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COMMITTATOR GROUPS OF MONOMIAL GROUPS

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COMMUTATOR GROUPS OF MONOMIAL GROUPS

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This paper is a study of the commutator groups of certain generalized permutation groups called complete monomial groups. In [2] Ore has shown that every element of the infinite permutation group is itsself a commutator of this group. Here it is shown that every element of the infinite complete monomial group is the product of at most two commutators of the infinite complete monomial group. The commutator subgroup of the infinite complete monomial group is itself, as is the case in the infinite symmetric group, [2]. The derived series is determined for a wide class of monomial groups.

Let H be an arbitrary group, and S a set of order B, $B \geqq d, d = \aleph_0$. Then one obtains a monomial group after the manner described in [1]. A monomial substitution over H is a linear transformation mapping each element x of S in a one-to-one manner onto some element of S multiplied by an element h of H, the multiplication being formal. The element h is termed a factor of the substitution. If substitution u maps x_i into $h_j x_j$, while substitution v maps x_j into $h_t x_t$, then the substitution uv maps x_i into $h_j h_t x_t$. A substitution all of whose factor are the identity e of H is called a permutation and the set of all permutations is a subgroup which is isomorphic to the symmetric group on B objects. A substitution which maps each element of S into itself multiplied by an element of H is called a multiplication. The set of all multiplications form a subgroup which is the strong direct product of groups H_α, each H_α isomorphic to H. Hereafter monomial substitutions which are permutations will be denoted by s, while those that are multiplications will be denoted by v. The monomial group whose elements are the monomial substitutions, restricted by the definitions of C and D as given below, will be denoted by $\sum(H; B, C, D)$, where the symbols in the name are to be interpreted as follows, H the given arbitrary group, B the order of the given set S, C a cardinal number such that the number of non-identity factors of any substitution of the group is less than C, D a cardinal number such that the number of elements of S being mapped into elements of S distinct from themselves by any substitution of the group is less than D. In the event $C = D = B^+, B^+$ the successor of B, the resulting monomial group is termed the complete monamial group generated by the given group H and the given set S. $S(B, M)$, $d \leqq M \leqq D$, will denote the subgroup of permutations which map fewer than M elements of S onto elements of S distinct from themselves, while $V(B, N)$,

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$d \leqq N \leqq C$, will denote the subgroup of multiplications which have fewer than N nonidentity factors. In particular $S(B, d)$ denotes the subgroup of finite permutations and $V(B, d)$ the subgroup consisting of those multiplications which have finitely many nonidentity factors. The concept of alternating as associated with permutation groups may be extended in an obvious manner to monomial groups. $A(B, d)$ will denote the alternating subgroup of the permutation group $S(B, d)$, while $\sum_A(H; B, d, d)$ will denote the alternating subgroup of the monomial group $\sum(H; B, d, d)$. Any substitution may be written as the product of a multiplication and a permutation. Hence we may write $\sum(H; B, C, D) = V(B, C) \cup S(B, D)$, where $\cup$ here and throughout will mean group generated by the set. G' will be used to denote the commutator subgroup of the group G.

THEOREM 1. *The commutator subgroup* $V'(B, C), d \leqq C \leqq B^+$, *of* $V(B, C)$ *is the set of all elements*

$$v' = (h'_1, h'_2, h'_3, \cdots), h'_i \in H' ,$$

where there exists an integer N *such that each* h'_i *is the product of* N *or fewer commutators of* H.

Proof. The theorem follows from the fact that $V(B, C)$ is the strong direct product, each of whose summands is isomorphic to H, together with the remark following the lemma page 308 of [2].

THEOREM 2. *The commutator subgroup* $S'(B, C), d < C \leqq B^+$, *of* $S(B, C)$ *is* $S(B, C)$. *The commutator subgroup* $S'(B, d)$ *of* $S(B, d)$ *is* $A(B, d)$.

The proof is contained in [2].

THEOREM 3. *The commutator subgroup* $\sum'(H; B, d, d)$ *of* $\sum(H; B, d, d)$ *is* $A(B, d) \cup V^+(B, d)$ *where* $V^+(B, d)$ *is the set of all elements of* $V(B, d)$ *whose product of factors is a member of* H'.

Proof. By reason of Theorem 2 we have
$\sum'(H; B, d, d) \supset A(B, d)$, and that
$\sum'(H; B, d, d) \supset V^+(B, d)$

will now be demonstrated.

If h_i is the only nonidentity factor of the multiplication v_i, then the commutator $v_i s v_i^{-1} s$, where $s = (x_i, x_j)$, is a multiplication whose only nonidentity factors are h_i and h_i^{-1}. It then follows that any multiplication v of $V^+(B, d)$ with n nonidentity factors can be written as the product of $n + 1$ multiplications, n of which are of the type of the commutator described above, and the remaining member having as its only nonidentity factor the product of the factors of v. But the first n members of the product belong to $\sum'(H; B, \mathrm{d}, \mathrm{d})$, while the other member of the product is an element of $V'(B, d)$, by reason of Theorem 1, and hence

$$\sum'(H; B, d, d) \supset V^{+}(B, d), \text{ since } V'(B, d) \subset \sum'(H; B, d, d).$$

Then

$$\sum'(H; B, d, d) \supset V^{+}(B, d) \cup A(B, d).$$

Since G/G' is abelian for any group G, and G' is the smallest group for which this is true, to demonstrate that

$$\sum(H; B, d, d)/V^{+}(B, d) \cup A(B, d)$$

is abelian will imply that

$$\sum'(H; B, d, d) \subset V^{+}(B, d) \cup A(B, d) ,$$

and the conclusion of the theorem will follow.

That $V^{+}(B, d) \supset V'(B, d)$ follows from the definition of $V^{+}(B, d)$, and hence $V(B, d)/V^{+}(B, d)$ is abelian. Therefore any two multiplications commute mod $V^{+}(B, d) \cup A(B, d)$. Since $A(B, d)$ consists of all even permutations there are but two cosets of $A(B, d)$ in $S(B, d)$, namely, $A(B, d)$ and $(x_1, x_2)A(B, d)$. Thus any element of the factor group $\sum(H; B, d, d)/V^{+}(B, d) \cup A(B, d)$ has one of the forms

$$v[V^{+}(B, d) \cup A(B, d)]$$

or

$$v(x_1, x_2)[V^{+}(B, d) \cup A(B, d)], \quad v \in V(B, d) .$$

But $v(x_1, x_2)v^{-1}(x_1, x_2)$ is the commutator $(h_1h_2^{-1}, h_2h_1^{-1}, e, \cdots)$ which belongs to $V^{+}(B, d)$. That is, (x_1, x_2) and v commute mod $[V^{+}(B, d) \cup A(B, d)]$, and hence $\sum(H; B, d, d)/V^{+}(B, d) \cup A(B, d)$ is abelian, which implies $\sum'(H; B, d, d) \subset V^{+}(B, d) \cup A(B, d)$, and we have

$$\sum'(H; B, d, d) = V^{+}(B, d) \cup A(B, d) .$$

The following theorem asserts that the derived series for $\sum(H; B, d, d)$ consists of but two distinct terms.

THEOREM 4. *The commutator subgroup* $\sum''(H; B, d, d)$ *of* $\sum'(H; B, d, d)$ *is* $\sum'(H; B, d, d)$.

Proof. $A(B, d) = A'(B, d)$, as was demonstrated in Theorem 7 of [2], and hence $\sum''(H; B, d, d)$ contains $A(B, d)$.

Consider elements v_1 and v_2 of $\sum'(H; B, d, d)$, where the factors of v_1 are all e except the first two and they are inverses of one another, and the factors of v_2 are all e except the first and third and they are inverses of one another. The commutator $v_1v_2v_1^{-1}v_2^{-1}$, which is an element of $\sum''(H; B, d, d)$, has as its first factor a commutator of H and all other factors e. It then follows that any element of $V'(B, d)$ is the product of elements of $\sum''(H; B, d, d)$ and hence is an element of $\sum''(H; B, d, d)$. That is $\sum''(H; B, d, d) \supset V'(B, d)$. Then one can in the manner described in the first part of Theorem 3 write any element v of $V^+(B, d)$ as the product of $n + 1$ elements, each member of the product being an element of $\sum''(H; B, d, d)$. That is $\sum''(H; B, d, d)$ contains $V^+(B, d)$, and hence $\sum''(H; B, d, d)$ contains $V^+(B, d) \cup A(B, d) = \sum'(H; B, d, d)$.

THEOREM 5. *The commutator subgroup*

$$\textstyle\sum_A'(H; B, d, d) \textit{ of } \sum_A(H; B, d, d) \text{ is } V^+(B, d) \cup A(B, d) .$$

This theorem together with Theorem 3 states that $\sum(H; B, d, d)$ has for its commutator subgroup $\sum_A(H; B, d, d)$. This is the analogue for monomial groups of the result Ore obtains for permutation groups in [2], and as stated in the second part of Theorem 2.

Proof. We have

$$\textstyle\sum'(H; B, d, d) \subset \sum_A(H; B, d, d) \subset \sum(H; B, d, d) ,$$

hence,

$$\textstyle\sum''(H; B, d, d) \subset \sum_A'(H; B, d, d) \subset \sum'(H; B, d, d).$$

Then by reason of Theorem 4,

$$\textstyle\sum'(H; B, d, d) = \sum''(H; B, d, d) = V^+(B, d) \cup A(B, d) .$$

Hence $\sum_A'(H; B, d, d) = V^+(B, d) \cup A(B, d)$.

THEOREM 6. *The commutator subgroup* $\sum'(H; B, C, D)$, $d < C \leqq D \leqq B^+$, *of* $\sum(H; B, C, D)$ *is* $\sum(H; B, C, D)$.

This theorem is also an analogue of a result Ore obtains in [2] for permutation groups as stated in the first part of Theorem 2.

Proof. It is shown in [2] that the commutator subgroup $S'(B, D)$ of $S(B, D)$ is $S(B, D)$. Hence $\sum'(H; B, C, D)$ contains $S(B, D)$. The conclusion of the theorem will then follow if it can be demonstrated that $\sum'(H; B, C, D) \supset V(B, C)$. Let

$$s = (\cdots, x_{-1}, x_0, x_1, \cdots)$$

and

$$v = (\cdots, h_{-1}, h_0, h_1, \cdots)$$

be elements of $\sum(H; B, C, D)$. Then the commutator $svs^{-1}v^{-1}$ an element of $\sum'(H; B, C, D)$ has the form

$$(\cdots, h_0 h_{-1}^{-1}, h_1 h_0^{-1}, h_2 h_1^{-1}, \cdots) .$$

Let

$$v_c = (\cdots, c_{-1}, c_0, c_1, \cdots)$$

be an arbitrary element of $V(B, C)$, and consider the following set of equations.

$$\cdots, h_0 h_{-1}^{-1} = c_{-1}, h_1 h_0^{-1} = c_0, h_2 h_1^{-1} = c_1, \cdots$$

This set of equations has solutions,

$$h_0 = c_{-1}, h_{-1} = e, h_n = c_{n-1} h_{n-1}, h_{-n} = \left[\prod_{i=2}^{n} c_{-i} \right]^{-1} .$$

Then if the factors of v be represented in terms of the factors of v_c as indicated above, we see that

$$svs^{-1}v^{-1} = v_c \in \sum'(H; B, C, D) ,$$

and hence $\sum'(H; B, C, D)$ contains $V(B, C)$, and therefore

$$\sum(H; B, C, D) = \sum'(H; B, C, D) .$$

COROLLARY 1. *Any element u of $\sum(H; B, C, D)$, $d < C \leqq D \leqq B^+$, is the product of at most two commutators.*

Proof. Every element of $S(B, D)$ is a commutator of $S(B, D)$, as was shown in [2]. Every element of $V(B, C)$ is a commutator of $\sum(H; B, C, D)$, as was shown in Theorem 6. Therefore any element of $\sum(H; B, C, D)$ which is either a multiplication or a permutation is a commutator. But every element of $\sum(H; B, C, D)$ maybe written as the product of a multiplication and a permutation and consequently may be written as the product of two commutators.

To see that the assertion that every element of $\sum(H; B, C, D)$ is the product of at most two commutators is the strongest possible, suppose every element of $\sum(H; B, C, D)$ is a commutator. Let

$$u \in \sum(H; B, d, d) \subset \sum(H; B, C, D) .$$

Then $u = u_1 u_2 u_1^{-1} u_2^{-1}$, u_1 and u_2 elements of $\sum(H; B, C, D)$. But since u belongs to $\sum(H; B, d, d)$ we can choose a u_1^* and u_2^* in $\sum(H; B, d, d)$ by causing u_1 and u_2 to become the map of x_i into ex_i except for those maps which yield the permutation and nonidentity factors of u. It then follows that u is an element of $\sum'(H; B, d, d)$, and hence $\sum(H; B, d, d) = \sum'(H; B, d, d)$. But this is a contradiction to Theorem 3.

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Pacific Journal of Mathematics

Vol. 10, No. 4 December, 1960

M. Altman, *An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space* 1107
Nesmith Cornett Ankeny, *Criterion for rth power residuacity* 1115
Julius Rubin Blum and David Lee Hanson, *On invariant probability measures I* 1125
Frank Featherstone Bonsall, *Positive operators compact in an auxiliary topology* 1131
Billy Joe Boyer, *Summability of derived conjugate series* 1139
Delmar L. Boyer, *A note on a problem of Fuchs* 1147
Hans-Joachim Bremermann, *The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces* 1149
Andrew Michael Bruckner, *Minimal superadditive extensions of superadditive functions* 1155
Billy Finney Bryant, *On expansive homeomorphisms* 1163
Jean W. Butler, *On complete and independent sets of operations in finite algebras* 1169
Lucien Le Cam, *An approximation theorem for the Poisson binomial distribution* 1181
Paul Civin, *Involutions on locally compact rings* 1199
Earl A. Coddington, *Normal extensions of formally normal operators* 1203
Jacob Feldman, *Some classes of equivalent Gaussian processes on an interval* 1211
Shaul Foguel, *Weak and strong convergence for Markov processes* 1221
Martin Fox, *Some zero sum two-person games with moves in the unit interval* 1235
Robert Pertsch Gilbert, *Singularities of three-dimensional harmonic functions* 1243
Branko Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes* 1257
Sidney Morris Harmon, *Regular covering surfaces of Riemann surfaces* 1263
Edwin Hewitt and Herbert S. Zuckerman, *The multiplicative semigroup of integers modulo m* 1291
Paul Daniel Hill, *Relation of a direct limit group to associated vector groups* 1309
Calvin Virgil Holmes, *Commutator groups of monomial groups* 1313
James Fredrik Jakobsen and W. R. Utz, *The non-existence of expansive homeomorphisms on a closed 2-cell* 1319
John William Jewett, *Multiplication on classes of pseudo-analytic functions* 1323
Helmut Klingen, *Analytic automorphisms of bounded symmetric complex domains* 1327
Robert Jacob Koch, *Ordered semigroups in partially ordered semigroups* 1333
Marvin David Marcus and N. A. Khan, *On a commutator result of Taussky and Zassenhaus* 1337
John Glen Marica and Steve Jerome Bryant, *Unary algebras* 1347
Edward Peter Merkes and W. T. Scott, *On univalence of a continued fraction* 1361
Shu-Teh Chen Moy, *Asymptotic properties of derivatives of stationary measures* 1371
John William Neuberger, *Concerning boundary value problems* 1385
Edward C. Posner, *Integral closure of differential rings* 1393
Marian Reichaw-Reichbach, *Some theorems on mappings onto* 1397
Marvin Rosenblum and Harold Widom, *Two extremal problems* 1409
Morton Lincoln Slater and Herbert S. Wilf, *A class of linear differential-difference equations* 1419
Charles Robson Storey, Jr., *The structure of threads* 1429
J. François Treves, *An estimate for differential polynomials in $\partial/\partial z_1$, , $\cdots$, ∂/∂z_n* 1447
J. D. Weston, *On the representation of operators by convolutions integrals* 1453
James Victor Whittaker, *Normal subgroups of some homeomorphism groups* 1469