PARTITIONS OF MASS-DISTRIBUTIONS AND OF CONVEX BODIES BY HYPERPLANES

BRANKO GRÜNBAUM
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B. GRÜNBAUM

1. Introduction. The following results are well-known (Neumann [7]; Eggleston [3], [4, p. 125–126], [5, p. 118]; Newman [8]):

(A) For any mass-distribution in the plane, such that the total mass contained in every half-plane is finite and depends continuously on the position of the half-plane, there exists a point $P$ such that each half-plane which contains $P$, contains at least $1/3$ of the total mass.

(B) For any convex body $K$ in the plane there exists a point $P$ such that for each half-plane $H$ containing $P$ the area of $H \cap K$ is at least $4/9$ of the area of $K$.

The main object of the present note is to generalize (A) and (B) to higher-dimensional Euclidean spaces.

In the following $m$ shall denote a fixed (non-negative) finite measure on the ring of subsets of $E^n$ generated by the closed half-spaces in $E^n$. (For the terminology and results on measures see, e.g., Halmos [6].)

For a real $\lambda$, $0 \leq \lambda \leq 1/2$, we define $\mathscr{C}(m, \lambda)$ as the subset of $E^n$ consisting of those points $P \in E^n$ which satisfy the condition: For any closed half-space $H \subset E^n$, with $P \in H$, the relation $m(H) \geq \lambda \cdot m(E^n)$ holds.

Obviously, $\mathscr{C}(m, \lambda)$ is a compact, convex (possibly empty) set.

Using the notation of $\mathscr{C}(m, \lambda)$, Theorem (A) may be extended as follows:

**Theorem 1.** $\mathscr{C}(m, 1/(n+1)) \neq \emptyset$ for any measure $m$ in $E^n$.

Let $V(S)$ denote the volume ($n$-dimensional Lebesgue measure) of the set $S \subset E^n$. For any convex body $K \subset E^n$, we denote by $m_K$ the measure (defined for all Lebesgue measurable subsets $S$ of $E^n$) obtained by taking $m_K(S) = V(S \cap K)$. We denote $\mathscr{C}(m_K, \lambda)$ by $\mathscr{C}(K, \lambda)$.

Theorem (B) may now be generalized as follows:

**Theorem 2.** If $K$ is any convex body in $E^n$ then

$$\mathscr{C}(K, \left(\frac{n}{n+1}\right)^n) \neq \emptyset.$$

We shall prove Theorems 1 and 2 in the following two sections.

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2. **Proof of Theorem 1.**\(^1\) If \(v\) is a unit vector (in \(E^n\)) and \(\alpha\) is a real number, let \(H(v, \alpha)\) be the closed half-space

\[
H(v, \alpha) = \{x \in E^n; (x, v) \leq \alpha\}.
\]

Let \(\alpha(v)\) be defined by

\[
\alpha(v) = \min \left\{ \alpha; m(H(v, \alpha)) \geq \frac{n}{n+1} m(E^n) \right\},
\]

(the minimum is attained since \(m(H(v, \alpha))\) is continuous to the right as a function of \(\alpha\)). Let \(H(v) = H(v, \alpha(v))\) and

\[
H^*(v) = \{x \in E^n; (x, v) \geq \alpha(v)\}.
\]

(Without loss of generality we shall in the sequel assume \(m(E^n) = 1\).) Obviously,

\[
\bigcap_v \left( m \left( \frac{1}{n+1} \right) \right) \subseteq \bigcap_v H(v);
\]

hence, if \(\bigcap_v H(v) \neq \emptyset\) the proof is complete. On the other hand, if \(\bigcap_v H(v) = \emptyset\), we shall show that

\[
C \left( m \left( \frac{1}{n+1} \right) \right) \neq \emptyset
\]

in the following way. The half-spaces \(H(v)\) are closed convex sets, and it is easily seen that a finite number of them may be found such that their intersection is compact. By Helly’s theorem on intersections of convex sets (see, e.g., Rademacher-Schoenberg [9]) the assumption \(\bigcap_v H(v) = \emptyset\) implies the existence of an \(n+1\) membered family of unit vectors \(v_i, \ 0 \leq i \leq n\), such that \(\bigcap_{i=0}^n H(v_i) = \emptyset\). Using an inductive argument it is easily seen that we may assume that every \(n\) of the vectors \(v_i\) are linearly independent. Therefore (denoting \(H_i = H(v_i)\) and \(H_i^* = H_i^*(v_i)\)) the set \(S = \bigcap_{i=0}^n H_i^*\) is a non-degenerate simplex whose faces are contained in the hyperplanes \(H_i \cap H_i^*\), \(0 \leq i \leq n\). By the definition of \(\alpha(v)\) we have \(m(H_i^*) \geq 1/(n + 1)\) and \(m(\text{Int } H_i^*) \leq 1/(n + 1)\) for all \(i\). Therefore \(m(H_j \cap \text{Int } H_i^*) \leq 1/(n + 1)\), and thus \(m(H_j \cap H_i) \geq (n - 1)/(n + 1)\) for all \(i \neq j\). Now, since \(\bigcap_{i=0}^n H_i = \emptyset\), we have

\[
\frac{n}{n+1} \geq m(H_i) \geq m \left[ H_i \cap \left( \bigcup_{j \neq i} H_j \right) \right] \geq \frac{1}{n-1} \sum_{0 \leq j \leq n, j \neq i} m(H_i \cap H_j) \geq \frac{1}{n-1} \cdot n \cdot \frac{n-1}{n+1} = \frac{n}{n+1}.
\]

\(^1\) The author is indebted to Professor B. M. Stewart for the correction of an error in the original proof.
Thus, for all $i$, equality signs hold throughout. In particular,

$$m \left( \bigcap_{0 \leq j \leq n} H_j \right) = \frac{1}{n + 1}$$

for all $i$ (i.e., the support of $m$ is contained in the "vertex-regions" of the simplex $S = \bigcap_i H_i$), and it is immediately verified that

$$\mathcal{E} \left( m; \frac{1}{(n + 1)} \right) \supset S \neq \phi .$$

This ends the proof of Theorem 1.

3. Proof of Theorem 2. Let $G_k$ denote the centroid of the convex body $K \subset E^n$. We shall prove Theorem 2 by establishing the stronger statement $G_k \in \mathcal{E}(K, \alpha_n)$, where $\alpha_n = (n/(n + 1))^n$. Assuming, to the contrary, that $G_k \notin \mathcal{E}(K, \alpha_n)$, there exists a hyperplane $L$ containing $G_k$ such that the volume of the part of the set $K$ contained in one of the half-spaces determined by $L$ is less than $\alpha_n \cdot V(K)$. We shall obtain a contradiction from this assumption.

Let $G_K$ be the origin of an orthogonal system of coordinates $(x_1, \ldots, x_n)$ of $E^n$, such that $L$ is the hyperplane determined by $x_1 = 0$.

Let $H^+$ be the half-space $\{(x_1, \ldots, x_n); x_1 \geq 0\}$ and $H^-$ the other closed half-space determined by $L$. We may assume that $V(K \cap H^-) < \alpha_n \cdot V(K)$. For any set $S \subset E^n$ we shall use the notations $S^- = S \cap H^-$ and $S^+ = S \cap H^+$. Let $\hat{K}$ be the set obtained from $K$ by spherical symmetrization ("Schwarzsche Abrundung", Bonnesen-Fenchel [1, p. 71]; "Schwarz rotation process", Eggleston [5, p. 100]) with respect to the $x_1$-axis (i.e., $\hat{K}$ is the union of the $(n - 1)$-dimensional spheres obtained by taking in each hyperplane $L_i = \{(x_1, \ldots, x_n); x_1 = t\}$ an $(n - 1)$-dimensional sphere with center $(t, 0, \ldots, 0)$ and $(n - 1)$-dimensional volume equal to that of $K \cap L_i$). It is well known that $\hat{K}$ is a convex body, and obviously $V(\hat{K}^-) = V(K^-)$, $V(\hat{K}^+) = V(K^+)$ and $\hat{G}_K = G_K$. Therefore $V(\hat{K}^-) < \alpha_n \cdot V(\hat{K})$ and $G_{\hat{K}} \notin \mathcal{E}(\hat{K}, \alpha_n)$. Let $C^-$ denote the (orthogonal) hypercone with base $\hat{K} \cap L$ and vertex $(c, 0, \ldots, 0) \in H^-$, where $c$ is chosen in such a way that $V(C^-) = V(\hat{K}^-)$. Let $C$ be the hypercone obtained by extending $C^-$ (along its generators) into $H^+$ in such a way that $V(C^+) = V(\hat{K}^+)$. With $C$ thus defined, it is easily verified that the $x_1$-coordinate of $G_{\hat{C}^-}$ (resp. $G_{\hat{C}^+}$) is not greater than that of $G_{\hat{K}^-}$ (resp. $G_{\hat{K}^+}$). Therefore, $G_{\hat{C}} \in H^-$, and thus the hyperplane $L^*$, parallel to $L$ and passing through $G_{\hat{C}}$, divides $C$ into two parts in such a way that the part contained in $H^-$ has a volume smaller than $\alpha_n \cdot V(C)$. But by a simple computation we find (since the centroid of a hypercone divides its height in the ratio $1:n$) that the volume in question equals $\alpha_n \cdot V(C)$. The contradiction reached proves the theorem.
4. Remarks. (i) It is very easy to find examples which show that the bounds in Theorems 1 and 2 are the best possible. From the proofs given, it is also easy to deduce that if \( \phi(K, a_n + \varepsilon) = \phi \) for all \( \varepsilon > 0 \) then \( K \) is a simplex, and that \( \phi(m, 1/(n+1) + \varepsilon) = \phi \) for all \( \varepsilon > 0 \) only if the support of \( m \) is contained in the "vertex-regions" of some (possibly degenerate) simplex, and all the "vertex-regions" have the same measure.

(ii) The proof of Theorem 1 may be somewhat simplified if the measure \( m \) is assumed to be continuous (as in Theorem (A)). The advantage of the more general form is that it includes, e.g., measures generated by finite point-sets, surface-area etc.

(iii) The following obvious corollary of Theorem 2 is interesting because of its independence on the dimension:
For any convex body \( K \subset E^n \) we have
\[
G_K \in \phi(K, e^{-1}) = C(K, 0.3678 \cdots).
\]

(iv) It would be interesting to find the analogue of Theorem 2 obtained by substituting the \((n-1)\)-dimensional surface area \( A(K) \) for the volume \( V(K) \) of \( K \subset E^n \). The problem seems to be unsolved even for \( n = 2 \).

(v) It is easily proved that for any continuous mass-distribution in the plane there exists a pair of orthogonal lines such that each "quadrant" determined by them contains 1/4 of the total mass. The analogous statement is not true for \( n \) mutually orthogonal hyperplanes in \( E^n \); does it become true if the condition of orthogonality is omitted?

(vi) It is well known (Buck and Buck [2]) that for any continuous mass-distribution in the plane there exist three concurrent straight lines such that each of the six "wedges" determined by them contains 1/6 of the total mass. Does this fact generalize to \( E^n \) when the three lines are replaced by \( n + 1 \) hyperplanes with a common \((n-2)\)-dimensional intersection?

Added in proof. After submitting the present note for publication, the following facts came to our attention:

(i) Theorems (A) and B are proved, and Theorem 1 suggested, in I. M. Jaglom—W. G. Boltjanski, Konvexe Figuren, Berlin, 1956, pp. 16, 18, 27, 104–106, 116, 135–136 (this is a translation of the Russian original, which appeared in 1951); Theorem (b) is there attributed (without references) to A. Winternitz.

(ii) A proof of Theorem 1 (using Brouwer’s fixed-point theorem), together with some related results, was given in B. J. Birch, On 3N points in a plane, Proc. Cambridge Philos. Soc., 55 (1959), 289–293.

(iii) A proof of Theorem 2, very similar to the one given in the
present paper, was found independently by P. C. Hammer; it is contained in a paper “Volumes cut from convex bodies by planes”, submitted to “Mathematika”.

(iv) The relation \( C(m, \frac{1}{2}) \neq \phi \) (resp. \( C(K, \frac{1}{2}) \neq \phi \)) holds for any distribution of masses (resp. convex body) with a center of symmetry. Obviously, \( C(m, \frac{1}{2}) \neq \phi \) is possible also for mass-distributions without a center. The conjecture (trivial for the plane) that \( C(K, \frac{1}{2}) \neq \phi \) characterizes centrally symmetric convex bodies was first established by Professor F. J. Dyson; it is hoped that a proof will be published soon.

(v) Results generalizing Theorem 1 were established by R. Rado in the paper, “A theorem on general measure”, J. London Math. Soc., 21 (1946), 291–300. Rado’s proof also uses Helley’s theorem, but in a fashion different from the one used in the present paper.

References


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