PARTITIONS OF MASS-DISTRIBUTIONS AND OF CONVEX BODIES BY HYPERPLANES

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1. Introduction. The following results are well-known (Neumann [7]; Eggleston [3], [4, p. 125–126], [5, p. 118]; Newman [8]:
(A) For any mass-distribution in the plane, such that the total mass contained in every half-plane is finite and depends continuously on the position of the half-plane, there exists a point \( P \) such that each half-plane which contains \( P \), contains at least \( 1/3 \) of the total mass.
(B) For any convex body \( K \) in the plane there exists a point \( P \) such that for each half-plane \( H \) containing \( P \) the area of \( H \cap K \) is at least \( 4/9 \) of the area of \( K \).

The main object of the present note is to generalize (A) and (B) to higher-dimensional Euclidean spaces.

In the following \( m \) shall denote a fixed (non-negative) finite measure on the ring of subsets of \( E^n \) generated by the closed half-spaces in \( E^n \). (For the terminology and results on measures see, e.g., Halmos [6].)

For a real \( \lambda, \ 0 \leq \lambda \leq 1/2 \), we define \( \mathcal{E}(m, \lambda) \) as the subset of \( E^n \) consisting of those points \( P \in E^n \) which satisfy the condition: For any closed half-space \( H \subset E^n \), with \( P \in H \), the relation \( m(H) \geq \lambda \cdot m(E^n) \) holds.

Obviously, \( \mathcal{E}(m, \lambda) \) is a compact, convex (possibly empty) set.

Using the notation of \( \mathcal{E}(m, \lambda) \), Theorem (A) may be extended as follows:

**Theorem 1.** \( \mathcal{E}(m, 1/(n+1)) \neq \emptyset \) for any measure \( m \) in \( E^n \).

Let \( V(S) \) denote the volume (\( n \)-dimensional Lebesgue measure) of the set \( S \subset E^n \). For any convex body \( K \subset E^n \), we denote by \( m_K \) the measure (defined for all Lebesgue measurable subsets \( S \) of \( E^n \)) obtained by taking \( m_K(S) = V(S \cap K) \). We denote \( \mathcal{E}(m_K, \lambda) \) by \( \mathcal{E}(K, \lambda) \).

Theorem (B) may now be generalized as follows:

**Theorem 2.** If \( K \) is any convex body in \( E^n \) then

\[
\mathcal{E}(K, \left(\frac{n}{n+1}\right)^n) \neq \emptyset .
\]

We shall prove Theorems 1 and 2 in the following two sections.

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The last section contains remarks and comments.

2. **Proof of Theorem I.** If $v$ is a unit vector (in $E^n$) and $\alpha$ is a real number, let $H(v, \alpha)$ be the closed half-space

$$H(v, \alpha) = \{x \in E^n; (x, v) \leq \alpha\}.$$ 

Let $\alpha(v)$ be defined by

$$\alpha(v) = \min \left\{ \alpha; m(H(v, \alpha)) \geq \frac{n}{n+1} m(E^n) \right\},$$

(the minimum is attained since $m(H(v, \alpha))$ is continuous to the right as a function of $\alpha$). Let $H(v) = H(v, \alpha(v))$ and

$$H^*(v) = \{x \in E^n; (x, v) \geq \alpha(v)\}.$$ 

(Without loss of generality we shall in the sequel assume $m(E^n) = 1$.) Obviously,

$$\mathcal{C} \left( m \left( \frac{1}{n+1} \right) \right) \supset \bigcap_{v} H(v);$$

hence, if $\bigcap_{v} H(v) \neq \phi$ the proof is complete. On the other hand, if $\bigcap_{v} H(v) = \phi$, we shall show that

$$\mathcal{C} \left( m \left( \frac{1}{n+1} \right) \right) \neq \phi$$

in the following way. The half-spaces $H(v)$ are closed convex sets, and it is easily seen that a finite number of them may be found such that their intersection is compact. By Helly’s theorem on intersections of convex sets (see, e.g., Rademacher-Schoenberg [9]) the assumption $\bigcap_{v} H(v) = \phi$ implies the existence of an $n + 1$ membered family of unit vectors $v_i$, $0 \leq i \leq n$, such that $\bigcap_{i=0}^{n} H(v_i) = \phi$. Using an inductive argument it is easily seen that we may assume that every $n$ of the vectors $v_i$ are linearly independent. Therefore (denoting $H_i = H(v_i)$ and $H_i^* = H_i^*(v_i)$) the set $S = \bigcap_{i=0}^{n} H_i^*$ is a non-degenerate simplex whose faces are contained in the hyperplanes $H_i \cap H_i^*$, $0 \leq i \leq n$. By the definition of $\alpha(v)$ we have $m(H_i^*) \geq 1/(n+1)$ and $m(\text{Int } H_i^*) \leq 1/(n+1)$ for all $i$. Therefore $m(H_i \cap \text{Int } H_i^*) \leq 1/(n+1)$, and thus $m(H_i \cap H_i) \geq (n-1)/(n+1)$ for all $i \neq j$. Now, since $\bigcap_{i=0}^{n} H_i = \phi$, we have

$$\frac{n}{n+1} \geq m(H_i) \geq m \left( H_i \cap \left( \bigcup_{j \neq i} H_j \right) \right) \geq \frac{1}{n-1} \sum_{j \neq i} m(H_i \cap H_j) \geq \frac{1}{n-1} \cdot n \cdot \frac{n-1}{n+1} = \frac{n}{n+1}.$$ 

\[1\] The author is indebted to Professor B. M. Stewart for the correction of an error in the original proof.
Thus, for all \( i \), equality signs hold throughout. In particular,

\[
m\left(\bigcap_{j \in J} H_j\right) = \frac{1}{n+1}
\]

for all \( i \) (i.e., the support of \( m \) is contained in the "vertex-regions" of the simplex \( S = \bigcap_i H_i^* \)), and it is immediately verified that

\[
\mathcal{C}\left(m; \frac{1}{n+1}\right) \supset S \neq \emptyset.
\]

This ends the proof of Theorem 1.

3. Proof of Theorem 2. Let \( G_k \) denote the centroid of the convex body \( K \subset E^n \). We shall prove Theorem 2 by establishing the stronger statement \( G_k \in \mathcal{C}(K, \alpha_n) \), where \( \alpha_n = (n/(n+1))^n \). Assuming, to the contrary, that \( G_k \notin \mathcal{C}(K, \alpha_n) \), there exists a hyperplane \( L \) containing \( G_k \) such that the volume of the part of \( K \) contained in one of the half-spaces determined by \( L \) is less than \( \alpha_n \cdot V(K) \). We shall obtain a contradiction from this assumption.

Let \( G_K \) be the origin of an orthogonal system of coordinates \((x_1, \ldots, x_n)\) of \( E^n \), such that \( L \) is the hyperplane determined by \( x_1 = 0 \).

Let \( H^+ \) be the half-space \( \{(x_1, \ldots, x_n); x_1 \geq 0\} \) and \( H^- \) the other closed half-space determined by \( L \). We may assume that \( V(K \cap H^-) < \alpha_n \cdot V(K) \). For any set \( S \subset E^n \) we shall use the notations \( S^- = S \cap H^- \) and \( S^+ = S \cap H^+ \). Let \( \hat{K} \) be the set obtained from \( K \) by spherical symmetrization ("Schwarzsche Abrundung", Bonnesen-Fenchel [1, p. 71], "Schwarz rotation process", Eggleston [5, p. 100]) with respect to the \( x_1 \)-axis (i.e., \( \hat{K} \) is the union of the \((n-1)\)-dimensional spheres obtained by taking in each hyperplane \( L_i = \{(x_1, \ldots, x_n); x_1 = t\} \) an \((n-1)\)-dimensional sphere with center \((t, 0, \ldots, 0)\) and \((n-1)\)-dimensional volume equal to that of \( K \cap L_i \)). It is well known that \( \hat{K} \) is a convex body, and obviously \( V(\hat{K}^-) = V(K^-) \), \( V(\hat{K}^+) = V(K^+) \) and \( G_{\hat{K}} = G_K \).

Therefore \( V(\hat{K}^-) < \alpha_n \cdot V(\hat{K}) \) and \( G_{\hat{K}} \notin \mathcal{C}(\hat{K}, \alpha_n) \). Let \( C^- \) denote the (orthogonal) hypercone with base \( \hat{K} \cap L \) and vertex \((c, 0, \ldots, 0) \in H^- \), where \( c \) is chosen in such a way that \( V(C^-) = V(\hat{K}) \). Let \( C \) be the hypercone obtained by extending \( C^- \) (along its generators) into \( H^+ \) in such a way that \( V(C^+) = V(\hat{K}^+) \). With \( C \) thus defined, it is easily verified that the \( x_1 \)-coordinate of \( G_0^- \) (resp. \( G_0^+ \)) is not greater than that of \( G_{\hat{K}}^- \) (resp. \( G_{\hat{K}}^+ \)). Therefore, \( G_0 \in H^- \), and thus the hyperplane \( L^* \), parallel to \( L \) and passing through \( G_0 \), divides \( C \) into two parts in such a way that the part contained in \( H^- \) has a volume smaller than \( \alpha_n \cdot V(C) \). But by a simple computation we find (since the centroid of a hypercone divides its height in the ratio \( 1:n \)) that the volume in question equals \( \alpha_n \cdot V(C) \). The contradiction reached proves the theorem.
4. Remarks. (i) It is very easy to find examples which show that the bounds in Theorems 1 and 2 are the best possible. From the proofs given, it is also easy to deduce that if $\mathcal{C}(K, \alpha_n + \varepsilon) = \phi$ for all $\varepsilon > 0$ then $K$ is a simplex, and that $\mathcal{C}(m, 1/(n + 1) + \varepsilon) = \phi$ for all $\varepsilon > 0$ only if the support of $m$ is contained in the "vertex-regions" of some (possibly degenerate) simplex, and all the "vertex-regions" have the same measure.

(ii) The proof of Theorem 1 may be somewhat simplified if the measure $m$ is assumed to be continuous (as in Theorem (A)). The advantage of the more general form is that it includes, e.g., measures generated by finite point-sets, surface-area etc.

(iii) The following obvious corollary of Theorem 2 is interesting because of its independence on the dimension:

For any convex body $K \subset E^n$ we have

$$G_K \in \mathcal{C}(K, e^{-1}) = C(K, 0.3678\ldots).$$

(iv) It would be interesting to find the analogue of Theorem 2 obtained by substituting the $(n - 1)$-dimensional surface area $A(K)$ for the volume $V(K)$ of $K \subset E^n$. The problem seems to be unsolved even for $n = 2$.

(v) It is easily proved that for any continuous mass-distribution in the plane there exists a pair of orthogonal lines such that each "quadrant" determined by them contains $1/4$ of the total mass. The analogous statement is not true for $n$ mutually orthogonal hyperplanes in $E^n$; does it become true if the condition of orthogonality is omitted?

(vi) It is well known (Buck and Buck [2]) that for any continuous mass-distribution in the plane there exist three concurrent straight lines such that each of the six "wedges" determined by them contains $1/6$ of the total mass. Does this fact generalize to $E^n$ when the three lines are replaced by $n + 1$ hyperplanes with a common $(n - 2)$-dimensional intersection?

Added in proof. After submitting the present note for publication, the following facts came to our attention:

(i) Theorems (A) and B are proved, and Theorem 1 suggested, in I. M. Jaglom—W. G. Boltjanski, Konvexe Figuren, Berlin, 1956, pp. 16, 18, 27, 104–106, 116, 135–136 (this is a translation of the Russian original, which appeared in 1951); Theorem (b) is there attributed (without references) to A. Winternitz.

(ii) A proof of Theorem 1 (using Brouwer’s fixed-point theorem), together with some related results, was given in B. J. Birch, On 3N points in a plane, Proc. Cambridge Philos. Soc., 55 (1959), 289–293.

(iii) A proof of Theorem 2, very similar to the one given in the
present paper, was found independently by P. C. Hammer; it is contained in a paper “Volumes cut from convex bodies by planes”, submitted to “Mathematika”.

(iv) The relation \( \mathcal{E}(m, \frac{1}{2}) \neq \phi \) (resp. \( \mathcal{E}(K, \frac{1}{2}) \neq \phi \)) holds for any distribution of masses (resp. convex body) with a center of symmetry. Obviously, \( \mathcal{E}(m, \frac{1}{2}) \neq \phi \) is possible also for mass-distributions without a center. The conjecture (trivial for the plane) that \( \mathcal{E}(K, \frac{1}{2}) \neq \phi \) characterizes centrally symmetric convex bodies was first established Professor F. J. Dyson; it is hoped that a proof will be published soon.

(v) Results generalizing Theorem 1 were established by R. Rado in the paper, “A theorem on general measure”, J. London Math. Soc., 21 (1946), 291–300. Rado’s proof also uses Helley’s theorem, but in a fashion different from the one used in the present paper.

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