RELATION OF A DIRECT LIMIT GROUP TO ASSOCIATED VECTOR GROUPS

PAUL DANIEL HILL
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A set $M$ with a binary, transitive relation $<$ is said to be directed if for each pair $a, b$ in $M$, there is a $c$ in $M$ such that $a < c, b < c$. Let $\{G_\alpha\}_{\alpha \in M}$ be a collection of groups indexed by a directed set $M = \{a, b, \cdots; <\}$, and for each $a < b$ in $M$ let $h_\alpha^\circ$ be a homomorphism of $G_a$ into $G_b$. The homomorphisms are assumed to satisfy the relations

(i) $h_\alpha^\circ h_\beta^\circ = h_\gamma^\circ$ if $a < b < c$ and
(ii) if $a < a$, then $h_a^\circ$ is the identity.

We call such a system a direct system of groups and define a direct limit group of this system in the following manner. Two elements $g_a \in G_a$ and $\bar{g}_b \in G_b$ are said to be equivalent if there is a $c > a, b$ such that $h_\alpha^\circ(g_a) = h_c^\circ(\bar{g}_b)$. Let $g^*_a$ denote the collection of elements which are equivalent to $g_a$. Now given any two equivalence classes $g^*_a$ and $\bar{g}^*_b$, there exists a $c$ and elements $g_c, \bar{g}_c$ in $G_c$ such that $g^*_a = g_c^*$ and $\bar{g}^*_b = \bar{g}_c^*$. We define $g^*_a \cdot \bar{g}^*_b = (g_c \bar{g}_c)^*$. This multiplication is a well defined binary operation on the set, $G^*$, of equivalence classes. And it may be shown that $G^*$ is a (multiplicative) group, which we define to be the direct limit group of the given system.

Let $G = \prod G_a$ be the restricted direct product of the given groups $G_a$, and consider the groups $G_a$ as subgroups of $G$. An element in $G$ of the form $g_a h_\alpha^\circ(g_a)$ is called a relation. Let $H$ be the subgroup generated by the relations of $G$. Note that the inverse of a relation is a relation. By a "last" element of $M$ we mean an element $b$ such that $a < b$ for all $a$ in $M$. If $M$ contains no last element, it is immediate that given $a_1, a_2, \cdots, a_k$ in $M$, there exists a $b \in M$ with the property $a_i < b, a_i \neq b$ for $i = 1, 2, \cdots, k$.

**Lemma 1.** If $M$ contains no last element, the commutator group $K$ of $G$ is contained in $H$.

**Proof.** Let $x = g_{a_1} g_{a_2} \cdots g_{a_k}$ and $y = \bar{g}_{b_1} \bar{g}_{b_2} \cdots \bar{g}_{b_j}$ be arbitrary elements of $G$, where $a_m = a_n$ or $b_m = b_n$ implies that $m = n$. First choose $a$ with the property that $a_i < a, a_i \neq a$, and $b_i \neq a$ for all $i$. Then choose $b$ such that $b_i < b, b_i \neq b, a_i \neq b$, and $a \neq b$. We have

$$xyx^{-1}y^{-1} = \prod_{i=1}^k g_{a_i} \prod_{i=1}^l \bar{g}_{b_i} \prod_{i=1}^k g_{a_i}^{-1} \prod_{i=1}^l \bar{g}_{b_i}^{-1}$$

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Thus \( xyx^{-1}y^{-1} \in H \). Since \( H \) is a group, the lemma follows.

**Corollary 1.** If \( M \) contains no last element, then \( H \) is a normal subgroup of \( G \).

The following example shows that \( H \) may not be normal in \( G \) if \( M \) contains a last element.

**Example.** Suppose that \( M = \{1, 2; \leq\} \). Let \( G \) be the symmetric group on the set \( \{1, 2, 3\} \), and let \( G_1 \) be the subgroup of \( G \) of those elements fixing 3. Define \( h_1 \) to be the identity isomorphism of \( G_1 \) into \( G \). Then \( H \), a cyclic group of order 2, is generated by \( \langle (1, 2), (1, 2) \rangle \). It is, therefore, not normal in \( G \).

**Lemma 2.** If \( g_a \) in \( G_a \) is in \( H \), then there exists a \( b \) such that \( h_b(g_a) \in K_b \), the commutator group of \( G_b \).

**Proof.** In general, if \( x_a \) in \( G_a \) is the product \( x_{a_1}x_{a_2} \cdots x_{a_n} \) where \( x_{a_i} \in G_{a_i} \) and if \( b > a, a_i \) for \( i = 1, 2, \ldots, n \), then \( h_b^{a}(x_a) \) can be written as the product of the elements \( h_b^{x_1}(x_{a_1}), h_b^{x_2}(x_{a_2}), \ldots, h_b^{x_n}(x_{a_n}) \) in some order. This fact is easily proved by induction on \( n \). If \( n > 1 \), by the induction hypothesis we may as well assume that the factors \( x_{a_i} \) are nontrivial. Thus two of the factors must be contained in a single group \( G_{a_i} \). And the product \( x_{a_1}x_{a_2} \cdots x_{a_n} \) can be contracted to a product of the same form with one less factor by taking one of the new factors to be the product of two of the old and letting the other factors remain unchanged (except, possibly, for the order in which they appear).

Since \( g_a \) is in \( H \), it can be written in the form \( \prod_{i=1}^{k} g_a^{-1} h_b^{a_i}(g_a) \). Choose \( b \) such that \( b > a, b_i \) for \( i = 1, 2, \ldots, k \). Then

\[
\begin{align*}
h_b^{g_a}(g_a)K_b & = \prod_{i=1}^{k} h_b^{a_i}(g_a^{-1})h_b^{a_i}(g_a)K_b \\
& = \prod_{i=1}^{k} h_b^{a_i}(g_a^{-1})h_b^{a_i}(g_a)K_b = 1_b K_b = K_b,
\end{align*}
\]

which proves the lemma.

**Theorem 1.** If \( H \) is a normal subgroup of \( G \), then \( G/H \) is a homomorphism image of \( G^* \), where the kernel of the homomorphism is contained in the commutator subgroup, \( K^* \).
REMARKS. The theorem is well known [1] in case the groups $G_a$ are abelian. In this case $H$ is necessarily normal and $K^* = 1$ is the identity. Thus $G^* \cong G/H$, and we have two equivalent definitions for the direct limit.

Proof of theorem. Let $f$ be the mapping of $G^*$ into $G/H$, defined by: $g^*_a \rightarrow g_aH$. In order to show that $f$ is single-valued, let $g^*_a = g^*_b$. There exists $c > a$, $b$ such that $h_b^c(g_a) = h_b^c(g_b)$. Thus $g_b^{-1}g_a = g_a^{-1}h_b^c(g_a)h_b^{-1}(g_b)^{-1}g_b$. Since $h_b^c(g_b)^{-1}g_b = g_bh_b^c(g_b)^{-1}$, we have $g_a^{-1}g_b \in H$, which implies that $f$ is independent of the representative of $g_a^*$. The multiplicative property of $f$ is immediate. We next show that $f$ is onto. Let $gH \in G/H$ and let $g = g_ag_a \cdots g_a^k$, where the $a_i$'s are distinct. Choose $b$ such that $a_i < b$ for $i = 1, 2, \cdots, k$. If $a_i = b$ for some $i$, we may as well assume that $i = k$ since the $g_{a_i}$'s commute. For each $i$, $g_i^{-1}h_b^c(g_a^i) \in H$. Thus

$$g = \prod_{i=k}^{1} g_i^{-1}h_b^c(g_a^i) = \prod_{i=k}^{1} g_i^{-1} \prod_{i=k}^{1} h_b^c(g_a^i)$$

is in $H$, which implies that $gH = gH$. But $gH = \prod_{i=k}^{1} h_b^c(g_a^i)$ is in $G_a$. Hence $f((g)_H) = gH$, and $f$ is onto. Since $g_a \in K_a$, the commutator group of $G_a$, implies that $g_a^* \in K^*$, it follows from Lemma 2 that the kernel of $f$ is contained in $K^*$.

Theorem 2. If $M$ contains no last element, then $G^*/K^* \cong G/H$.

Proof. By Corollary 1, $H$ is normal in $G$. Thus by Theorem 1, we need only show that the kernel of $f$ is the whole commutator group, $K^*$. However, if $g_a^*$ is a commutator of $G^*$, then there exist a $b$ and a commutator $g_b \in K_b$ such that $g_a^* = g_b^*$. Since $K_b \subseteq K$, by Lemma 1 $g_b \in H$. Thus $f(g_b^*) = H$, and the theorem follows.

The limit group $G^*$ is abelian if and only if for every $a$ in $M$ and for every commutator $g_a$ of the group $G_a$, there exists a $b > a$ (depending on $g_a$) such that $h_b^c(g_a) = 1$. Also, under this condition the commutator subgroups, $K_a$, of the groups $G_a$ are contained in $H$, and $H$ is normal in $G$ since the conjugate of a generator of $H$ transformed by a general element of $G$

$$\{x_o\}_{o \in G} a^{-1}h_b^c(g_a)(x_o^{-1}) = x_a x_b a^{-1}h_b^c(g_a) x_a^{-1} x_b^{-1} = x_a a^{-1} x_a a^{-1} h_b^c(g_a) h_b^c(g_a)^{-1} x_b h_b^c(g_b) x_b^{-1}$$

remains in $H$.

Corollary 2. If the limit group $G^*$ is abelian, then $G^* \cong G/H$. Moreover, the converse holds if $M$ contains no last element.
A directed set $M = \{a, b, \cdots; <\}$ is said to be completely directed if for every $a$ in $M$ all but a finite number of $b$’s in $M$ satisfy the relation $a < b$. In particular, the positive integers are completely directed by $<$. Letting $G' = \prod G_a$ be the complete direct product of the given groups $G_a$, we have

**Lemma 3.** If $M$ is completely directed and has no last element, then $G^*$ is contained (in the sense of isomorphism) in the factor group $G'/G$.

**Proof.** Define a mapping $h$ of $G^*$ into $G'/G$ by: $g^*_a \rightarrow \{x_a\}_{a \in a}G$, where $x_a = g_a$ and $a < b$ implies that $x_b = h(x_a)$. The coordinate $x_b$ may be chosen as an arbitrary element of $G_b$ if $b$ fails to satisfy $a \leq b$. It may be shown that $h$ is a homomorphism with trivial kernel, which proves the lemma.

Letting $F$ be the inverse image of $h(G^*)$ under the natural homomorphism of $G'$ onto $G'/G$, we observe

**Corollary 3.** Let $M$ satisfy the conditions of Lemma 3, and let $G^*$ be abelian. Then in the chain

$$G' \supseteq F \supseteq G \supseteq H \supseteq 1$$

we have $F/G \cong G/H \cong G^*$.

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