ON A COMMUTATOR RESULT OF TAUSSKY AND ZASSENHAUS

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1. Introduction and results. Let $M_n$ denote the set of $n$-square matrices over a field $F$. For $A, B$ in $M_n$ let $[A, B] = AB - BA'$, where $A'$ is the transpose of $A$ and define inductively

$$[A, B]_k = [A, [A, B]_{k-1}] .$$

If $P^{-1}JP = A$, then

$$[A, X] = [P^{-1}JP, X] = P^{-1}[J, PXP'](P^{-1})' ,$$

and similarly

$$[A, X]_k = P^{-1}[J, PXP']_k(P^{-1})' .$$

Now for a fixed $A$ let $T$ be the linear map of $M_n$ into itself defined by

$$T(Y) = [A, Y]$$

and (1.1) implies that

$$T^k(Y) = [A, Y]_k .$$

In a recent paper [1], Taussky and Zassenhaus showed that $A$ is non-derogatory if and only if any nonsingular $X$ in the null space of $T$ is symmetric. In this note we investigate the structure of the null space of both $T$ and $T^2$ for arbitrary $A$.

Enlarge the field $F$ to include $\lambda_i, i = 1, \cdots, p$, the distinct eigenvalues of $A$, and let $(x - \lambda_i)^{e_{ij}}$, $j = 1, \cdots, n$, $e_{ij} > \cdots > e_{in}$, $i = 1, \cdots, p$ be the distinct elementary divisors of $A$ where $(x - \lambda_i)^{e_{ij}}$ appears with multiplicity $r_{ij}$. Set $m_i = \sum_{j=1}^{n_i} r_{ij}$, the algebraic multiplicity of $\lambda_i$. Let $\gamma(T)$ denote the null space of $T$, $\sigma(T)$ denote the subspace of symmetric matrices in $\gamma(T)$, and $\gamma(T)$ denote the subspace of skew-symmetric matrices in $\gamma(T)$. We show that

$$\dim \gamma(T) = \sum_{i=1}^{p} \left[ \sum_{j=1}^{n_i} \left( r_{ij}^2 e_{ij} + 2 r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right],$$

$$\dim \sigma(T) = \frac{1}{2} \sum_{i=1}^{p} \left[ \sum_{j=1}^{n_i} \left( r_{ij} (r_{ij} + 1) e_{ij} + 2 r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right].$$

Received December 17, 1959. The work of this author was supported by U.S. National Science Foundation Grant, NSF-G5416. The second author is a Postdoctorate Fellow of the National Research Council of Canada. The authors are grateful to Professor O. Taussky for her helpful suggestions.
(1.6) \[ \dim \gamma(T^o) = \sum_{\ell=1}^{n} \left[ \sum_{j=1}^{n_{\ell}} \left\{ r^\ell_{ij}(2e_{ij} - 1) + 4r^\ell_{ij} \sum_{k=j+1}^{n_{\ell}} r_{ik}e_{tk} \right\} \right] , \]

(1.7) \[ \dim \sigma(T^o) = \frac{1}{2} \sum_{\ell=1}^{n} \left[ \sum_{j=1}^{n_{\ell}} \left\{ r^\ell_{ij}(2e_{ij} - 1) + r_{ij} + 4r^\ell_{ij} \sum_{k=j+1}^{n_{\ell}} r_{ik}e_{tk} \right\} \right] . \]

In case \( A \) is nonderogatory, \( n_i = 1, r_{ij} = 1, i = 1, \ldots, p \) and (1.4) and (1.5) reduce to

\[ \dim \gamma(T) = n = \dim \sigma(T) . \]

Thus every matrix \( X \) satisfying

(1.8) \[ AX = XA' \]

where \( A \) is non-derogatory is symmetric, the result in [1]. Moreover, if every matrix \( X \) satisfying (1.8) is symmetric then \( \dim \gamma(T) = \dim \sigma(T) \).

Using the formulas (1.4) and (1.5) we see that this condition implies that

\[ \sum_{\ell=1}^{n} \sum_{j=1}^{n_{\ell}} (r^\ell_{ij} - r_{ij}) e_{ij} + 2 \sum_{\ell=1}^{n} r^\ell_{ij} \sum_{j=1}^{n_{\ell}} r_{ik}e_{tk} = 0 . \]

Now since \( r_{ij}, e_{ij} \) and \( n_i \) are all positive integers we conclude that \( r_{ij} = 1, j = 1, \ldots, n_i \) and \( n_i = 1 \). That is, there is only one elementary divisor corresponding to each eigenvalue. Hence, if every matrix \( X \) satisfying (1.8) is symmetric then \( A \) is non-derogatory, a result also found in [1].

We also show in this case that \( \gamma(T) \) consists of matrices of the form \( PXP' \) where \( P \) is fixed (depending on \( A \)) and \( X \) is persymmetric, (i.e. all the entries of \( X \) on each line perpendicular to the main diagonal are equal).

We next note that \( \gamma(T) = \sigma(T) + \gamma(T) \) (direct) and \( \gamma(T^o) = \sigma(T^o) + \gamma(T^o) \) (direct). The first statement is easy to show; we indicate the brief proof of the second statement:

Since \( X = \frac{X + X'}{2} + \frac{X - X'}{2}, \) if \( X \in \gamma(T^o) \), then

\[ T^o(X + X') = [A, [A, X + X']] \]
\[ = [A, [A, X]] + [A, X']] \]
\[ = [A, [A, X]] + [A, [A, X']] \]
\[ = T^o(X) - [A, [A, X']] \]
\[ = [A, [A, X]]' \]
\[ = (T^o(X))' = 0 . \]

Similarly, \( T^o(X - X') = 0 \). Thus any \( X \in \gamma(T^o) \) is expressible uniquely as a sum of two elements, one in \( \sigma(T^o) \) and the other in \( \gamma(T^o) \). Hence
(1.9) \[ \dim \gamma(T) = \dim \eta(T) - \dim \sigma(T) \]
\[ = \frac{1}{2} \sum_{i=1}^{p} \left[ \sum_{j=1}^{n_i} \left( r_{ij}(r_{ij} - 1)e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right] , \]

(1.10) \[ \dim \gamma(T^2) = \dim \eta(T^2) - \dim \sigma(T^2) \]
\[ = \frac{1}{2} \sum_{i=1}^{p} \left[ \sum_{j=1}^{n_i} \left( r_{ij}(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right] . \]

In case \( A \) is non-derogatory, (1.6), (1.7) and (1.10) reduce to
\[ \dim \gamma(T^2) = 2n - p , \]
\[ \dim \sigma(T^2) = n , \]
\[ \dim \gamma(T^2) = n - p . \]

We thus conclude that unless all the eigenvalues of \( A \) are distinct \( (p = n) \) there exist skew-symmetric matrices \( X \) satisfying
\[ A^2X - 2AXA' + X(A')^2 = 0 . \]

If \( p = n, \) and \( A \) is non-derogatory
\[ \dim \gamma(T^2) = n = \dim \sigma(T^2) \]
and any matrix \( X \) satisfying (1.11) is symmetric.

On the other hand suppose
\[ \dim \gamma(T^2) = \dim \sigma(T^2) . \]

From (1.6) and (1.7) we conclude that
\[ \sum_{i=1}^{p} \left[ \sum_{j=1}^{n_i} \left( r_{ij}(2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right) \right] = 0 . \]

Hence \( n_i = 1, \) \( r_{ij} = 1, \) \( e_{ik} = 1 \) and we conclude that \( p = n. \) That is, if every matrix \( X \) satisfying (1.11) is symmetric then the eigenvalues of \( A \) are distinct.

We show finally (Theorem 2) that if \( A \) is an \( n \)-square matrix with \( p \) distinct eigenvalues then both \( \dim \gamma(T) \) and \( \dim \gamma(T^2) \) are at most \( \frac{1}{2}(n - p)(n - p + 1) \). Moreover, for each \( p \) this bound is best possible.

Thus if there exists a skew-symmetric solution of (1.8) or (1.11), then \( A \) has multiple eigenvalues, without the assumption that \( A \) is non-derogatory.

II. Proofs. Let \( E_{ij} \in M_n \) be the matrix with 1 in position \( i, j \) and 0 elsewhere. With respect to this basis, ordered lexicographically, it may be checked that \( T \) has the matrix representaion
(2.1) \[ T = I \otimes A - A \otimes I \]

where \( \otimes \) indicates Kronecker product.

From (1.2) we may take \( A \) to be in Jordan canonical form \( J \), since \([A, X]_k = 0\) if and only if \([J, PXP']_k = 0\) and \( PXP' \) is symmetric if and only if \( X \) is. We write

(2.2) \[ J = \sum_{s=1}^{p} \cdot J_s \]

where

(2.3) \[ J_s = \lambda_s I_{m_s} + \sum_{t=1}^{r_{st}} \sum_{i} U_{e_{st}} ; \]

\( \sum \cdot \) indicates direct sum, \( I_s \) is a \( t \)-square identity matrix, \( U_t \) is \( t \)-square auxiliary unit matrix (i.e. 1 in the superdiagonal and 0 elsewhere) and \( \sum_{t=1}^{r_{st}} U_{e_{st}} \) is the direct sum of \( U_{e_{st}} \) with itself \( r_{st} \) times.

By a routine computation we see that

\[ T^k(Y) = 0 \]

if and only if

(2.4) \[ \sum_{a=0}^{k} \binom{k}{a} (-1)^a J_s^{k-a} Y_s(J_t)^a = 0 , \quad s, t = 1, \ldots, p , \]

where \( Y = (Y_{st}) \), \( s, t = 1, \ldots, p \) is a partitioning of \( Y \) conformal with the partitioning of \( J \) given by (2.2).

For \( s \neq t \), it is clear that the matrix representation of (2.4),

\[ (I_{m_s} \otimes J_s - J_t \otimes I_{m_t})^k \]

has the single nonzero eigenvalue \( (\lambda_s - \lambda_t)^k \) and thus \( Y_{st} = 0 \). Hence we need only consider the equation (2.4) for \( s = t \). We may again partition \( Y_{ss} \) conformally with \( J_s \) in (2.3). We are thus led to consider the null space of the mapping

(2.5) \[ (I_{e_{st}} \otimes U_{e_{sf}} - U_{e_{st}} \otimes I_{e_{sf}})^k . \]

**Lemma 1.** Let \( T = I_m \otimes U_n - U_m \otimes I_n \). Then

(2.6) \[ \dim \gamma(T) = \min (m, n) , \]

(2.7) \[ \dim \gamma(T^2) = \begin{cases} 2 \min (m, n) , & \text{if } m \neq n \ 
2n - 1 , & \text{if } m = n . \end{cases} \]

**Proof.** Suppose \( n \leq m \) and that \( T(X) = 0 \). Let \( x_1, \ldots, x_m \) be the column \( n \)-vectors of \( X \). Then we have
(2.8) \[ U_n x_j - x_{j+1} = 0, \quad j = 1, 2, \ldots, m - 1, \]
\[ U_n x_m = 0. \]

For \( r = 1, 2, \ldots, n - 1 \) consider the \((r - j + 1)\) coordinate of (2.8) for \( j = 1, \ldots, r \) and we conclude that
\[ x_{r+1,1} = x_{r,2} = \cdots = x_{r,r} = e_{r+1}. \]

Next consider the \((n - j + 1)\) coordinate of (2.8) for \( j = 1, \ldots, n \) to obtain
\[ 0 = x_{n+1,1} = x_{n-1,1} = \cdots = x_{1,n+1}. \]

Similarly we see that the remaining elements of \( X \) are zero. Hence we find that the \( j \)th column of the \( n \times m \) matrix \( X \) is the transpose of the \( n \)-vector
\[ [c_j, c_{j+1}, \ldots, c_n, 0, \ldots, 0] \]
for \( j = 1, 2, \ldots, n \). The other \( m - n \) columns are zero.

In case \( n \geq m \), it is easy to check that the \( j \)th row of \( X \) is the \( m \)-vector
\[ [c_j, c_{j+1}, \ldots, c_m, 0, \ldots, 0] \]
for \( j = 1, 2, \ldots, m \). The other \( n - m \) rows are zero.

This establishes (2.6). To prove (2.7) let \( T^2(X) = 0 \) and \( x_1, x_2, \ldots, x_m \) be the column \( n \)-vectors of \( X \). Let us consider the following cases:

(i) \( m = n \).
We have
\[ U_n^2 x_n = 0, \quad U_n^2 x_{n-1} = 2 U_n x_n \]
and
\[ U_n^2 x_j - 2 U_n x_{j+1} + x_{j+2} = 0, \quad j = 1, 2, \ldots, n - 2. \]

Solving these equations recursively we find that the 1st, 2nd and \( j \)th rows of \( X \) are respectively
\[ [x_{11}, x_{12}, \cdots, x_{1,n-2}, x_{1,n-1}, x_{1n}] , \]
\[ [x_{21}, x_{22}, \cdots, x_{2,n-2}, x_{2,n-1}, 0] \]
and
\[ (j - 1)[x_{2,j-1}, x_{2,j}, \cdots, x_{2,n-1}, 0, \ldots, 0] \]
\[ - (j - 2)[x_{1,j}, x_{1,j+1}, \cdots, x_{1,n}, 0, \ldots, 0] , \]
for \( j = 3, 4, \ldots, n \).

The number of arbitrary parameters in \( X \) is \( 2n - 1 \).
(ii) $n < m$.

Here we have the following equations:

\begin{equation}
U_n^2 x_j - 2 U_n x_{j+1} + x_{j+2} = 0, \quad j = 1, 2, \ldots, m - 2
\end{equation}

\begin{equation}
U_n^2 x_{m-1} - 2 U_n x_m = 0
\end{equation}

\begin{equation}
U_n^2 x_m = 0
\end{equation}

and by solving recursively again we find that the 1st, 2nd and $j$th rows of $X$ are respectively the $m$-vectors

\begin{align*}
[x_{11}, \cdots, x_{1,n-1}, x_{1,n}, n x_{n,2}, 0, \cdots, 0], \\
[x_{21}, \cdots, x_{2,n-1}, (n - 1)x_{n,2}, 0, 0, \cdots, 0]
\end{align*}

and

\begin{align*}
[(j - 1)x_{2,j-1}, \cdots, (j - 1)x_{2,n-1}, (n - j + 1)x_{n,2}, 0, \cdots, 0] \\
- (j - 2)[x_{1,j}, \cdots, x_{1,n}, 0, 0, \cdots, 0]
\end{align*}

for $j = 3, 4, \cdots, n$.

In case $n > m$, by similar computation we find that the 1st, 2nd and $j$th rows of $X$ are respectively

\begin{align*}
[x_{11}, \cdots, x_{1,m-2}, x_{1,m-1}, x_{1m}], \\
[x_{21}, \cdots, x_{2,m-2}, x_{2,m-1}, x_{2m}]
\end{align*}

and

\begin{align*}
(j - 1)[x_{2,j-1}, \cdots, x_{2,m-1}, x_{2m}, 0, \cdots, 0] \\
- (j - 2)[x_{1,j}, \cdots, x_{1,m}, 0, 0, \cdots, 0]
\end{align*}

for $j = 3, 4, \cdots, m + 1$. The remaining $n - m - 1$ rows are zero.

From case (ii), we observe that the number of parameters in $X$ is $2 \min(m, n)$.

We now state and prove the following

**Lemma 2.** Let $A$ be an $n$-square matrix with the single eigenvalue $\lambda$ and let $(x - \lambda)^{n_i}$ be an elementary divisor of $A$ of multiplicity $r_i$, $i = 1, \cdots, p$, $n_1 > \cdots > n_p$. Then the most general matrix $X$ satisfying (1.11) has

\begin{equation}
\sum_{i=1}^{p} r_i (2n_i - 1) + 4 \sum_{j=1}^{p} r_j e_j
\end{equation}

arbitrary parameters.

Moreover if $X$ is symmetric it contains
(2.11) \[ \frac{1}{2} \sum_{i=1}^{p} \left[ r_i^2(2n_i - 1) + r_i + 4r_i \sum_{j=i+1}^{p} r_j n_j \right] \]

parameters.

Proof. Without any loss of generality we can assume that

(2.12) \[ A = \sum_{i=1}^{p} \sum_{j=1}^{r_i} U_i \]

where \( \sum_{\cdot} U_i \) indicates the direct sum of \( U_i \) with itself \( r_i \) times. We partition \( X \) conformally with \( A \) in (2.12) and observe that the equation

\[ U_iX_{i3} - 2U_iX_{i3}U'_j + X_{i3}(U'_j)^2 = 0 \]

determines the structure of any block \( X_{i3} \) in the partitioning of \( X \).

From case (i) of Lemma 1, we conclude that any block \( X_{i3} \) corresponding to equal \( U_i \)'s contains \( 2n_i - 1 \) arbitrary parameters and there are \( r_i \) such blocks. Also from case (ii) any block in \( X \) that corresponds to \( U_i \) and \( U_j, i < j \), contains \( 2n_j \) arbitrary parameters. Hence the total number of parameters in \( X \) is given by (2.10).

In order to find the number of parameters in a symmetric \( X \) we first consider a diagonal block. Its structure has been discussed in Lemma 1, case (i). We observe that if this matrix is symmetric, the number of parameters in it reduces from \( 2n_i - 1 \) to \( n_i \).

Then we consider two symmetrically placed off-diagonal blocks \( X_{i3} \) and \( X_{ji} \) of orders \( n_i \times n_j \) and \( n_j \times n_i \) respectively. If \( X \) is to be symmetric then by equating the terms of \( X_{i3} \) and \( X_{ji} \) which are symmetrically placed about the main diagonal of \( X \), the number of arbitrary parameters in \( X_{i3} \) and \( X_{ji} \) reduces from \( 2(2n_j) \) to \( 2n_j \). If \( X_{i3} \) and \( X_{ji} \) are of order \( n_i \times n_i \) then the number of parameters reduces from \( 2(2n_i - 1) \) to \( 2n_i - 1 \).

We are now in a position to sum the number of parameters in \( X \) if it is symmetric and satisfies (1.11). There are \( r_i \) blocks in the main diagonal, each of order \( n_i, i = 1, \ldots , p \). The number of parameters in each of these blocks is \( n_i \). There are \( r_i(r_i - 1)/2 \) other square blocks of order \( n_i \). Each of them contains \( (2n_i - 1) \) parameters. Thus

\[ \frac{1}{2} \sum_{i=1}^{p} \{ r_i^2(2n_i - 1) + r_i \} \]

is the number of parameters in all those blocks of \( X \) which are square. Since any block of order \( n_i \times n_j \) where \( n_i > n_j \) contains \( 2n_j \) parameters, and since we are considering \( X \) to be symmetric, we conclude that the total number of arbitrary parameters in \( X \) is given by (2.11).

We can similarly prove the following

**Lemma 3.** Let \( A \) be the matrix given in Lemma 2. Then the most
general matrix \( X \) satisfying (1.8) has

\[
\sum_{i=1}^{p} \left( r_i n_i + 2r_i \sum_{j=i+1}^{p} r_j n_j \right)
\]

arbitrary parameters.

Moreover if \( X \) is symmetric, it contains

\[
\frac{1}{2} \sum_{i=1}^{p} \left[ r_i (r_i + 1)n_i + 2r_i \sum_{j=i+1}^{p} r_j n_j \right]
\]

parameters.

We now state and prove the following

**THEOREM 1.** Let \( A \) be an \( n \)-square matrix with distinct eigenvalues \( \lambda_1, \ldots, \lambda_p \) and let \( (x - \lambda_i)^{e_{ij}}, j = 1, \ldots, n_i, e_{ii} > \cdots > e_{in_i} \) be the elementary divisors of \( A \) corresponding to \( \lambda_i \), where each \( (x - \lambda_i)^{e_{ij}} \) has been repeated \( r_{ij} \) times. Then (1.4), (1.5), (1.6) and (1.7) hold.

**Proof.** It was pointed out earlier that if \( Y = (Y_{rs}), r, s = 1, \ldots, p \) is the partitioning of \( Y \) conformal with the partitioning of \( J \) in (2.2), then all the off-diagonal blocks are zero. Hence we have simply to find the number of parameters in \( Y_{ii}, i = 1, \ldots, p \).

As proved in Lemma 2, the number of parameters in \( Y_{ii} \) is

\[
\sum_{j=1}^{n_i} \left[ r_{ij}^2 (2e_{ij} - 1) + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right].
\]

Summing the above with respect to \( i \) we obtain the formula (1.6). In case \( Y \) is symmetric, the number of parameters in \( Y_{ii} \) is

\[
\frac{1}{2} \sum_{j=1}^{n_i} \left[ r_{ij}^2 (2e_{ij} - 1) + r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right].
\]

Summing the above on \( i \) we obtain (1.7).

Similarly, we can make use of Lemma 3 in proving (1.4) and (1.5).

We now prove

**THEOREM 2.** Let \( A \) be as given in Theorem 1. Then the maximum number of linearly independent skew-symmetric matrices satisfying (1.8) or (1.11) is

\[
\frac{1}{2} (n - p)(n - p + 1)
\]

**Proof.** In order to prove our result for \( \dim \gamma(T^2) \), let \( m_i = \sum_{j=1}^{n_i} r_{ij} e_{ij} \) and consider
\[ m_i^2 - m_i - \sum_{j=1}^{n_i} \left[ r_{ij}^2 (2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \]

\[
= \sum_{j=1}^{n_i} \left[ r_{ij}^2 e_{ij}^2 + 2r_{ij} e_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} - r_{ij} e_{ij} \right] \\
- \sum_{j=1}^{n_i} \left[ r_{ij} (2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \\
= \sum_{j=1}^{n_i} \left[ r_{ij} (e_{ij} - 1)^2 - r_{ij} (e_{ij} - 1) + 2r_{ij} (e_{ij} - 2) \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right].
\]

Now, it is clear that \( r_{ij}^2 (e_{ij} - 1) \geq r_{ij} (e_{ij} - 1) \). The last term in the above expression will be negative only when \( e_{ij} = 1 \). But we know that \( e_{i1} > e_{i2} > \cdots > e_{in_i} \), so that \( e_{ij} \) will be 1 only for \( j = n_i \). In that case \( \sum_{k=j+1}^{n_i} \) does not appear, and we have

\[ \frac{1}{2} \sum_{j=1}^{n_i} \left[ r_{ij}^2 (2e_{ij} - 1) - r_{ij} + 4r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \leq \frac{1}{2} (m_i^2 - m_i). \]

This holds for \( i = 1, \ldots, p \).

To determine a bound on \( \gamma(T) \), consider

\[ m_i^2 - m_i - \sum_{j=1}^{n_i} \left[ r_{ij} (r_{ij} - 1) e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \]

\[ = \sum_{j=1}^{n_i} \left[ r_{ij} e_{ij} (e_{ij} - 1) + 2r_{ij} (e_{ij} - 1) \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \]

\[ \geq 0, \text{ since } e_{ij} \geq 1. \]

Thus we have

\[ \frac{1}{2} \sum_{j=1}^{n_i} \left[ r_{ij} (r_{ij} - 1) e_{ij} + 2r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik} \right] \leq \frac{1}{2} (m_i^2 - m_i). \]

It may be observed that the upper bound is attained for \( r_{i1} = m_i, e_{i1} = 1 \) and the remaining \( e \)'s and \( r \)'s all zero.

We have thus proved that

\[ \dim \gamma(T^2) \leq \frac{1}{2} \sum_{i=1}^{p} (m_i^2 - m_i) \]

and

\[ \dim \gamma(T) \leq \frac{1}{2} \sum_{i=1}^{p} (m_i^2 - m_i), \]

where \( m_i \) is the multiplicity of the eigenvalue \( \lambda_i \) of \( A \).

Now we have to maximize \( \sum_{i=1}^{p} (m_i^2 - m_i) \) under the condition that
$m_1 + \cdots + m_p = n$, the order of $A$. Note that

$$m_i^2 - m_i = (m_i - 1)^2 + (m_i - 1)$$

and each $m_i - 1 \geq 0$. Hence, we have

$$\sum_{i=1}^{p} (m_i - 1)^2 \leq \left( \sum_{i=1}^{p} (m_i - 1) \right)^2 = (n - p)^2.$$ 

Thus the maximum value of both $\dim \gamma(T^2)$ and $\dim \gamma(T)$ is

$$\frac{1}{2} [(n - p)^2 + (n - p)].$$

The bounds are achieved when $m_1 = \cdots = m_{p-1} = 1$ and $m_p = n - p + 1$.

**Reference**


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The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
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