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ON A COMMUTATOR RESULT OF TAUSSKY AND ZASSENHAUS

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1. Introduction and results. Let M_n denote the set of n-square matrices over a field F. For A, B in M_n let $[A, B] = AB - BA'$, where A' is the transpose of A and define inductively

$$[A, B]_k = [A, [A, B]_{k-1}] \,. \tag{1.1}$$

If $P^{-1}JP = A$, then

$$[A, X] = [P^{-1}JP, X] = P^{-1}[J, PXP'](P^{-1})' \,,$$

and similarly

$$[A, X]_k = P^{-1}[J, PXP']_k(P^{-1})' \,. \tag{1.2}$$

Now for a fixed A let T be the linear map of M_n into itself defined by

$$T(Y) = [A, Y] \tag{1.3}$$

and (1.1) implies that

$$T^k(Y) = [A, Y]_k \,.$$

In a recent paper [1], Taussky and Zassenhaus showed that *A is nonderogatory if and only if any nonsingular X in the null space of T is symmetric.* In this note we investigate the structure of the null space of both T and T^2 for arbitrary A.

Enlarge the field F to include λ_i, $i = 1, \cdots, p$, the distinct eigenvalues of A, and let $(x - \lambda_i)^{e_{ij}}$, $j = 1, \cdots, n_i$, $e_{i1} > \cdots > e_{in_i}$, $i = 1, \cdots, p$ be the distinct elementary divisors of A where $(x - \lambda_i)^{e_{ij}}$ appears with multiplicity r_{ij}. Set $m_i = \sum_{j=1}^{n_i} r_{ij}e_{ij}$, the algebraic multiplicity of λ_i. *Let $\eta(T)$ denote the null space of T, $\sigma(T)$ denote the subspace of symmetric matrices in $\eta(T)$, and $\gamma(T)$ denote the subspace of skew-symmetric matrices in $\eta(T)$.* We show that

$$\dim \eta(T) = \sum_{i=1}^{p}\left[\sum_{j=1}^{n_i}\left(r_{ij}^2 e_{ij} + 2r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right)\right], \tag{1.4}$$

$$\dim \sigma(T) = \frac{1}{2}\sum_{i=1}^{p}\left[\sum_{j=1}^{n_i}\left\{r_{ij}(r_{ij}+1)e_{ij} + 2r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right\}\right], \tag{1.5}$$

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$$\dim \eta(T^2) = \sum_{i=1}^{p}\left[\sum_{j=1}^{n_i}\left\{r_{ij}^2(2e_{ij}-1) + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right\}\right], \tag{1.6}$$

$$\dim \sigma(T^2) = \frac{1}{2}\sum_{i=1}^{p}\left[\sum_{j=1}^{n_i}\left\{r_{ij}^2(2e_{ij}-1) + r_{ij} + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right\}\right]. \tag{1.7}$$

In case A is nonderogatory, $n_i = 1$, $r_{ij} = 1$, $i=1, \cdots, p$ and (1.4) and (1.5) reduce to

$$\dim \eta(T) = n = \dim \sigma(T) .$$

Thus every matrix X satisfying

$$AX = XA' \tag{1.8}$$

where A is non-derogatory is symmetric, the result in [1]. Moreover, if every matrix X satisfying (1.8) is symmetric then $\dim \eta(T) = \dim \sigma(T)$. Using the formulas (1.4) and (1.5) we see that this condition implies that

$$\sum_{i=1}^{p}\sum_{j=1}^{n_i}(r_{ij}^2 - r_{ij})e_{ij} + 2\sum_{i=1}^{p} r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik} = 0 .$$

Now since r_{ij}, e_{ij} and n_i are all positive integers we conclude that $r_{ij} = 1$, $j = 1, \cdots, n_i$ and $n_i = 1$. That is, there is only one elementary divisor corresponding to each eigenvalue. Hence, *if every matrix X satisfying* (1.8) *is symmetric then A is non-derogatory*, a result also found in [1].

We also show in this case that $\eta(T)$ *consists of matrices of the form PXP' where P is fixed* (depending on A) *and X is persymmetric*, (i.e. all the entries of X on each line perpendicular to the main diagonal are equal).

We next note that $\eta(T) = \sigma(T) + \gamma(T)$ (direct) and $\eta(T^2) = \sigma(T^2) + \gamma(T^2)$ (direct). The first statement is easy to show; we indicate the brief proof of the second statement:

Since $X = \dfrac{X + X'}{2} + \dfrac{X - X'}{2}$, if $X \in \eta(T^2)$, then

$$\begin{aligned} T^2(X + X') &= [A, [A, X + X']] \\ &= [A, [A, X] + [A, X']] \\ &= [A, [A, X]] + [A, [A, X']] \\ &= T^2(X) - [A, [A, X]'] \\ &= [A, [A, X]]' \\ &= (T^2(X))' = 0 . \end{aligned}$$

Similarly, $T^2(X - X') = 0$. Thus any $X \in \eta(T^2)$ is expressible uniquely as a sum of two elements, one in $\sigma(T^2)$ and the other in $\gamma(T^2)$. Hence

$$\dim \gamma(T) = \dim \eta(T) - \dim \sigma(T) = \frac{1}{2}\sum_{i=1}^{p}\left[\sum_{j=1}^{n_i}\left\{r_{ij}(r_{ij}-1)e_{ij} + 2r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right\}\right], \tag{1.9}$$

$$\dim \gamma(T^2) = \dim \eta(T^2) - \dim \sigma(T^2) = \frac{1}{2}\sum_{i=1}^{p}\left[\sum_{j=1}^{n_i}\left\{r_{ij}^2(2e_{ij}-1) - r_{ij} + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right\}\right]. \tag{1.10}$$

In case A is non-derogatory, (1.6), (1.7) and (1.10) reduce to

$$\dim \eta(T^2) = 2n - p\,,$$

$$\dim \sigma(T^2) = n\,,$$

$$\dim \gamma(T^2) = n - p\,.$$

We thus conclude that *unless all the eigenvalues of A are distinct ($p = n$) there exist skew-symmetric matrices X satisfying*

$$A^2X - 2AXA' + X(A')^2 = 0\,. \tag{1.11}$$

If $p = n$, and A is non-derogatory

$$\dim \eta(T^2) = n = \dim \sigma(T^2)$$

and any matrix X satisfying (1.11) *is symmetric.*

On the other hand suppose

$$\dim \eta(T^2) = \dim \sigma(T^2)\,.$$

From (1.6) and (1.7) we conclude that

$$\sum_{i=1}^{p}\left[\sum_{j=1}^{n_i}\left\{r_{ij}^2(2e_{ij}-1) - r_{ij} + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right\}\right] = 0\,.$$

Hence $n_i = 1$, $r_{ij} = 1$, $e_{ik} = 1$ and we conclude that $p = n$. That is, *if every matrix X satisfying* (1.11) *is symmetric then the eigenvalues of A are distinct.*

We show finally (Theorem 2) that *if A is an n-square matrix with p distinct eigenvalues then both* $\dim \gamma(T)$ *and* $\dim \gamma(T^2)$ *are at most* $\frac{1}{2}(n-p)(n-p+1)$. *Moreover, for each p this bound is best possible.*

Thus *if there exists a skew-symmetric solution of* (1.8) *or* (1.11), *then A has multiple eigenvalues, without the assumption that A is non-derogatory.*

II. *Proofs.* Let $E_{ij} \in M_n$ be the matrix with 1 in position i, j and 0 elsewhere. With respect to this basis, ordered lexicographically, it may be checked that T has the matrix representaion

$$T = I \otimes A - A \otimes I \tag{2.1}$$

where $\otimes$ indicates Kronecker product.

From (1.2) we may take A to be in Jordan canonical form J, since $[A, X]_k = 0$ if and only if $[J, PXP']_k = 0$ and PXP' is symmetric if and only if X is. We write

$$J = \sum_{s=1}^{p}{}^{\boldsymbol{\cdot}} J_s \tag{2.2}$$

where

$$J_s = \lambda_s I_{m_s} + \sum_{t=1}^{n_s}{}^{\boldsymbol{\cdot}} \sum_{1}^{r_{st}}{}^{\boldsymbol{\cdot}} U_{e_{st}} ; \tag{2.3}$$

$\sum^{\boldsymbol{\cdot}}$ indicates direct sum, I_t is a t-square identity matrix, U_t is t-square auxiliary unit matrix (i.e. 1 in the superdiagonal and 0 elsewhere) and $\sum_1^{r_{ji}}{}^{\boldsymbol{\cdot}} U_{e_{st}}$ is the direct sum of $U_{e_{st}}$ with itself r_{ij} times.

By a routine computation we see that

$$T^k(Y) = 0$$

if and only if

$$\sum_{\alpha=0}^{k} \binom{k}{\alpha} (-1)^\alpha J_s^{k-\alpha} Y_{st} (J_t')^\alpha = 0 , \qquad s, t = 1, \cdots, p , \tag{2.4}$$

where $Y = (Y_{st})$, $s, t = 1, \cdots, p$ is a partitioning of Y conformal with the partitioning of J given by (2.2).

For $s \neq t$, it is clear that the matrix representation of (2.4),

$$(I_{m_t} \otimes J_s - J_t \otimes I_{m_s})^k$$

has the single nonzero eigenvalue $(\lambda_s - \lambda_t)^k$ and thus $Y_{st} = 0$. Hence we need only consider the equation (2.4) for $s = t$. We may again partition Y_{ss} conformally with J_s in (2.3). We are thus led to consider the null space of the mapping

$$(I_{e_{si}} \otimes U_{e_{sj}} - U_{e_{si}} \otimes I_{e_{sj}})^k . \tag{2.5}$$

LEMMA 1. *Let* $T = I_m \otimes U_n - U_m \otimes I_n$. *Then*

$$\dim \eta(T) = \min(m, n) , \tag{2.6}$$

$$\dim \eta(T^2) = \begin{cases} 2 \min(m, n) , & \text{if } m \neq n \\ 2n - 1, & \text{if } m = n . \end{cases} \tag{2.7}$$

Proof. Suppose $n \leqq m$ and that $T(X) = 0$. Let $x_1, \cdots, x_m$ be the column n-vectors of X. Then we have

$$U_n x_j - x_{j+1} = 0 \,, \quad j = 1, 2, \cdots, m-1 \,, \tag{2.8}$$
$$U_n x_m = 0 \,.$$

For $r = 1, 2, \cdots, n-1$ consider the $(r - j + 1)$ coordinate of (2.8) for $j = 1, \cdots, r$ and we conclude that

$$x_{r+1,1} = x_{r,2} = \cdots = x_{1,r+1} = c_{r+1} \,.$$

Next consider the $(n - j + 1)$ coordinate of (2.8) for $j = 1, \cdots, n$ to obtain

$$0 = x_{n2} = x_{n-1,3} = \cdots = x_{1,n+1} \,.$$

Similarly we see that the remaining elements of X are zero. Hence we find that the jth column of the $n \times m$ matrix X is the transpose of the n-vector

$$[c_j, c_{j+1}, \cdots, c_n, 0, \cdots, 0]$$

for $j = 1, 2, \cdots, n$. The other $m - n$ columns are zero.

In case $n \geqq m$, it is easy to check that the jth row of X is the m-vector

$$[c_j, c_{j+1}, \cdots, c_m, 0, \cdots, 0]$$

for $j = 1, 2, \cdots, m$. The other $n - m$ rows are zero.

This establishes (2.6). To prove (2.7) let $T^2(X) = 0$ and $x_1, x_2, \cdots, x_m$ be the column n-vectors of X. Let us consider the following cases:

(i) $m = n$.

We have

$$U_n^2 x_n = 0, \;\; U_n^2 x_{n-1} = 2U_n x_n$$

and

$$U_n^2 x_j - 2U_n x_{j+1} + x_{j+2} = 0, \; j = 1, 2, \cdots, n-2 \,.$$

Solving these equations recursively we find that the lst, 2nd and jth rows of X are respectively

$$[x_{11}, x_{12}, \cdots, x_{1,n-2}, x_{1,n-1}, x_{1n}] \,,$$
$$[x_{21}, x_{22}, \cdots, x_{2,n-2}, x_{2,n-1}, 0]$$

and

$$(j-1)[x_{2,j-1}, x_{2,j}, \cdots, x_{2,n-1}, 0, \cdots, 0]$$
$$-(j-2)[x_{1,j}, x_{1,j+1}, \cdots, x_{1,n}, 0, \cdots, 0] \,,$$

for $j = 3, 4, \cdots, n$.

The number of arbitrary parameters in X is $2n - 1$.

(ii) $n < m$.

Here we have the following equations:

$$U_n^2 x_j - 2U_n x_{j+1} + x_{j+2} = 0,\ j = 1, 2, \cdots, m-2 \tag{2.9}$$
$$U_n^2 x_{m-1} - 2U_n x_m = 0$$
$$U_n^2 x_m = 0$$

and by solving recursively again we find that the 1st, 2nd and jth rows of X are respectively the m-vectors

$$[x_{11}, \cdots, x_{1,n-1}, x_{1,n}, nx_{n,2}, 0, \cdots, 0]\ ,$$
$$[x_{21}, \cdots, x_{2,n-1}, (n-1)x_{n,2}, 0, 0, \cdots, 0]$$

and

$$[(j-1)x_{2,j-1}, \cdots, (j-1)x_{2,n-1}, (n-j+1)x_{n,2}, 0, \cdots, 0]$$
$$-(j-2)[x_{1,j}, \cdots, x_{1,n}, 0, 0, \cdots, 0]$$

for $j = 3, 4, \cdots, n$.

In case $n > m$, by similar computation we find that the 1st, 2nd and jth rows of X are respectively

$$[x_{11}, \cdots, x_{1,m-2}, x_{1,m-1}, x_{1m}]\ ,$$
$$[x_{21}, \cdots, x_{2,m-2}, x_{2,m-1}, x_{2m}]$$

and

$$(j-1)[x_{2,j-1}, \cdots, x_{2,m-1}, x_{2m}, 0, \cdots, 0]$$
$$-(j-2)[x_{1,j}, \cdots, x_{1,m}, 0, 0, \cdots, 0]$$

for $j = 3, 4, \cdots, m+1$. The remaining $n - m - 1$ rows are zero.

From case (ii), we observe that the number of parameters in X is $2 \min (m, n)$.

We now state and prove the following

LEMMA 2. *Let A be an n-square matrix with the single eigenvalue λ and let $(x-\lambda)^{n_i}$ be an elementary divisor of A of multiplicity r_i, $i = 1, \cdots, p, n_1 > \cdots > n_p$. Then the most general matrix X satisfying* (1.11) *has*

$$\sum_{i=1}^{p}\left[r_i^2(2n_i - 1) + 4r_i \sum_{j=i+1}^{p} r_j e_j\right] \tag{2.10}$$

arbitrary parameters.

Moreover if X is symmetric it contains

$$\tag{2.11} \frac{1}{2}\sum_{i=1}^{p}\left[r_i^2(2n_i-1)+r_i+4r_i\sum_{j=i+1}^{p}r_jn_j\right]$$

parameters.

Proof. Without any loss of generality we can assume that

$$\tag{2.12} A=\sum_{i=1}^{p}{}^{\cdot}\sum_{j=1}^{r_i}{}^{\cdot}U_i$$

where $\sum^{\cdot}U_i$ indicates the direct sum of U_i with itself r_i times. We partition X conformally with A in (2.12) and observe that the equation

$$U_i^2X_{ij}-2U_iX_{ij}U_j'+X_{ij}(U_j')^2=0$$

determines the structure of any block X_{ij} in the partitioning of X.

From case (i) of Lemma 1, we conclude that any block X_{ij} corresponding to equal U_i's contains $2n_i-1$ arbitrary parameters and there are r_i^2 such blocks. Also from case (ii) any block in X that corresponds to U_i and U_j, $i<j$, contains $2n_j$ arbitrary parameters. Hence the total number of parameters in X is given by (2.10).

In order to find the number of parameters in a symmetric X we first consider a diagonal block. Its structure has been discussed in Lemma 1, case (i). We observe that if this matrix is symmetric, the number of parameters in it reduces from $2n_i-1$ to n_i.

Then we consider two symmetrically placed off-diagonal blocks X_{ij} and X_{ji} of orders $n_i\times n_j$ and $n_j\times n_i$ respectively. If X is to be symmetric then by equating the terms of X_{ij} and X_{ji} which are symmetrically placed about the main diagonal of X, the number of arbitrary parameters in X_{ij} and X_{ji} reduces from $2(2n_j)$ to $2n_j$. If X_{ij} and X_{ji} are of order $n_i\times n_i$ then the number of parameters reduces from $2(2n_i-1)$ to $2n_i-1$.

We are now in a position to sum the number of parameters in X if it is symmetric and satisfies (1.11). There are r_i blocks in the main diagonal, each of order n_i, $i=1,\cdots,p$. The number of parameters in each of these blocks is n_i. There are $r_i(r_i-1)/2$ other square blocks of order n_i. Each of them contains $(2n_i-1)$ parameters. Thus

$$\frac{1}{2}\sum_{i=1}^{p}\{r_i^2(2n_i-1)+r_i\}$$

is the number of parameters in all those blocks of X which are square. Since any block of order $n_i\times n_j$ where $n_i>n_j$ contains $2n_j$ parameters, and since we are considering X to be symmetric, we conclude that the total number of arbitrary parameters in X is given by (2.11).

We can similarly prove the following

LEMMA 3. *Let A be the matrix given in Lemma 2. Then the most* *general matrix X satisfying* (1.8) *has*

$$\sum_{i=1}^{p}\left(r_i^2 n_i + 2r_i \sum_{j=i+1}^{p} r_j n_j\right)$$

arbitrary parameters.

Moreover if X is symmetric, it contains

$$\frac{1}{2}\sum_{i=1}^{p}\left[r_i(r_i + 1)n_i + 2r_i \sum_{j=i+1}^{p} r_j n_j\right]$$

parameters.

We now state and prove the following

THEOREM 1. *Let A be an n-square matrix with distinct eigenvalues $\lambda_1, \cdots, \lambda_p$ and let $(x - \lambda_i)^{e_{ij}}$, $j = 1, \cdots, n_i$, $e_{i1} > \cdots > e_{in_i}$ be the elementary divisors of A corresponding to λ_i, where each $(x - \lambda_i)^{e_{ij}}$ has been repeated r_{ij} times. Then* (1.4), (1.5), (1.6) *and* (1.7) *hold.*

Proof. It was pointed out earlier that if $Y = (Y_{rs})$, $r, s = 1, \cdots, p$ is the partitioning of Y conformal with the partitioning of J in (2.2), then all the off-diagonal blocks are zero. Hence we have simply to find the number of parameters in Y_{ii}, $i = 1, \cdots, p$.

As proved in Lemma 2, the number of parameters in Y_{ii} is

$$\sum_{j=1}^{n_i}\left[r_{ij}^2(2e_{ij} - 1) + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right].$$

Summing the above with respect to i we obtain the formula (1.6). In case Y is symmetric, the number of parameters in Y_{ii} is

$$\frac{1}{2}\sum_{j=1}^{n_i}\left[r_{ij}^2(2e_{ij} - 1) + r_{ij} + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right].$$

Summing the above on i we obtain (1.7).

Similarly, we can make use of Lemma 3 in proving (1.4) and (1.5).

We now prove

THEOREM 2. *Let A be as given in Theorem* 1. *Then the maximum number of linearly independent skew-symmetric matrices satisfying* (1.8) *or* (1.11) *is*

$$\frac{1}{2}(n - p)(n - p + 1)\ .$$

Proof. In order to prove our result for $\dim \gamma(T^2)$, let $m_i = \sum_{j=1}^{n_i} r_{ij}e_{ij}$ and consider

$$\begin{aligned}
m_i^2 - m_i &- \sum_{j=1}^{n_i}\left[r_{ij}^2(2e_{ij}-1) - r_{ij} + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right] \\
&= \sum_{j=1}^{n_i}\left[r_{ij}^2e_{ij}^2 + 2r_{ij}e_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik} - r_{ij}e_{ij}\right] \\
&\quad - \sum_{j=1}^{n_i}\left[r_{ij}^2(2e_{ij}-1) - r_{ij} + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right] \\
&= \sum_{j=1}^{n_i}\left[r_{ij}^2(e_{ij}-1)^2 - r_{ij}(e_{ij}-1) + 2r_{ij}(e_{ij}-2)\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right].
\end{aligned}$$

Now, it is clear that $r_{ij}^2(e_{ij}-1) \geqq r_{ij}(e_{ij}-1)$. The last term in the above expression will be negative only when $e_{ij}=1$. But we know that $e_{i1} > e_{i2} > \cdots > e_{in_i}$, so that e_{ij} will be 1 only for $j = n_i$. In that case $\sum_{k=j+1}^{n_i}$ does not appear, and we have

$$\frac{1}{2}\sum_{j=1}^{n_i}\left[r_{ij}^2(2e_{ij}-1) - r_{ij} + 4r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right] \leqq \frac{1}{2}(m_i^2 - m_i)\ .$$

This holds for $i = 1, \cdots, p$.

To determine a bound on $\gamma(T)$, consider

$$\begin{aligned}
m_i^2 - m_i &- \sum_{j=1}^{n_i}\left[r_{ij}(r_{ij}-1)e_{ij} + 2r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right] \\
&= \sum_{j=1}^{n_i}\left[r_{ij}^2e_{ij}(e_{ij}-1) + 2r_{ij}(e_{ij}-1)\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right] \\
&\geqq 0, \text{ since } e_{ij} \geqq 1.
\end{aligned}$$

Thus we have

$$\frac{1}{2}\sum_{j=1}^{n_i}\left[r_{ij}(r_{ij}-1)e_{ij} + 2r_{ij}\sum_{k=j+1}^{n_i} r_{ik}e_{ik}\right] \leqq \frac{1}{2}(m_i^2 - m_i)\ .$$

It may be observed that the upper bound is attained for $r_{i1} = m_i$, $e_{i1} = 1$ and the remaining e's and r's all zero.

We have thus proved that

$$\dim \gamma(T^2) \leqq \frac{1}{2}\sum_{i=1}^{p}(m_i^2 - m_i)$$

and

$$\dim \gamma(T) \leqq \frac{1}{2}\sum_{i=1}^{p}(m_i^2 - m_i)\ ,$$

where m_i is the multiplicity of the eigenvalue λ_i of A.

Now we have to maximize $\sum_{i=1}^{p}(m_i^2 - m_i)$ under the condition that

$m_1 + \cdots + m_p = n$, the order of A. Note that

$$m_i^2 - m_i = (m_i - 1)^2 + (m_i - 1)$$

and each $m_i - 1 \geqq 0$. Hence, we have

$$\sum_{i=1}^{p} (m_i - 1)^2 \leqq \left[\sum_{i=1}^{p} (m_i - 1)\right]^2 = (n - p)^2 .$$

Thus the maximum value of both $\dim \gamma(T^2)$ and $\dim \gamma(T)$ is

$$\frac{1}{2}[(n - p)^2 + (n - p)] .$$

The bounds are achieved when $m_1 = \cdots = m_{p-1} = 1$ and $m_p = n - p + 1$.

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