiv 0 ; \\ \phi_2(t_1, t_2) &= t_1(1 - t_1)t_2 ; \\ \phi_3(t_1, t_2, t_3) &= \begin{cases} 0 & \text{if } \min(t_1, 1 - t_1) \leq t_2 \leq \max(t_1, 1 - t_1) ; \\ t_2(1 - t_2)(t_1 - t_2) \left| t_1 - \frac{1}{2} \right| & \text{if } t_2 < t_1 < \frac{1}{2} \\ & \text{or } \frac{1}{2} < t_1 < t_2 ; \\ t_2(1 - t_2)[t_1 - (1 - t_2)] \left| t_1 - \frac{1}{2} \right| & \text{if } \frac{1}{2} \leq t_1 < 1 - t_2 \\ & \text{or } 1 - t_2 < t_1 \leq \frac{1}{2} ; \end{cases} \end{aligned}$$

$$f(t_1, t_2, t_3, t_4) = |t_1 - t_4| - 10 |t_2 - t_3| .$$

Assume $t_2 \neq 0$ or 1. Then, $\phi_3(t_1, t_2, t_3) > 0$ for $\min(t_2, 1 - t_2) < t_1 < 1/2$ while $\phi_3(t_1, t_2, t_3) < 0$ for $1/2 < t_1 < \max(t_2, 1 - t_2)$. On the other hand, $\phi_3(t_1, t_2, t_3) = 0$ otherwise.

Let F be any strategy for player I. Fix $\epsilon > 0$ and let $\delta \in (0, \epsilon)$ be sufficiently small so that $F_1(0, \delta] + F_1[1 - \delta, 1) < \epsilon$. Let

$$y_2(\alpha) = \begin{cases} \frac{1}{2} & \text{if } \alpha = 0 ; \\ \frac{1}{4} & \text{if } \alpha > 0 ; \\ \frac{3}{4} & \text{if } \alpha < 0 . \end{cases}$$

Let G assign probability 1/2 to each of the pure strategies (δ, y_2) and $(1 - \delta, y_2)$. Then,

$$\begin{aligned}
M(F, G) &\leq \int_{[0, \delta]} \left(\frac{1}{2} - t_1 \right) F_1(dt_1) + \int_{[1-\delta, 1]} \left(t_1 - \frac{1}{2} \right) F_1(dt_1) \\
&\quad + \int_{(\delta, 1/2)} \left| t_1 - \frac{1}{4} \right| F_1(dt_1) + \int_{(1/2, 1-\delta)} \left| t_1 - \frac{3}{4} \right| F_1(dt_1) \\
&\quad - 10[F_1\{0\} + F_1\{1\}] \left[\frac{1}{2} \int |\delta - t_3| F_{2,0}(dt_3) \right. \\
&\quad \quad \quad \left. + \frac{1}{2} \int |1 - \delta - t_3| F_{2,0}(dt_3) \right] \\
&< \frac{1}{2} [F_1\{0\} + F_1\{1\}] + \frac{1}{2} \varepsilon + \frac{1}{4} [1 - \varepsilon - F_1\{0\} - F_1\{1\}] \\
&\quad - 5[F_1\{0\} + F_1\{1\}] \left[\left(\frac{1}{2} - \delta \right) + \left(1 - \delta - \frac{1}{2} \right) \right] \\
&= \frac{1}{4} + \frac{1}{4} \varepsilon - [F_1\{0\} + F_1\{1\}] \left[5(1 - 2\delta) - \frac{1}{4} \right] \\
&< \frac{1}{4} + 11\varepsilon
\end{aligned}$$

so that $\sup_F \inf_G M(F, G) \leq 1/4$.

Let G be any strategy for player II. Fix $\varepsilon > 0$ and let $\delta \in (0, \varepsilon) \cap (0, 1/2)$ be sufficiently small so that $G_{1,0}(0, \delta) + G_{1,0}(1 - \delta, 1) < \varepsilon$. Let $x_2(\alpha) = \alpha/[\delta(1 - \delta)]$ and let F assign probability $1/2$ to each of the pure strategies (δ, x_2) and $(1 - \delta, x_2)$. When player I uses the strategy F the value of the nonpositive term in f will always be zero. Thus,

$$\begin{aligned}
M(F, G) &\geq \left[1 - G_{1,0}(0, \delta) - G_{1,0}(1 - \delta, 1) \right] \\
&\quad \times \left[\frac{1}{2} \int |\delta - t_4| G_{2,0}(dt_4) + \frac{1}{2} \int |1 - \delta - t_4| G_{2,0}(dt_4) \right] \\
&> \frac{1}{2} (1 - \varepsilon) \left[\left(\frac{1}{2} - \delta \right) + \left(1 - \delta - \frac{1}{2} \right) \right] \\
&> \frac{1}{2} - \frac{3}{2} \varepsilon
\end{aligned}$$

so that $\inf_G \sup_F M(F, G) \geq 1/2$ and the game has no value.

Here again the primary competition between the players is to make their first moves as close to the endpoints as possible without actually choosing the endpoints. If player I is successful in choosing a point t_1 at least as close to one of the endpoints as is player II's choice, then player II will have less information about t_1 than would be the case otherwise. Player I is prevented from choosing an endpoint by the fact

that if he does so he will get no information about his opponent's first move so that he cannot guarantee that he can keep the negative term close to zero. Player II is prevented from choosing an endpoint by the fact that when he does so the function ϕ_3 will take the value zero no matter what his opponent does so that he will have no information about player I's first move.

5. The case of information sets which are unions of generalized intervals. The case to be considered here is that in which each ϕ_i takes only a finite number of values and each is constant only on sets which are finite unions of i -dimensional generalized intervals. This is the only case considered in this paper in which n remains arbitrary.

Let the values of ϕ_i be $1, 2, \dots, m_i$. Let $P_j\phi_i^{-1}(k)$ be the projection on the j th coordinate of $\phi_i^{-1}(k)$ where $j = 1, 2, \dots, i$. The interval $[0, 1]$ can be subdivided into disjoint sets $B_{j1}, B_{j2}, \dots, B_{j\ell_j}$ such that for each $B_{j\ell}$ there exist i_1, i_2, \dots, i_r and k_1, k_2, \dots, k_u , all integers, such that $t \in B_{j\ell}$ if, and only if, $t \in P_j\phi_i^{-1}(k)$ whenever $i \in \{i_1, i_2, \dots, i_r\}$ and $k \in \{k_1, k_2, \dots, k_u\}$ while $t \notin P_j\phi_i^{-1}(k)$ otherwise. Suppose j is even so that player II makes the j th move. Let $y = (y_1, y_2, \dots, y_{[n/2]})$ and $y' = (y'_1, y'_2, \dots, y'^{[n/2]})$ be any strategies for player II such that $y_i = y'_i$ for $i \neq j/2$ and if $y_{j/2}(k) \in B_{j\ell}$, then $y'_{j/2}(k) \in B_{j\ell}$. For any pure strategy x for player I we have $t_i(x, y) = t_i(x, y')$ for $i = 1, 2, \dots, j - 1$ since for these values of i player II's moves are unchanged. If $t_j(x, y) \in B_{j\ell}$, then $t_j(x, y') \in B_{j\ell}$. Hence,

$$\phi_j(t_1(x, y), t_2(x, y), \dots, t_j(x, y)) = \phi_j(t_1(x, y'), t_2(x, y'), \dots, t_j(x, y'))$$

so that $t_{j+1}(x, y) = t_{j+1}(x, y')$. Suppose that $t_i(x, y) = t_i(x, y')$ for $i = j + 1, j + 2, \dots, i_0$. Then, $\phi_{i_0}(t_1(x, y), t_2(x, y), \dots, t_{i_0}(x, y)) = \phi_{i_0}(t_1(x, y'), t_2(x, y'), \dots, t_{i_0}(x, y'))$ so that $t_{i_0+1}(x, y) = t_{i_0+1}(x, y')$. Thus, $t_i(x, y) = t_i(x, y')$ for all $i \neq j$.

For each $j = 1, 2, \dots, n - 1$ fix $\delta_j > 0$ and select points $t_{j1}, t_{j2}, \dots, t_{jv_j}$ such that for any $t_j \in B_{j\ell}$ there exists $t_{jv} \in B_{j\ell}$ such that for any $t_1, t_2, \dots, t_{j-1}, t_{j+1}, \dots, t_n$ we have

$$\begin{aligned} &|f(t_1, t_2, \dots, t_{j-1}, t_j, t_{j+1}, \dots, t_n) \\ &\quad - f(t_1, t_2, \dots, t_{j-1}, t_{jv}, t_{j+1}, \dots, t_n)| < \delta_j. \end{aligned}$$

Select the t_{jv} in such a way that as $\delta_j \downarrow$ the set of all the t_{jv} increases monotonically.

Let the game $\mathcal{G}(\delta_1, \delta_2, \dots, \delta_i) = (X(\delta_1, \delta_2, \dots, \delta_i), Y(\delta_1, \delta_2, \dots, \delta_i), M_{\delta_1, \delta_2, \dots, \delta_i})$ be our original game with the j th move for $j = 1, 2, \dots, i$ restricted to $t_{j1}, t_{j2}, \dots, t_{jv_j}$. In $\mathcal{G}(\delta_1, \delta_2, \dots, \delta_{n-1})$ the player who makes the $(n - 1)$ st move has only a finite number of strategies so that $\mathcal{G}(\delta_1, \delta_2, \dots, \delta_{n-1})$ has a value (see Wald [2]).

Suppose $\mathcal{G}(\delta_1, \delta_2, \dots, \delta_{i-1}, \delta_i)$ has a value for all $\delta_i > 0$. It follows, by a proof similar to Ville's, that $\mathcal{G}(\delta_1, \delta_2, \dots, \delta_{i-1})$ has a value. Thus, by induction, \mathcal{G} will also have a value.

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Vol. 10, No. 4

December, 1960

M. Altman, <i>An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space</i>	1107
Nesmith Cornett Ankeny, <i>Criterion for rth power residuacity</i>	1115
Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures I</i>	1125
Frank Featherstone Bonsall, <i>Positive operators compact in an auxiliary topology</i>	1131
Billy Joe Boyer, <i>Summability of derived conjugate series</i>	1139
Delmar L. Boyer, <i>A note on a problem of Fuchs</i>	1147
Hans-Joachim Bremermann, <i>The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces</i>	1149
Andrew Michael Bruckner, <i>Minimal superadditive extensions of superadditive functions</i>	1155
Billy Finney Bryant, <i>On expansive homeomorphisms</i>	1163
Jean W. Butler, <i>On complete and independent sets of operations in finite algebras</i>	1169
Lucien Le Cam, <i>An approximation theorem for the Poisson binomial distribution</i>	1181
Paul Civin, <i>Involutions on locally compact rings</i>	1199
Earl A. Coddington, <i>Normal extensions of formally normal operators</i>	1203
Jacob Feldman, <i>Some classes of equivalent Gaussian processes on an interval</i>	1211
Shaul Foguel, <i>Weak and strong convergence for Markov processes</i>	1221
Martin Fox, <i>Some zero sum two-person games with moves in the unit interval</i>	1235
Robert Pertsch Gilbert, <i>Singularities of three-dimensional harmonic functions</i>	1243
Branko Grünbaum, <i>Partitions of mass-distributions and of convex bodies by hyperplanes</i>	1257
Sidney Morris Harmon, <i>Regular covering surfaces of Riemann surfaces</i>	1263
Edwin Hewitt and Herbert S. Zuckerman, <i>The multiplicative semigroup of integers modulo m</i>	1291
Paul Daniel Hill, <i>Relation of a direct limit group to associated vector groups</i>	1309
Calvin Virgil Holmes, <i>Commutator groups of monomial groups</i>	1313
James Fredrik Jakobsen and W. R. Utz, <i>The non-existence of expansive homeomorphisms on a closed 2-cell</i>	1319
John William Jewett, <i>Multiplication on classes of pseudo-analytic functions</i>	1323
Helmut Klingen, <i>Analytic automorphisms of bounded symmetric complex domains</i>	1327
Robert Jacob Koch, <i>Ordered semigroups in partially ordered semigroups</i>	1333
Marvin David Marcus and N. A. Khan, <i>On a commutator result of Tausky and Zassenhaus</i>	1337
John Glen Marica and Steve Jerome Bryant, <i>Unary algebras</i>	1347
Edward Peter Merkes and W. T. Scott, <i>On univalence of a continued fraction</i>	1361
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James Victor Whittaker, <i>Normal subgroups of some homeomorphism groups</i>	1469