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**ON A COMMUTATOR RESULT OF TAUSSKY AND  
ZASSENHAUS**

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**1. Introduction and results.** Let  $M_n$  denote the set of  $n$ -square matrices over a field  $F$ . For  $A, B$  in  $M_n$  let  $[A, B] = AB - BA'$ , where  $A'$  is the transpose of  $A$  and define inductively

$$(1.1) \quad [A, B]_k = [A, [A, B]_{k-1}].$$

If  $P^{-1}JP = A$ , then

$$[A, X] = [P^{-1}JP, X] = P^{-1}[J, PXP'](P^{-1})',$$

and similarly

$$(1.2) \quad [A, X]_k = P^{-1}[J, PXP']_k(P^{-1})'.$$

Now for a fixed  $A$  let  $T$  be the linear map of  $M_n$  into itself defined by

$$(1.3) \quad T(Y) = [A, Y]$$

and (1.1) implies that

$$T^k(Y) = [A, Y]_k.$$

In a recent paper [1], Taussky and Zassenhaus showed that  $A$  is non-derogatory if and only if any nonsingular  $X$  in the null space of  $T$  is symmetric. In this note we investigate the structure of the null space of both  $T$  and  $T^2$  for arbitrary  $A$ .

Enlarge the field  $F$  to include  $\lambda_i, i = 1, \dots, p$ , the distinct eigenvalues of  $A$ , and let  $(x - \lambda_i)^{e_{ij}}, j = 1, \dots, n_i, e_{i1} > \dots > e_{in_i}, i = 1, \dots, p$  be the distinct elementary divisors of  $A$  where  $(x - \lambda_i)^{e_{ij}}$  appears with multiplicity  $r_{ij}$ . Set  $m_i = \sum_{j=1}^{n_i} r_{ij}e_{ij}$ , the algebraic multiplicity of  $\lambda_i$ . Let  $\eta(T)$  denote the null space of  $T$ ,  $\sigma(T)$  denote the subspace of symmetric matrices in  $\eta(T)$ , and  $\gamma(T)$  denote the subspace of skew-symmetric matrices in  $\eta(T)$ . We show that

$$(1.4) \quad \dim \gamma(T) = \sum_{i=1}^p \left[ \sum_{j=1}^{n_i} \left( r_{ij}^2 e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right],$$

$$(1.5) \quad \dim \sigma(T) = \frac{1}{2} \sum_{i=1}^p \left[ \sum_{j=1}^{n_i} \left\{ r_{ij}(r_{ij} + 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right\} \right],$$

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$$(1.6) \quad \dim \eta(T^2) = \sum_{i=1}^p \left[ \sum_{j=1}^{n_i} \left\{ r_{ij}^2(2e_{ij} - 1) + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right],$$

$$(1.7) \quad \dim \sigma(T^2) = \frac{1}{2} \sum_{i=1}^p \left[ \sum_{j=1}^{n_i} \left\{ r_{ij}^2(2e_{ij} - 1) + r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right].$$

In case  $A$  is nonderogatory,  $n_i = 1, r_{ij} = 1, i=1, \dots, p$  and (1.4) and (1.5) reduce to

$$\dim \eta(T) = n = \dim \sigma(T).$$

Thus every matrix  $X$  satisfying

$$(1.8) \quad AX = XA'$$

where  $A$  is non-derogatory is symmetric, the result in [1]. Moreover, if every matrix  $X$  satisfying (1.8) is symmetric then  $\dim \eta(T) = \dim \sigma(T)$ . Using the formulas (1.4) and (1.5) we see that this condition implies that

$$\sum_{i=1}^p \sum_{j=1}^{n_i} (r_{ij}^2 - r_{ij})e_{ij} + 2 \sum_{i=1}^p r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} = 0.$$

Now since  $r_{ij}, e_{ij}$  and  $n_i$  are all positive integers we conclude that  $r_{ij} = 1, j = 1, \dots, n_i$  and  $n_i = 1$ . That is, there is only one elementary divisor corresponding to each eigenvalue. Hence, *if every matrix  $X$  satisfying (1.8) is symmetric then  $A$  is non-derogatory*, a result also found in [1].

We also show in this case that  $\eta(T)$  consists of matrices of the form  $PXP'$  where  $P$  is fixed (depending on  $A$ ) and  $X$  is persymmetric, (i.e. all the entries of  $X$  on each line perpendicular to the main diagonal are equal).

We next note that  $\eta(T) = \sigma(T) + \gamma(T)$  (direct) and  $\eta(T^2) = \sigma(T^2) + \gamma(T^2)$  (direct). The first statement is easy to show; we indicate the brief proof of the second statement:

Since  $X = \frac{X + X'}{2} + \frac{X - X'}{2}$ , if  $X \in \eta(T^2)$ , then

$$\begin{aligned} T^2(X + X') &= [A, [A, X + X']] \\ &= [A, [A, X] + [A, X']] \\ &= [A, [A, X]] + [A, [A, X']] \\ &= T^2(X) - [A, [A, X']] \\ &= [A, [A, X]]' \\ &= (T^2(X))' = 0. \end{aligned}$$

Similarly,  $T^2(X - X') = 0$ . Thus any  $X \in \eta(T^2)$  is expressible uniquely as a sum of two elements, one in  $\sigma(T^2)$  and the other in  $\gamma(T^2)$ . Hence

$$(1.9) \quad \begin{aligned} \dim \gamma(T) &= \dim \eta(T) - \dim \sigma(T) \\ &= \frac{1}{2} \sum_{i=1}^p \left[ \sum_{j=1}^{n_i} \left\{ r_{ij}(r_{ij} - 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right], \end{aligned}$$

$$(1.10) \quad \begin{aligned} \dim \gamma(T^2) &= \dim \eta(T^2) - \dim \sigma(T^2) \\ &= \frac{1}{2} \sum_{i=1}^p \left[ \sum_{j=1}^{n_i} \left\{ r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right]. \end{aligned}$$

In case  $A$  is non-derogatory, (1.6), (1.7) and (1.10) reduce to

$$\dim \eta(T^2) = 2n - p,$$

$$\dim \sigma(T^2) = n,$$

$$\dim \gamma(T^2) = n - p.$$

We thus conclude that *unless all the eigenvalues of  $A$  are distinct ( $p = n$ ) there exist skew-symmetric matrices  $X$  satisfying*

$$(1.11) \quad A^2X - 2AXA' + X(A')^2 = 0.$$

*If  $p = n$ , and  $A$  is non-derogatory*

$$\dim \gamma(T^2) = n = \dim \sigma(T^2)$$

*and any matrix  $X$  satisfying (1.11) is symmetric.*

On the other hand suppose

$$\dim \gamma(T^2) = \dim \sigma(T^2).$$

From (1.6) and (1.7) we conclude that

$$\sum_{i=1}^p \left[ \sum_{j=1}^{n_i} \left\{ r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right\} \right] = 0.$$

Hence  $n_i = 1$ ,  $r_{ij} = 1$ ,  $e_{ik} = 1$  and we conclude that  $p = n$ . That is, *if every matrix  $X$  satisfying (1.11) is symmetric then the eigenvalues of  $A$  are distinct.*

We show finally (Theorem 2) that *if  $A$  is an  $n$ -square matrix with  $p$  distinct eigenvalues then both  $\dim \gamma(T)$  and  $\dim \gamma(T^2)$  are at most  $\frac{1}{2}(n - p)(n - p + 1)$ . Moreover, for each  $p$  this bound is best possible.*

*Thus if there exists a skew-symmetric solution of (1.8) or (1.11), then  $A$  has multiple eigenvalues, without the assumption that  $A$  is non-derogatory.*

II. *Proofs.* Let  $E_{ij} \in M_n$  be the matrix with 1 in position  $i, j$  and 0 elsewhere. With respect to this basis, ordered lexicographically, it may be checked that  $T$  has the matrix representaion

$$(2.1) \quad T = I \otimes A - A \otimes I$$

where  $\otimes$  indicates Kronecker product.

From (1.2) we may take  $A$  to be in Jordan canonical form  $J$ , since  $[A, X]_k = 0$  if and only if  $[J, PXP']_k = 0$  and  $PXP'$  is symmetric if and only if  $X$  is. We write

$$(2.2) \quad J = \sum_{s=1}^p J_s$$

where

$$(2.3) \quad J_s = \lambda_s I_{m_s} + \sum_{t=1}^{n_s} \sum_1^{r_{st}} U_{e_{st}} ;$$

$\sum$  indicates direct sum,  $I_t$  is a  $t$ -square identity matrix,  $U_t$  is  $t$ -square auxiliary unit matrix (i.e. 1 in the superdiagonal and 0 elsewhere) and  $\sum_1^{r_{st}} U_{e_{st}}$  is the direct sum of  $U_{e_{st}}$  with itself  $r_{st}$  times.

By a routine computation we see that

$$T^k(Y) = 0$$

if and only if

$$(2.4) \quad \sum_{\alpha=0}^k \binom{k}{\alpha} (-1)^\alpha J_s^{k-\alpha} Y_{st} (J_t)^\alpha = 0, \quad s, t = 1, \dots, p,$$

where  $Y = (Y_{st})$ ,  $s, t = 1, \dots, p$  is a partitioning of  $Y$  conformal with the partitioning of  $J$  given by (2.2).

For  $s \neq t$ , it is clear that the matrix representation of (2.4),

$$(I_{m_t} \otimes J_s - J_t \otimes I_{m_s})^k$$

has the single nonzero eigenvalue  $(\lambda_s - \lambda_t)^k$  and thus  $Y_{st} = 0$ . Hence we need only consider the equation (2.4) for  $s = t$ . We may again partition  $Y_{ss}$  conformally with  $J_s$  in (2.3). We are thus led to consider the null space of the mapping

$$(2.5) \quad (I_{e_{st}} \otimes U_{e_{sj}} - U_{e_{st}} \otimes I_{e_{sj}})^k.$$

LEMMA 1. Let  $T = I_m \otimes U_n - U_m \otimes I_n$ . Then

$$(2.6) \quad \dim \eta(T) = \min(m, n),$$

$$(2.7) \quad \dim \eta(T^2) = \begin{cases} 2 \min(m, n), & \text{if } m \neq n \\ 2n - 1, & \text{if } m = n. \end{cases}$$

*Proof.* Suppose  $n \leq m$  and that  $T(X) = 0$ . Let  $x_1, \dots, x_m$  be the column  $n$ -vectors of  $X$ . Then we have

$$(2.8) \quad \begin{aligned} U_n x_j - x_{j+1} &= 0, \quad j = 1, 2, \dots, m - 1, \\ U_n x_m &= 0. \end{aligned}$$

For  $r = 1, 2, \dots, n - 1$  consider the  $(r - j + 1)$  coordinate of (2.8) for  $j = 1, \dots, r$  and we conclude that

$$x_{r+1,1} = x_{r,2} = \dots = x_{1,r+1} = c_{r+1}.$$

Next consider the  $(n - j + 1)$  coordinate of (2.8) for  $j = 1, \dots, n$  to obtain

$$0 = x_{n2} = x_{n-1,3} = \dots = x_{1,n+1}.$$

Similarly we see that the remaining elements of  $X$  are zero. Hence we find that the  $j$ th column of the  $n \times m$  matrix  $X$  is the transpose of the  $n$ -vector

$$[c_j, c_{j+1}, \dots, c_n, 0, \dots, 0]$$

for  $j = 1, 2, \dots, n$ . The other  $m - n$  columns are zero.

In case  $n \geq m$ , it is easy to check that the  $j$ th row of  $X$  is the  $m$ -vector

$$[c_j, c_{j+1}, \dots, c_m, 0, \dots, 0]$$

for  $j = 1, 2, \dots, m$ . The other  $n - m$  rows are zero.

This establishes (2.6). To prove (2.7) let  $T^2(X) = 0$  and  $x_1, x_2, \dots, x_m$  be the column  $n$ -vectors of  $X$ . Let us consider the following cases:

(i)  $m = n$ .

We have

$$U_n^2 x_n = 0, \quad U_n^2 x_{n-1} = 2U_n x_n$$

and

$$U_n^2 x_j - 2U_n x_{j+1} + x_{j+2} = 0, \quad j = 1, 2, \dots, n - 2.$$

Solving these equations recursively we find that the 1st, 2nd and  $j$ th rows of  $X$  are respectively

$$[x_{11}, x_{12}, \dots, x_{1,n-2}, x_{1,n-1}, x_{1n}],$$

$$[x_{21}, x_{22}, \dots, x_{2,n-2}, x_{2,n-1}, 0]$$

and

$$\begin{aligned} &(j - 1)[x_{2,j-1}, x_{2,j}, \dots, x_{2,n-1}, 0, \dots, 0] \\ &- (j - 2)[x_{1,j}, x_{1,j+1}, \dots, x_{1,n}, 0, \dots, 0], \end{aligned}$$

for  $j = 3, 4, \dots, n$ .

The number of arbitrary parameters in  $X$  is  $2n - 1$ .

(ii)  $n < m$ .

Here we have the following equations:

$$(2.9) \quad \begin{aligned} U_n^2 x_j - 2U_n x_{j+1} + x_{j+2} &= 0, \quad j = 1, 2, \dots, m - 2 \\ U_n^2 x_{m-1} - 2U_n x_m &= 0 \\ U_n^2 x_m &= 0 \end{aligned}$$

and by solving recursively again we find that the 1st, 2nd and  $j$ th rows of  $X$  are respectively the  $m$ -vectors

$$\begin{aligned} [x_{11}, \dots, x_{1,n-1}, x_{1,n}, nx_{n,2}, 0, \dots, 0], \\ [x_{21}, \dots, x_{2,n-1}, (n-1)x_{n,2}, 0, 0, \dots, 0] \end{aligned}$$

and

$$\begin{aligned} [(j-1)x_{2,j-1}, \dots, (j-1)x_{2,n-1}, (n-j+1)x_{n,2}, 0, \dots, 0] \\ - (j-2)[x_{1,j}, \dots, x_{1,n}, 0, 0, \dots, 0] \end{aligned}$$

for  $j = 3, 4, \dots, n$ .

In case  $n > m$ , by similar computation we find that the 1st, 2nd and  $j$ th rows of  $X$  are respectively

$$\begin{aligned} [x_{11}, \dots, x_{1,m-2}, x_{1,m-1}, x_{1m}], \\ [x_{21}, \dots, x_{2,m-2}, x_{2,m-1}, x_{2m}] \end{aligned}$$

and

$$\begin{aligned} (j-1)[x_{2,j-1}, \dots, x_{2,m-1}, x_{2m}, 0, \dots, 0] \\ - (j-2)[x_{1,j}, \dots, x_{1,m}, 0, 0, \dots, 0] \end{aligned}$$

for  $j = 3, 4, \dots, m + 1$ . The remaining  $n - m - 1$  rows are zero.

From case (ii), we observe that the number of parameters in  $X$  is  $2 \min(m, n)$ .

We now state and prove the following

**LEMMA 2.** *Let  $A$  be an  $n$ -square matrix with the single eigenvalue  $\lambda$  and let  $(x - \lambda)^{n_i}$  be an elementary divisor of  $A$  of multiplicity  $r_i$ ,  $i = 1, \dots, p$ ,  $n_1 > \dots > n_p$ . Then the most general matrix  $X$  satisfying (1.11) has*

$$(2.10) \quad \sum_{i=1}^p \left[ r_i^2(2n_i - 1) + 4r_i \sum_{j=i+1}^p r_j e_j \right]$$

arbitrary parameters.

Moreover if  $X$  is symmetric it contains

$$(2.11) \quad \frac{1}{2} \sum_{i=1}^p \left[ r_i^2(2n_i - 1) + r_i + 4r_i \sum_{j=i+1}^p r_j n_j \right]$$

parameters.

*Proof.* Without any loss of generality we can assume that

$$(2.12) \quad A = \sum_{i=1}^p \sum_{j=1}^{r_i} U_i$$

where  $\sum \cdot U_i$  indicates the direct sum of  $U_i$  with itself  $r_i$  times. We partition  $X$  conformally with  $A$  in (2.12) and observe that the equation

$$U_i^2 X_{ij} - 2U_i X_{ij} U_j' + X_{ij} (U_j')^2 = 0$$

determines the structure of any block  $X_{ij}$  in the partitioning of  $X$ .

From case (i) of Lemma 1, we conclude that any block  $X_{ij}$  corresponding to equal  $U_i$ 's contains  $2n_i - 1$  arbitrary parameters and there are  $r_i^2$  such blocks. Also from case (ii) any block in  $X$  that corresponds to  $U_i$  and  $U_j, i < j$ , contains  $2n_j$  arbitrary parameters. Hence the total number of parameters in  $X$  is given by (2.10).

In order to find the number of parameters in a symmetric  $X$  we first consider a diagonal block. Its structure has been discussed in Lemma 1, case (i). We observe that if this matrix is symmetric, the number of parameters in it reduces from  $2n_i - 1$  to  $n_i$ .

Then we consider two symmetrically placed off-diagonal blocks  $X_{ij}$  and  $X_{ji}$  of orders  $n_i \times n_j$  and  $n_j \times n_i$  respectively. If  $X$  is to be symmetric then by equating the terms of  $X_{ij}$  and  $X_{ji}$  which are symmetrically placed about the main diagonal of  $X$ , the number of arbitrary parameters in  $X_{ij}$  and  $X_{ji}$  reduces from  $2(2n_j)$  to  $2n_j$ . If  $X_{ij}$  and  $X_{ji}$  are of order  $n_i \times n_i$  then the number of parameters reduces from  $2(2n_i - 1)$  to  $2n_i - 1$ .

We are now in a position to sum the number of parameters in  $X$  if it is symmetric and satisfies (1.11). There are  $r_i$  blocks in the main diagonal, each of order  $n_i, i = 1, \dots, p$ . The number of parameters in each of these blocks is  $n_i$ . There are  $r_i(r_i - 1)/2$  other square blocks of order  $n_i$ . Each of them contains  $(2n_i - 1)$  parameters. Thus

$$\frac{1}{2} \sum_{i=1}^p \{r_i^2(2n_i - 1) + r_i\}$$

is the number of parameters in all those blocks of  $X$  which are square. Since any block of order  $n_i \times n_j$  where  $n_i > n_j$  contains  $2n_j$  parameters, and since we are considering  $X$  to be symmetric, we conclude that the total number of arbitrary parameters in  $X$  is given by (2.11).

We can similarly prove the following

LEMMA 3. *Let  $A$  be the matrix given in Lemma 2. Then the most*



general matrix  $X$  satisfying (1.8) has

$$\sum_{i=1}^p \left( r_i^2 n_i + 2r_i \sum_{j=i+1}^p r_j n_j \right)$$

arbitrary parameters.

Moreover if  $X$  is symmetric, it contains

$$\frac{1}{2} \sum_{i=1}^p \left[ r_i(r_i + 1)n_i + 2r_i \sum_{j=i+1}^p r_j n_j \right]$$

parameters.

We now state and prove the following

**THEOREM 1.** *Let  $A$  be an  $n$ -square matrix with distinct eigenvalues  $\lambda_1, \dots, \lambda_p$  and let  $(x - \lambda_i)^{e_{ij}}$ ,  $j = 1, \dots, n_i$ ,  $e_{i1} > \dots > e_{in_i}$  be the elementary divisors of  $A$  corresponding to  $\lambda_i$ , where each  $(x - \lambda_i)^{e_{ij}}$  has been repeated  $r_{ij}$  times. Then (1.4), (1.5), (1.6) and (1.7) hold.*

*Proof.* It was pointed out earlier that if  $Y = (Y_{rs})$ ,  $r, s = 1, \dots, p$  is the partitioning of  $Y$  conformal with the partitioning of  $J$  in (2.2), then all the off-diagonal blocks are zero. Hence we have simply to find the number of parameters in  $Y_{ii}$ ,  $i = 1, \dots, p$ .

As proved in Lemma 2, the number of parameters in  $Y_{ii}$  is

$$\sum_{j=1}^{n_i} \left[ r_{ij}^2(2e_{ij} - 1) + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right].$$

Summing the above with respect to  $i$  we obtain the formula (1.6). In case  $Y$  is symmetric, the number of parameters in  $Y_{ii}$  is

$$\frac{1}{2} \sum_{j=1}^{n_i} \left[ r_{ij}^2(2e_{ij} - 1) + r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right].$$

Summing the above on  $i$  we obtain (1.7).

Similarly, we can make use of Lemma 3 in proving (1.4) and (1.5).

We now prove

**THEOREM 2.** *Let  $A$  be as given in Theorem 1. Then the maximum number of linearly independent skew-symmetric matrices satisfying (1.8) or (1.11) is*

$$\frac{1}{2}(n - p)(n - p + 1).$$

*Proof.* In order to prove our result for  $\dim \gamma(T^2)$ , let  $m_i = \sum_{j=1}^{n_i} r_{ij} e_{ij}$  and consider

$$\begin{aligned}
 m_i^2 - m_i - \sum_{j=1}^{n_i} & \left[ r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \\
 &= \sum_{j=1}^{n_i} \left[ r_{ij}^2 e_{ij}^2 + 2r_{ij}e_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} - r_{ij}e_{ij} \right] \\
 &\quad - \sum_{j=1}^{n_i} \left[ r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \\
 &= \sum_{j=1}^{n_i} \left[ r_{ij}^2(e_{ij} - 1)^2 - r_{ij}(e_{ij} - 1) + 2r_{ij}(e_{ij} - 2) \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right].
 \end{aligned}$$

Now, it is clear that  $r_{ij}^2(e_{ij} - 1) \geq r_{ij}(e_{ij} - 1)$ . The last term in the above expression will be negative only when  $e_{ij} = 1$ . But we know that  $e_{i1} > e_{i2} > \dots > e_{in_i}$ , so that  $e_{ij}$  will be 1 only for  $j = n_i$ . In that case  $\sum_{k=j+1}^{n_i}$  does not appear, and we have

$$\frac{1}{2} \sum_{j=1}^{n_i} \left[ r_{ij}^2(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \leq \frac{1}{2}(m_i^2 - m_i).$$

This holds for  $i = 1, \dots, p$ .

To determine a bound on  $\gamma(T)$ , consider

$$\begin{aligned}
 m_i^2 - m_i - \sum_{j=1}^{n_i} & \left[ r_{ij}(r_{ij} - 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \\
 &= \sum_{j=1}^{n_i} \left[ r_{ij}^2 e_{ij}(e_{ij} - 1) + 2r_{ij}(e_{ij} - 1) \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \\
 &\geq 0, \text{ since } e_{ij} \geq 1.
 \end{aligned}$$

Thus we have

$$\frac{1}{2} \sum_{j=1}^{n_i} \left[ r_{ij}(r_{ij} - 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik}e_{ik} \right] \leq \frac{1}{2}(m_i^2 - m_i).$$

It may be observed that the upper bound is attained for  $r_{i1} = m_i, e_{i1} = 1$  and the remaining  $e$ 's and  $r$ 's all zero.

We have thus proved that

$$\dim \gamma(T^2) \leq \frac{1}{2} \sum_{i=1}^p (m_i^2 - m_i)$$

and

$$\dim \gamma(T) \leq \frac{1}{2} \sum_{i=1}^p (m_i^2 - m_i),$$

where  $m_i$  is the multiplicity of the eigenvalue  $\lambda_i$  of  $A$ .

Now we have to maximize  $\sum_{i=1}^p (m_i^2 - m_i)$  under the condition that

$m_1 + \cdots + m_p = n$ , the order of  $A$ . Note that

$$m_i^2 - m_i = (m_i - 1)^2 + (m_i - 1)$$

and each  $m_i - 1 \geq 0$ . Hence, we have

$$\sum_{i=1}^p (m_i - 1)^2 \leq \left[ \sum_{i=1}^p (m_i - 1) \right]^2 = (n - p)^2.$$

Thus the maximum value of both  $\dim \gamma(T^2)$  and  $\dim \gamma(T)$  is

$$\frac{1}{2}[(n - p)^2 + (n - p)].$$

The bounds are achieved when  $m_1 = \cdots = m_{p-1} = 1$  and  $m_p = n - p + 1$ .

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