ON A COMMUTATOR RESULT OF TAUSKY AND ZASSENHAUS

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1. Introduction and results. Let $M_n$ denote the set of $n$-square matrices over a field $F$. For $A, B$ in $M_n$ let $[A, B] = AB - BA'$, where $A'$ is the transpose of $A$ and define inductively

$$[A, B]_k = [A, [A, B]_{k-1}].$$

If $P^{-1}JP = A$, then

$$[A, X] = [P^{-1}JP, X] = P^{-1}[J, PXP'](P^{-1})',$n

and similarly

$$[A, X]_k = P^{-1}[J, PXP']_k(P^{-1})'.$n

Now for a fixed $A$ let $T$ be the linear map of $M_n$ into itself defined by

$$T(Y) = [A, Y],$$

and (1.1) implies that

$$T^k(Y) = [A, Y]_k.$$

In a recent paper [1], Taussky and Zassenhaus showed that $A$ is non-derogatory if and only if any nonsingular $X$ in the null space of $T$ is symmetric. In this note we investigate the structure of the null space of both $T$ and $T^2$ for arbitrary $A$.

Enlarge the field $F$ to include $\lambda_i, i = 1, \ldots, p$, the distinct eigenvalues of $A$, and let $(x - \lambda_i)^{r_{ij}}, j = 1, \ldots, n_i, e_{ij} > \cdots > e_{i_1}, i = 1, \ldots, p$ be the distinct elementary divisors of $A$ where $(x - \lambda_i)^{r_{ij}}$ appears with multiplicity $r_{ij}$. Set $m_i = \sum_j r_{ij}e_{ij}$, the algebraic multiplicity of $\lambda_i$. Let $\gamma(T)$ denote the null space of $T$, $\sigma(T)$ denote the subspace of symmetric matrices in $\gamma(T)$, and $\eta(T)$ denote the subspace of skew-symmetric matrices in $\gamma(T)$. We show that

$$\dim \gamma(T) = \sum_{i=1}^{p} \left( \sum_{j=1}^{n_i} \left( r_{ij}^2 e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right),$$

$$\dim \sigma(T) = \frac{1}{2} \sum_{i=1}^{p} \left( \sum_{j=1}^{n_i} \left( r_{ij}(r_{ij} + 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right).$$

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(1.6) \[ \dim \gamma(T) = \sum_{i=1}^{p} \left( \sum_{j=1}^{n_i} \left( r_{ij}^2 (2e_{ij} - 1) + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right), \]

(1.7) \[ \dim \sigma(T) = \frac{1}{2} \sum_{i=1}^{p} \left( \sum_{j=1}^{n_i} \left( r_{ij}^2 (2e_{ij} - 1) + r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right). \]

In case \( A \) is nonderogatory, \( n_i = 1, r_{ij} = 1, i = 1, \ldots, p \) and (1.4) and (1.5) reduce to

\[ \dim \gamma(T) = n = \dim \sigma(T). \]

Thus every matrix \( X \) satisfying

(1.8) \[ AX = XA' \]

where \( A \) is non-derogatory is symmetric, the result in [1]. Moreover, if every matrix \( X \) satisfying (1.8) is symmetric then \( \dim \gamma(T) = \dim \sigma(T) \). Using the formulas (1.4) and (1.5) we see that this condition implies that

\[ \sum_{i=1}^{p} \sum_{j=1}^{n_i} (r_{ij}^2 - r_{ij})e_{ij} + 2 \sum_{i=1}^{p} r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} = 0. \]

Now since \( r_{ij}, e_{ij} \) and \( n_i \) are all positive integers we conclude that \( r_{ij} = 1, j = 1, \ldots, n_i \) and \( n_i = 1 \). That is, there is only one elementary divisor corresponding to each eigenvalue. Hence, if every matrix \( X \) satisfying (1.8) is symmetric then \( A \) is non-derogatory, a result also found in [1].

We also show in this case that \( \gamma(T) \) consists of matrices of the form \( PXP' \) where \( P \) is fixed (depending on \( A \)) and \( X \) is persymmetric, (i.e. all the entries of \( X \) on each line perpendicular to the main diagonal are equal).

We next note that \( \gamma(T) = \sigma(T) + \gamma(T) \) (direct) and \( \gamma(T') = \sigma(T') + \gamma(T') \) (direct). The first statement is easy to show; we indicate the brief proof of the second statement:

Since \( X = \frac{X + X'}{2} + \frac{X - X'}{2} \), if \( X \in \gamma(T) \), then

\[ T'(X + X') = \left[ A, [A, X + X'] \right] \]
\[ = \left[ A, [A, X] + [A, X'] \right] \]
\[ = \left[ A, [A, X] + [A, X'] \right] \]
\[ = T'(X) - [A, [A, X']] \]
\[ = \left[ A, [A, X] \right]' \]
\[ = (T'(X))' = 0. \]

Similarly, \( T'(X - X') = 0 \). Thus any \( X \in \gamma(T') \) is expressible uniquely as a sum of two elements, one in \( \sigma(T') \) and the other in \( \gamma(T') \). Hence
(1.9) \[ \dim \gamma(T) = \dim \eta(T) - \dim \sigma(T) \]
\[ = \frac{1}{2} \sum_{i=1}^{p} \left[ \sum_{j=1}^{n_i} \left\{ r_{ij} (r_{ij} - 1) e_{i,j} + 2 r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right\} \right], \]

(1.10) \[ \dim \gamma(T^2) = \dim \eta(T^2) - \dim \sigma(T^2) \]
\[ = \frac{1}{2} \sum_{i=1}^{p} \left[ \sum_{j=1}^{n_i} \left\{ r_{ij}^2 (2 e_{i,j} - 1) - r_{ij} + 4 r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right\} \right]. \]

In case \( A \) is non-derogatory, (1.6), (1.7) and (1.10) reduce to
\[ \dim \gamma(T^2) = 2n - p, \]
\[ \dim \sigma(T^2) = n, \]
\[ \dim \gamma(T^2) = n - p. \]

We thus conclude that unless all the eigenvalues of \( A \) are distinct (\( p = n \)) there exist skew-symmetric matrices \( X \) satisfying

(1.11) \[ A^2 X - 2AXA' + X(A')^2 = 0. \]

If \( p = n, \) and \( A \) is non-derogatory
\[ \dim \gamma(T^2) = n = \dim \sigma(T^2) \]
and any matrix \( X \) satisfying (1.11) is symmetric.

On the other hand suppose
\[ \dim \gamma(T^2) = \dim \sigma(T^2). \]

From (1.6) and (1.7) we conclude that
\[ \sum_{i=1}^{p} \left[ \sum_{j=1}^{n_i} \left\{ r_{ij} (2 e_{i,j} - 1) - r_{ij} + 4 r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right\} \right] = 0. \]

Hence \( n_i = 1, \) \( r_{ij} = 1, \) \( e_{ik} = 1 \) and we conclude that \( p = n. \) That is, if every matrix \( X \) satisfying (1.11) is symmetric then the eigenvalues of \( A \) are distinct.

We show finally (Theorem 2) that if \( A \) is an \( n \)-square matrix with \( p \) distinct eigenvalues then both \( \dim \gamma(T) \) and \( \dim \gamma(T^2) \) are at most \( \frac{1}{2}(n - p)(n - p + 1). \) Moreover, for each \( p \) this bound is best possible.

Thus if there exists a skew-symmetric solution of (1.8) or (1.11), then \( A \) has multiple eigenvalues, without the assumption that \( A \) is non-derogatory.

II. Proofs. Let \( E_{ij} \in M_n \) be the matrix with 1 in position \( i, j \) and 0 elsewhere. With respect to this basis, ordered lexicographically, it may be checked that \( T \) has the matrix representatio

\[ \cdots \]
(2.1) \[ T = I \otimes A - A \otimes I \]
where \( \otimes \) indicates Kronecker product.

From (1.2) we may take \( A \) to be in Jordan canonical form \( J \), since \([A, X]_k = 0\) if and only if \([J, PXP']_k = 0\) and \( PXP' \) is symmetric if and only if \( X \) is. We write

(2.2) \[ J = \sum_{s=1}^{p} J_s \]

where

(2.3) \[ J_s = \lambda_s I_{m_s} + \sum_{t=1}^{r} \sum_{r_{st}} U_{st} \cdot \]

\( \sum' \) indicates direct sum, \( I_t \) is a \( t \)-square identity matrix, \( U_t \) is \( t \)-square auxiliary unit matrix (i.e. 1 in the superdiagonal and 0 elsewhere) and \( \sum_r U_{st} \) is the direct sum of \( U_{st} \) with itself \( r_{st} \) times.

By a routine computation we see that

(2.4) \[ T^s(Y) = 0 \]

if and only if

(2.5) \[ \sum_{s=0}^{k} \binom{k}{s} (-1)^s J_s^s - Y_{st}(J_t')^s = 0, \quad s, t = 1, \ldots, p, \]

where \( Y = (Y_{st}) \), \( s, t = 1, \ldots, p \) is a partitioning of \( Y \) conformal with the partitioning of \( J \) given by (2.2).

For \( s \neq t \), it is clear that the matrix representation of (2.4),

(2.6) \[ (I_{m_s} \otimes J_s - J_t \otimes I_{m_t})^k \]

has the single nonzero eigenvalue \((\lambda_s - \lambda_t)^k\) and thus \( Y_{st} = 0 \). Hence we need only consider the equation (2.4) for \( s = t \). We may again partition \( Y_{ss} \) conformally with \( J_s \) in (2.3). We are thus led to consider the null space of the mapping

(2.7) \[ \dim \gamma(T) = \begin{cases} 2 \min(m, n), & \text{if } m \neq n \\ 2n - 1, & \text{if } m = n. \end{cases} \]

**Proof.** Suppose \( n \leq m \) and that \( T(X) = 0 \). Let \( x_1, \ldots, x_m \) be the column \( n \)-vectors of \( X \). Then we have
For $r = 1, 2, \cdots, n - 1$ consider the $(r - j + 1)$ coordinate of (2.8) for $j = 1, \cdots, r$ and we conclude that

$$x_{r+1} = x_{r,2} = \cdots = x_{r,r+1} = e_{r+1}.$$ 

Next consider the $(n - j + 1)$ coordinate of (2.8) for $j = 1, \cdots, n$ to obtain

$$0 = x_{n^2} = x_{n-1,3} = \cdots = x_{1,n+1}.$$ 

Similarly we see that the remaining elements of $X$ are zero. Hence we find that the $j$th column of the $n \times m$ matrix $X$ is the transpose of the $n$-vector

$$[e_j, e_{j+1}, \cdots, e_n, 0, \cdots, 0]$$

for $j = 1, 2, \cdots, n$. The other $m - n$ columns are zero.

In case $n \geq m$, it is easy to check that the $j$th row of $X$ is the $m$-vector

$$[e_j, e_{j+1}, \cdots, e_m, 0, \cdots, 0]$$

for $j = 1, 2, \cdots, m$. The other $n - m$ rows are zero.

This establishes (2.6). To prove (2.7) let $\Gamma(X) = 0$ and $x_1, x_2, \cdots, x_m$ be the column $n$-vectors of $X$. Let us consider the following cases:

(i) $m = n.$

We have

$$U_n^2 x_n = 0, \quad U_n^2 x_{n-1} = 2U_n x_n$$

and

$$U_n^3 x_j - 2U_n x_{j+1} + x_{j+2} = 0, \quad j = 1, 2, \cdots, n - 2.$$ 

Solving these equations recursively we find that the 1st, 2nd and $j$th rows of $X$ are respectively

$$[x_{11}, x_{12}, \cdots, x_{1,n-2}, x_{1,n-1}, x_{1,n}],$$

$$[x_{21}, x_{22}, \cdots, x_{2,n-2}, x_{2,n-1}, 0]$$

and

$$(j - 1)[x_{2,j-1}, x_{2,j}, \cdots, x_{2,n-1}, 0, \cdots, 0]$$

$$- (j - 2)[x_{1,j}, x_{1,j+1}, \cdots, x_{1,n}, 0, \cdots, 0],$$

for $j = 3, 4, \cdots, n$.

The number of arbitrary parameters in $X$ is $2n - 1$. 

(2.8) \quad U_n^2 x_j - x_{j+1} = 0, \quad j = 1, 2, \cdots, m - 1,

$$U_n x_m = 0.$$
(ii) \( n < m \).

Here we have the following equations:

\begin{align*}
U_{n}^{2}x_{j} - 2U_{n}x_{j+1} + x_{j+2} &= 0, \quad j = 1, 2, \ldots, m - 2 \\
U_{n}^{2}x_{m-1} - 2U_{n}x_{m} &= 0 \\
U_{n}^{2}x_{m} &= 0
\end{align*}

and by solving recursively again we find that the 1st, 2nd and \( j \)th rows of \( X \) are respectively the \( m \)-vectors

\begin{align*}
[x_{11}, \ldots, x_{1,n-1}, x_{1,n}, nx_{n,2}, 0, \ldots, 0], \\
[x_{21}, \ldots, x_{2,n-1}, (n-1)x_{n,2}, 0, \ldots, 0]
\end{align*}

and

\begin{align*}
[(j - 1)x_{2,j-1}, \ldots, (j - 1)x_{2,n-1}, (n - j + 1)x_{n,2}, 0, \ldots, 0] \\
- (j - 2)[x_{1,j}, \ldots, x_{1,n}, 0, \ldots, 0]
\end{align*}

for \( j = 3, 4, \ldots, n \).

In case \( n > m \), by similar computation we find that the 1st, 2nd and \( j \)th rows of \( X \) are respectively

\begin{align*}
[x_{11}, \ldots, x_{1,m-2}, x_{1,m-1}, x_{1m}], \\
[x_{21}, \ldots, x_{2,m-2}, x_{2,m-1}, x_{2m}]
\end{align*}

and

\begin{align*}
(j - 1)[x_{2,j-1}, \ldots, x_{2,m-2}, x_{2m}, 0, \ldots, 0] \\
- (j - 2)[x_{1,j}, \ldots, x_{1,m}, 0, \ldots, 0]
\end{align*}

for \( j = 3, 4, \ldots, m + 1 \). The remaining \( n - m - 1 \) rows are zero.

From case (ii), we observe that the number of parameters in \( X \) is \( 2 \min (m, n) \).

We now state and prove the following

**Lemma 2.** Let \( A \) be an \( n \)-square matrix with the single eigenvalue \( \lambda \) and let \( (x - \lambda)^{n_i} \) be an elementary divisor of \( A \) of multiplicity \( r_i \), \( i = 1, \ldots, p, n_1 > \cdots > n_p \). Then the most general matrix \( X \) satisfying (1.11) has

\begin{equation}
\sum_{i=1}^{p} r_i^2(2n_i - 1) + 4r_i \sum_{j=1}^{p} r_j e_j
\end{equation}

arbitrary parameters.

Moreover if \( X \) is symmetric it contains
\[
(2.11) \quad \frac{1}{2} \sum_{i=1}^{p} r_i^2 (2n_i - 1) + r_i + 4r_i \sum_{j=1}^{n} r_j n_j
\]
parameters.

**Proof.** Without any loss of generality we can assume that

\[
(2.12) \quad A = \sum_{i=1}^{r} \sum_{j=1}^{r_i} U_i
\]
where \( \sum_i U_i \) indicates the direct sum of \( U_i \) with itself \( r_i \) times. We partition \( X \) conformally with \( A \) in (2.12) and observe that the equation

\[
U_i X_{ij} - 2U_i X_{ij} U_j' + X_{ij}(U')^2 = 0
\]
determines the structure of any block \( X_{ij} \) in the partitioning of \( X \).

From case (i) of Lemma 1, we conclude that any block \( X_{ij} \) corresponding to equal \( U_i \)'s contains \( 2n_i - 1 \) arbitrary parameters and there are \( r_i \) such blocks. Also from case (ii) any block in \( X \) that corresponds to \( U_i \) and \( U_j, i < j \), contains \( 2n_j \) arbitrary parameters. Hence the total number of parameters in \( X \) is given by (2.10).

In order to find the number of parameters in a symmetric \( X \) we first consider a diagonal block. Its structure has been discussed in Lemma 1, case (i). We observe that if this matrix is symmetric, the number of parameters in it reduces from \( 2n_i - 1 \) to \( n_i \).

Then we consider two symmetrically placed off-diagonal blocks \( X_{ij} \) and \( X_{ji} \) of orders \( n_i \times n_j \) and \( n_j \times n_i \) respectively. If \( X \) is to be symmetric then by equating the terms of \( X_{ij} \) and \( X_{ji} \), which are symmetrically placed about the main diagonal of \( X \), the number of arbitrary parameters in \( X_{ij} \) and \( X_{ji} \) reduces from \( 2(2n_j) \) to \( 2n_j \). If \( X_{ij} \) and \( X_{ji} \) are of order \( n_i \times n_j \) then the number of parameters reduces from \( 2(2n_i - 1) \) to \( 2n_i - 1 \).

We are now in a position to sum the number of parameters in \( X \) if it is symmetric and satisfies (1.11). There are \( r_i \) blocks in the main diagonal, each of order \( n_i, i = 1, \ldots, p \). The number of parameters in each of these blocks is \( n_i \). There are \( r_i(r_i - 1)/2 \) other square blocks of order \( n_i \). Each of them contains \( (2n_i - 1) \) parameters. Thus

\[
\frac{1}{2} \sum_{i=1}^{p} \{r_i^2(2n_i - 1) + r_i\}
\]
is the number of parameters in all those blocks of \( X \) which are square. Since any block of order \( n_i \times n_j \) where \( n_i > n_j \) contains \( 2n_j \) parameters, and since we are considering \( X \) to be symmetric, we conclude that the total number of arbitrary parameters in \( X \) is given by (2.11).

We can similarly prove the following

**Lemma 3.** Let \( A \) be the matrix given in Lemma 2. Then the most
general matrix $X$ satisfying (1.8) has
\[ \sum_{i=1}^{p} \left( r_i^2 n_i + 2r_i \sum_{j=i+1}^{p} r_j n_j \right) \]
arbitrary parameters.
Moreover if $X$ is symmetric, it contains
\[ \frac{1}{2} \sum_{i=1}^{p} \left[ r_i(r_i + 1)n_i + 2r_i \sum_{j=i+1}^{p} r_j n_j \right] \]
parameters.

We now state and prove the following

**Theorem 1.** Let $A$ be an $n$-square matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ and let $(x - \lambda_i)^{n_{ij}}, j = 1, \ldots, n_{ii}, e_{ii} > \cdots > e_{iu}$ be the elementary divisors of $A$ corresponding to $\lambda_i$, where each $(x - \lambda_i)^{n_{ij}}$ has been repeated $r_{ij}$ times. Then (1.4), (1.5), (1.6) and (1.7) hold.

**Proof.** It was pointed out earlier that if $Y = (Y_{rs})$, $r, s = 1, \ldots, p$ is the partitioning of $Y$ conformal with the partitioning of $J$ in (2.2), then all the off-diagonal blocks are zero. Hence we have simply to find the number of parameters in $Y_{ii}, i = 1, \ldots, p$.

As proved in Lemma 2, the number of parameters in $Y_{ii}$ is
\[ \sum_{i=1}^{n_i} \left[ r_i^2 (2e_{ij} - 1) + 4r_i \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right]. \]
Summing the above with respect to $i$ we obtain the formula (1.6). In case $Y$ is symmetric, the number of parameters in $Y_{ii}$ is
\[ \frac{1}{2} \sum_{j=1}^{n_i} \left[ r_i^2 (2e_{ij} - 1) + r_{ij} + 4r_i \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right]. \]
Summing the above on $i$ we obtain (1.7).

Similarly, we can make use of Lemma 3 in proving (1.4) and (1.5).

We now prove

**Theorem 2.** Let $A$ be as given in Theorem 1. Then the maximum number of linearly independent skew-symmetric matrices satisfying (1.8) or (1.11) is
\[ \frac{1}{2} (n - p)(n - p + 1). \]

**Proof.** In order to prove our result for $\dim \gamma(T^2)$, let $m_i = \sum_{j=1}^{n_i} r_{ij} e_{ij}$ and consider
Now, it is clear that \( r_i^2(e_{ij} - 1) \geq r_i(e_{ij} - 1) \). The last term in the above expression will be negative only when \( e_{ij} = 1 \). But we know that \( e_{i1} > e_{i2} > \cdots > e_{in_i} \), so that \( e_{ij} \) will be 1 only for \( j = n_i \). In that case \( \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \) does not appear, and we have

\[
\frac{1}{2} \sum_{j=1}^{n_i} \left[ r_i^2(2e_{ij} - 1) - r_i + 4r_i \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \leq \frac{1}{2}(m_i^2 - m_i).
\]

This holds for \( i = 1, \ldots, p \).

To determine a bound on \( \gamma(T) \), consider

\[
m_i^2 - m_i - \sum_{j=1}^{n_i} \left[ r_i^2(r_{ij} - 1)e_{ij} + 2r_i \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \]

\[
= \sum_{j=1}^{n_i} \left[ r_i^2(e_{ij} - 1) + 2r_i(e_{ij} - 1) \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \geq 0, \text{ since } e_{ij} \geq 1.
\]

Thus we have

\[
\frac{1}{2} \sum_{j=1}^{n_i} \left[ r_i^2(2e_{ij} - 1) - r_i + 4r_i \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \leq \frac{1}{2}(m_i^2 - m_i).
\]

It may be observed that the upper bound is attained for \( r_{i1} = m_i, e_{i1} = 1 \) and the remaining \( e \)'s and \( r \)'s all zero.

We have thus proved that

\[
\dim \gamma(T^a) \leq \frac{1}{2} \sum_{i=1}^{n} (m_i^2 - m_i)
\]

and

\[
\dim \gamma(T) \leq \frac{1}{2} \sum_{i=1}^{n} (m_i^2 - m_i),
\]

where \( m_i \) is the multiplicity of the eigenvalue \( \lambda_i \) of \( A \).

Now we have to maximize \( \sum_{i=1}^{n} (m_i^2 - m_i) \) under the condition that
$m_1 + \cdots + m_p = n$, the order of $A$. Note that

$$m_i^2 - m_i = (m_i - 1)^2 + (m_i - 1)$$

and each $m_i - 1 \geq 0$. Hence, we have

$$\sum_{i=1}^{p} (m_i - 1)^2 \leq \left[ \sum_{i=1}^{p} (m_i - 1) \right]^2 = (n - p)^2.$$

Thus the maximum value of both $\dim \gamma(T^2)$ and $\dim \gamma(T)$ is

$$\frac{1}{2} [(n - p)^2 + (n - p)].$$

The bounds are achieved when $m_1 = \cdots = m_{p-1} = 1$ and $m_p = n-p+1$.

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