

# Pacific Journal of Mathematics

**ANALYTIC AUTOMORPHISMS OF BOUNDED SYMMETRIC  
COMPLEX DOMAINS**

HELMUT KLINGEN

# ANALYTIC AUTOMORPHISMS OF BOUNDED SYMMETRIC COMPLEX DOMAINS

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In a former paper [2] I determined the full group of one-to-one analytic mappings of a bounded symmetric Cartan domain [1]. Those investigations were incomplete, because it was impossible to treat the second Cartan-type of  $n(n-1)/2$  complex dimensions for odd  $n$  by this method. The present note is devoted to a new shorter proof of the former result ( $n$  even), which furthermore covers the remaining case of odd  $n$ .

Take the complex  $n(n-1)/2$ -dimensional space of skew symmetric  $n$ -rowed matrices  $Z$ . The irreducible bounded symmetric Cartan space in question is the set  $\mathcal{E}_n$  of those matrices  $Z$ , for which

$$I + Z\bar{Z} > 0, \quad Z' = -Z,$$

is positive definite. Here  $I$  is the  $n$  by  $n$  unit matrix. Obviously  $\mathcal{E}_2$  is the unit circle. It is easy to see that analytic automorphisms of  $\mathcal{E}_n$  are described by the group  $\phi$  of the mappings

$$(1) \quad W = (AZ + B)(-\bar{B}Z + \bar{A})^{-1},$$

where the  $n$ -rowed matrices  $A, B$  fulfill

$$M^*KM = K \quad \text{with} \quad M = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \quad K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Here  $M^*$  denotes the conjugate transpose of  $M$ . For  $n = 4$

$$W = \tilde{Z}$$

is a further analytic automorphism, where  $\tilde{Z}$  arises from  $Z$  by interchanging the elements  $z_{14}$  and  $z_{23}$ ,

$$\tilde{Z} = \begin{pmatrix} 0 & z_{12} & z_{13} & z_{23} \\ -z_{12} & 0 & z_{14} & z_{24} \\ -z_{13} & -z_{14} & 0 & z_{34} \\ -z_{23} & -z_{24} & -z_{34} & 0 \end{pmatrix}.$$

For  $W\bar{W}$  and  $\tilde{Z}\bar{\tilde{Z}}$  have the same characteristic roots. But this mapping is not contained in  $\phi$ , since  $CZ = \tilde{Z}D$  cannot be satisfied identically in  $Z$  by non-singular constant matrices  $C, D$ . On the other hand the following theorem holds.

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**THEOREM.** *Each analytic automorphism of  $\mathcal{E}_n$  can be written as  $W = f(Z)$  or  $W = f(\bar{Z})$  (only for  $n = 4$ ) with  $f \in \phi$ .*

Therefore the group  $\phi$  is already the full group of analytic automorphisms for  $n \neq 4$ . Only in the exceptional case  $n = 4$  there are the further mappings  $W = f(\bar{Z})$ , which together with  $\phi$  form the full group of analytic automorphisms. The proof of this theorem consists of two parts. The first analytic part is a reproduction of my former proof [2], which will be given here again for completeness, the second part is of algebraic character.

The group  $\phi$  acts transitively on  $\mathcal{E}_n$ . For take an arbitrary point  $Z_1$  of  $\mathcal{E}_n$ , choose the matrix  $A$  such that

$$A(I + Z_1\bar{Z}_1)A^* = I$$

and define  $B = -AZ_1$ . Then (1) maps  $Z$  into 0. Therefore it is sufficient to investigate the stability group of the zero matrix.

First we show that each analytic one-to-one mapping  $W = W(Z)$  of  $\mathcal{E}_n$  with the fixed point 0 is linear. For an arbitrary point  $Z_1 \in \mathcal{E}_n$  let  $r_1, \dots, r_n, 0 \leq r_1 \leq \dots \leq r_n < 1$ , be the characteristic roots of  $Z_1 Z_1^*$ . Then also  $tZ_1$  belongs to  $\mathcal{E}_n$ , if  $t$  is a complex number with  $t\bar{t}r_n < 1$ . Consequently there exists a power series expansion

$$(2) \quad W(tZ_1) = \sum_{k=1}^{\infty} t^k W_k(Z_1), \quad t\bar{t}r_n < 1.$$

The elements of the skew-symmetric matrices  $W_k(Z_1)$  are homogeneous polynomials of degree  $k$  in the independent elements of  $Z_1$ . Because of  $I + W(tZ_1)\bar{W}(tZ_1) > 0$  for  $\bar{t}t = 1$ , one obtains from (2)

$$(3) \quad \frac{1}{2\pi i} \int_{|t|=1} (I + W(tZ_1)\bar{W}(tZ_1)) \frac{dt}{t} = I + \sum_{k=1}^{\infty} W_k(Z_1)\bar{W}_k(Z_1) > 0$$

and in particular  $I + \bar{W}_1(Z_1)W_1(Z_1) > 0$ . Therefore the linear function  $W_1(Z)$  is an analytic mapping of  $\mathcal{E}_n$  into itself. Its determinant  $D$  is at the same time the Jacobian of the function  $W(Z)$  with respect to  $Z$ . By interchanging  $Z$  and  $W$  it can be assumed  $D\bar{D} \geq 1$ . Consequently  $W(Z)$  is an analytic automorphism of  $\mathcal{E}_n$  and even maps the boundary onto itself. Take now in particular

$$(4) \quad Z_1 = U'PU, \quad P = [(0), p_1 F, \dots, p_m F], \quad F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with an unitary matrix  $U, m = [n/2]$ .  $P$  shall be the matrix, which is built up by the two-rowed blocks  $p_1 F, \dots, p_m F$  and possibly by the element 0 along the main diagonal.  $Z_1$  belongs to the interior of  $\mathcal{E}_n$ , if  $-1 < p_k < 1$  ( $k = 1, \dots, m$ ), and to the boundary, if  $-1 \leq p_k \leq 1$  ( $k =$

$1, \dots, m$ ) and  $p_k = \pm 1$  for at least one  $k$ . Now  $|I + W_1(Z_1)\bar{W}_1|$  is a polynomial in  $p_1, \dots, p_m$  of total degree  $4m$  and on the other hand (see [2], Lemma 4) the square of a polynomial. As  $|I + W_1(Z_1)\bar{W}_1|$  vanishes on the boundary of  $\mathcal{E}_n$ , this polynomial is divisible by

$$|I + Z_1\bar{Z}_1| = \prod_{k=1}^m (1 - p_k^2)^2 .$$

Because the constant terms and the degrees of both polynomials are equal, one obtains

$$(5) \quad |I + W_1(Z_1)\bar{W}_1| = |I + Z_1\bar{Z}_1|$$

even identically in  $Z_1$ ; for each skew-symmetric matrix  $Z_1$  permits a representation (4) (see [2], Lemma 3). On account of (5) and the linearity of  $W_1$  the matrices  $W_1\bar{W}_1$  and  $Z\bar{Z}$  always have the same characteristic roots and this implies

$$(6) \quad W_1(Z) = U'ZU$$

with unitary  $U$ , which for the present still depends on  $Z$ .

Put now

$$Z = uX, \quad X = U'_1 [e^{i\zeta_1}F, \dots, e^{i\zeta_r}F, (0)]U_1, \quad 0 \leq u \leq 1,$$

with real variables  $\zeta_1, \dots, \zeta_r$ . Then  $Z \in \mathcal{E}_n$  and by (6)

$$W_1W_1^* = u^2U'U'_1 \begin{pmatrix} I^{(n-1)} & 0 \\ 0 & (0) \end{pmatrix} \bar{U}_1\bar{U}$$

for all  $u$  between 0 and 1. Because of (3) one obtains

$$\bar{U}_1\bar{U}(I + W_1\bar{W}_1 + W_k\bar{W}_k)U'U'_1 > 0 \quad (k = 2, 3, \dots) .$$

If  $u$  tends to 1, one gets

$$\begin{pmatrix} 0 & 0 \\ 0 & (1) \end{pmatrix} + \bar{U}_1\bar{U}W_k\bar{W}_kU'U'_1 > 0 ,$$

hence  $W_k(X) = 0$ . As  $W_k$  is a polynomial,  $W_k(Z)$  even vanishes identically in  $Z$ . Therefore the stability group of  $\mathcal{E}_n$  is linear.

The investigation of  $W = W_1(Z)$  is now a purely algebraic problem. The representation (6) shows that  $\text{rank } W = \text{rank } Z$  and beyond this the equality of the characteristic roots of  $W\bar{W}$  and  $Z\bar{Z}$ . These properties will be used in order to determine  $W(Z)$  explicitly. We have to prove

$$(7) \quad W(Z) = U'ZU \quad \text{or} \quad W(Z) = U'\tilde{Z}U$$

with unitary constant  $U$ , where the second type only occurs for  $n = 4$ . The proof of this fact will be given by induction. The assertion (7) is trivial for the unit circle ( $n = 2$ ). Let us assume its correctness for  $2, 3, \dots, n - 1$  and consider  $\mathcal{E}_n$ . Write the linear mapping  $W(Z)$  of  $\mathcal{E}_n$  onto itself as

$$W = \sum_{k < l} z_{kl} A_{kl}$$

with constant skew-symmetric  $n$  by  $n$  matrices  $A_{kl}$ . Because of the equality of the characteristic roots of  $WW^*$  and  $ZZ^*$  the hermitian matrix  $A_{kl}A_{kl}^*$  has  $1, 1, 0, \dots, 0$  as characteristic roots. Therefore after unitary transformation of  $W$  we can assume  $A_{12} = E_{12}$ , where in general  $E_{kl}$  denotes the skew-symmetric matrix the elements of which are all zero besides the element in the  $k$ th row and  $l$ th column and the element in the  $l$ th row and  $k$ th column, which are  $1$  respectively  $-1$ . Since  $\text{tr}(A_{12}\bar{A}_{kl}) = 0$  for  $(k, l) \neq (1, 2)$ , one obtains

$$A_{kl} = \begin{pmatrix} 0^{(2)} & * \\ * & * \end{pmatrix} \quad (k, l) \neq (1, 2).$$

$A_{12} = E_{12}$  does not change, if  $W$  is transformed by

$$\begin{pmatrix} U^{(2)} & 0 \\ 0 & V \end{pmatrix}$$

with unitary  $U, V, |U| = 1$ . Therefore

$$A_{13} = \begin{pmatrix} 0^{(2)} & B \\ -B' & C \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

can be assumed. From  $\text{rank } W = \text{rank } Z$  identically in  $Z$  one obtains possibly after unitary transformation  $A_{13} = E_{13}$ .

For  $A_{14} = (a_{kl})$  we get two possibilities. First the equation  $\text{tr}(A_{12}\bar{A}_{14}) = \text{tr}(A_{13}\bar{A}_{14}) = 0$  implies  $a_{12} = a_{13} = 0$ . After unitary transformation all the elements of the first row besides  $a_{14}$  are zero. Then take only the elements  $z_{12}, z_{13}, z_{14}$  of  $Z$  distinct from zero; from  $\text{rank } W = \text{rank } Z = 2$  one sees

$$A_{14} = E_{14} \quad \text{or} \quad A_{14} = E_{23}.$$

By a similar consideration  $A_{1\nu}$  turns out to be  $E_{1\nu}$  or  $E_{23}$ . But actually for  $\nu > 4$  the second possibility  $A_{1\nu} = E_{23}$  may not occur. For  $A_{14} = A_{1\nu} = E_{23}$  is impossible because of  $\text{tr}(A_{14}\bar{A}_{1\nu}) = 0$ . If  $A_{14} = E_{14}, A_{1\nu} = E_{23}$ , choose only the elements  $z_{1\nu}, z_{14} \neq 0$ , then  $\text{rank } Z = 2$  but  $\text{rank } W = 4$ . Therefore  $A_{1\nu} = E_{1\nu}$  ( $\nu \neq 4$ ),  $A_{14} = E_{14}$  or  $E_{23}$ . Furthermore  $A_{14} = E_{23}$  may only happen if  $n = 4$ . For assume  $A_{14} = E_{23}, A_{15} = E_{15}$  and take only the elements  $z_{14}, z_{15} \neq 0$ . This implies  $\text{rank } Z = 2$  but  $\text{rank } W = 4$ .

Let us summarize our results. After a suitable unitary transformation  $W$  can be written as

$$W = \begin{pmatrix} 0 & z' \\ -z & L(QZ_0) \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & z' \\ -z & Z_0 \end{pmatrix},$$

besides the exceptional case  $n = 4, A_{14} = E_{23}$ . Now  $L(Z_0)$  is an analytic automorphism of  $\mathcal{E}_{n-1}$  with the fixed point 0. For  $n = 3$  we know  $L(Z_1) = e^{i\zeta}Z_1$  with a real constant  $\zeta$ . Therefore  $W = U'ZU$  with a constant unitary matrix  $U$ , which is the theorem for  $n = 3$ . For  $n > 5$  the induction hypothesis shows

$$W = \begin{pmatrix} 0 & z'U' \\ -Uz & Z_0 \end{pmatrix}$$

with constant unitary  $U$ . From the equality

$$\text{rank } W = \text{rank } Z$$

$U$  turns out to be a diagonal matrix. Finally consider the sum of the two-rowed principal minors of  $W\bar{W}$  and  $Z\bar{Z}$ . These two quantities are equal identically in  $Z$  because of the fact that  $W\bar{W}$  and  $Z\bar{Z}$  have the same characteristic roots. By this identity one obtains  $U = aI$  with a complex number  $a$  of absolute value 1, which again proves our theorem.

There still remain the cases  $n = 4$  and 5. For  $n = 4, A_{14} = E_{14}$  we can use the reasoning above. Let  $A_{14} = E_{23}$ ; since

$$\text{tr}(A_{1\nu}\bar{A}_{23}) = \text{tr}(A_{1\nu}\bar{A}_{24}) = \text{tr}(A_{1\nu}\bar{A}_{34}) = 0 \quad (\nu = 2, 3, 4)$$

$W$  only differs from  $\tilde{Z}$  in the last row, where a linear combination of  $z_{23}, z_{24}, z_{34}$  appears. The identity between the ranks of  $Z$  and  $W$  shows  $w_{14} = a_1z_{23}, w_{24} = a_2z_{24}, w_{34} = a_3z_{34}$ . Now it is easy to compute the sum of the two-rowed principal minors of  $W\bar{W}$  and  $Z\bar{Z}$ . This computation shows again the assertion for  $n = 4$ .

For  $n = 5$  we know by the induction hypothesis

$$L(Z_0) = U'Z_0U \quad \text{or} \quad L(Z_0) = U'\tilde{Z}_0U$$

with constant unitary  $U$ . The first case can be treated as above. In the second case one obtains

$$W = \begin{pmatrix} 0 & z'U' \\ -Uz & Z_0 \end{pmatrix}.$$

Choose once only  $z_{14}, z_{24} \neq 0$ , then only  $z_{14}, z_{34}, z_{45} \neq 0$ . In any case  $\text{rank } Z = 2$ , hence  $\text{rank } W = 2$ . But this implies that all the elements of the third column of  $U$  vanish, which is a contradiction to the unitary character of  $U$ . This final remark completes the proof.

## REFERENCES

1. E. Cartan, *Sur les domaines bornés homogènes de l'espace de  $n$  variables complexes*, Oeuvres, partie 1, 1259–1304.
2. H. Klingen, *Diskontinuierliche Gruppen in symmetrischen Räumen*, Mathematische Annalen **129** (1955), 470–488.

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