

Pacific Journal of Mathematics

ON UNIVALENCE OF A CONTINUED FRACTION

EDWARD PETER MERKES AND W. T. SCOTT

ON UNIVALENCE OF A CONTINUED FRACTION

E. P. MERKES AND W. T. SCOTT

1. Introduction. For a fixed positive integer α let K_α denote the class of functions $f(z)$ which are regular at $z = 0$ and which have C -fraction expansions of the form

$$(1.1) \quad f(z) \sim \frac{z}{1} + \frac{a_1 z^\alpha}{1} + \frac{a_2 z^{2\alpha}}{1} + \dots + \frac{a_n z^{n\alpha}}{1} + \dots, |a_n| \leq 1/4.$$

From an elementary convergence theorem for continued fractions [4, p.42], it follows that each function of the class K_α is regular for $|z| < 1$. This and the one-to-one correspondence between C -fractions and power series [4, p. 400] permit a replacement of the correspondence symbol in (1.1) by equality for $|z| < 1$.

The purpose of this paper is to determine for K_α the radius of univalence, $U(\alpha)$, and bounds for the starlike radius, $S(\alpha)$, and the radius of convexity, $C(\alpha)$. In the case of S -fractions it was shown by Thale [3] that $U(1) \geq 12\sqrt{2}-16$ and Perron [2] established the fact that actual equality holds. This result is a special case of Theorem 2.1 whose proof employs value region techniques similar to those used by Thale and Perron. Moreover, the result $S(1) \geq 8/9$ in [3] is improved in Theorem 4.2.

The developments in this depend on the following value region theorem which is an immediate consequence of a result of Paydon and Wall [1]:

THEOREM 1.1. *If $f(z) \in K_\alpha$ and $|z|^\alpha = \rho^\alpha \leq 4r(1-r)$, $0 \leq r \leq 1/2$, then*

$$(1.2) \quad \left| \frac{f(z)}{z} - \frac{1}{1-r^2} \right| \leq \frac{r}{1-r^2}.$$

Moreover, for $z = \sqrt[\alpha]{4r(1-r)} e^{im\pi/\alpha}$, ($m = 1, 2, \dots, \alpha$), there is a value of $f(z)/z$ on the boundary of the disc (1.2) if and only if there exists a φ , $0 \leq \varphi < 2\pi$, such that $f(z) \equiv f(z; \varphi)$, where

$$(1.3) \quad f(z; \varphi) = \frac{z}{1} + \frac{\frac{1}{4}e^{i\varphi}z^\alpha}{1} + \frac{\frac{1}{4}z^\alpha}{1} + \dots + \frac{\frac{1}{4}z^\alpha}{1} + \dots.$$

2. Determination of $U(\alpha)$. For $f(z) \in K_\alpha$ and for a fixed positive integer n put

Received April 27, 1959, and in revised form July 31, 1959.

$$(2.1) \quad \begin{aligned} f_{0,n}(z) &= z, \\ f_{p+1,n}(z) &= \frac{z}{1 + a_{n-p}z^{\alpha-1}f_{p,n}(z)}, \quad (p = 0, 1, \dots, n-1), \end{aligned}$$

where the numbers a_j are the coefficients in the C -fraction expansion (1.1) of $f(z)$. It is easily seen that $f_{p,n}(z)$ is the approximant of (1.1) of order $n + 1$, and that $f_{p,n}(z) \in K_\alpha$ for each p .

For non-negative integers s, t , and for non-zero numbers z_1, z_2 , (2.1) may be used to show that

$$(2.2) \quad \begin{aligned} & z_1^s z_2^t f_{p+1,n}(z_1) - z_1^t z_2^s f_{p+1,n}(z_2) \\ &= \frac{f_{p+1,n}(z_1)f_{p+1,n}(z_2)}{z_1 z_2} \{ z_1^{s+1} z_2^t - z_1^t z_2^{s+1} - a_{n-p} [z_1^{t+\alpha-1} z_2^{s+1} f_{p,n}(z_1) \\ & \quad - z_1^{s+1} z_2^{t+\alpha-1} f_{p,n}(z_2)] \}, \quad (p = 0, 1, \dots, n-1). \end{aligned}$$

This identity plays a fundamental role in the proof of the following theorem.

THEOREM 2.1. *The radius of univalence of K_α is given by*

$$(2.3) \quad \begin{aligned} U(2) &= 2\sqrt{2/3}, \\ [U(\alpha)]^\alpha &= \left[\frac{6\sqrt{\alpha^2 - 2\alpha + 9} - 2(\alpha + 7)}{(\alpha - 2)^2} \right], \quad (\alpha = 1, 3, 4, \dots). \end{aligned}$$

There is no larger region, containing the disc $|z| < U(\alpha)$, in which all functions of K_α are univalent.

Proof. For $f(z) \in K_\alpha$ and for a fixed positive odd integer $n = 2m + 1$ it follows from (2.2) that

$$(2.4) \quad \begin{aligned} & f_{n,n}(z_1) - f_{n,n}(z_2) \\ &= \frac{f_{n,n}(z_1)f_{n,n}(z_2)}{z_1 z_2} \{ z_1 - z_2 - a_1 [z_1^{\alpha-1} z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1} f_{n-1,n}(z_2)] \}. \end{aligned}$$

Repeated application of (2.2) yields

$$(2.5) \quad \begin{aligned} & a_1 [z_1^{\alpha-1} z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1} f_{n-1,n}(z_2)] \\ &= \sum_{j=1}^{m+1} (z_1 z_2)^{j-1} (z_1^{\alpha-1} - z_2^{\alpha-1}) \prod_{p=1}^{2j-1} a_p \frac{f_{n-p,n}(z_1)f_{n-p,n}(z_2)}{z_1 z_2} \\ & \quad - \sum_{j=1}^m (z_1 z_2)^j (z_1 - z_2) \prod_{p=1}^{2j} a_p \frac{f_{n-p,n}(z_1)f_{n-p,n}(z_2)}{z_1 z_2}. \end{aligned}$$

For z_1 and z_2 in the disc $|z| < 1$, r can be chosen with $0 < r < 1/2$ such that $|z_i|^\alpha \leq 4r(1-r)$, ($i = 1, 2$), and by Theorem 1.1, $|f_{p,n}(z_i)/z_i| \leq 1/(1-r)$, ($i = 1, 2; p = 0, 1, \dots, n$). When the triangle inequality is applied to the right member of (2.5) and the indicated bounds are used, there

results

$$\begin{aligned} & | \alpha_1 | | z_1^{\alpha-1} z_2 f_{n-1,n}(z_1) - z_1 z_2^{\alpha-1} f_{n-1,n}(z_2) | \\ \cong & | z - z_2 | \left[\sum_{j=1}^{m+1} (\alpha - 1) \left(\frac{r}{1-r} \right)^{2j-1} + \sum_{j=1}^m \left(\frac{r}{1-r} \right)^{2j} \right] \\ < & | z_1 - z_2 | \frac{r}{1-2r} [\alpha - 1 - (\alpha - 2)r] . \end{aligned}$$

This inequality and (2.4) give

$$(2.6) \quad \begin{aligned} & | f_{n,n}(z_1) - f_{n,n}(z_2) | \\ \cong & \frac{| f_{n,n}(z_1) f_{n,n}(z_2) |}{| z_1 z_2 |} | z_1 - z_2 | \left\{ 1 - \frac{r[\alpha - 1 - (\alpha - 2)r]}{1 - 2r} \right\} . \end{aligned}$$

Since Theorem 1.1 shows that neither of the factors $| f_{n,n}(z_i) / z_i |$, ($i=1, 2$), is zero, it follows from (2.6) that $f_{n,n}(z_1) \neq f_{n,n}(z_2)$ for $z_1 \neq z_2$ if r is such that $1 - 2r > r[\alpha - 1 - (\alpha - 2)r]$. This is equivalent to the condition $r < r_0(\alpha)$ where

$$\begin{aligned} r_0(2) &= 1/3 \\ r_0(\alpha) &= \frac{\alpha + 1 - \sqrt{\alpha^2 - 2\alpha + 9}}{2(\alpha - 2)} , \quad (\alpha = 1, 3, 4, \dots) , \end{aligned}$$

and it is easily seen that $f_{2m+1,2m+1}(z)$ is univalent for $|z|^\alpha < [U(\alpha)]^\alpha = 4r_0(\alpha)[1 - r_0(\alpha)]$.

If the function $f(z)$ has a non-terminating C -fraction (1.1), the univalence of $f(z)$ for $|z| < U(\alpha)$ is an immediate consequence of the fact that $f(z)$ is the uniform limit of its sequence of even approximants, $f_{2m+1,2m+1}(z)$, for $|z| \leq \rho < 1$. The case where $f(z)$ has a C -fraction expansion (1.1) terminating with an odd number of partial quotients may be reduced to the previously considered case for even approximants by adding a partial quotient, $a_{2m} z^\alpha / 1$ with $a_{2m} = 0$, and noting that $f_{2m-1,2m-1}(z) = f_{2m,2m}(z)$ in this case.

In order to complete the proof that the radius of univalence of K_α is the value $U(\alpha)$ given in (2.3), it suffices to exhibit a function of K_α which is not univalent in $|z| < \rho$ for any $\rho > U(\alpha)$. Such a function is the function $f(z, \pi)$ of (1.3), that is,

$$f(z, \pi) = \frac{2z}{3 - \sqrt{1 + z^\alpha}} ,$$

where the branch of the radical with positive real part for $|z| < 1$ is used. This function is not univalent at the points $e^{im\pi/\alpha} U(\alpha)$, ($m = 1, 2, \dots, \alpha$), where its derivative vanishes.

The final statement in Theorem 2.1 may be verified by applying to the function $f(z, \pi)$ the observation that, for every real θ , $e^{-i\theta} f(e^{i\theta} z) \in K_\alpha$

whenever $f(z) \in K_\alpha$.

3. A covering theorem. The value region inequality (1.2) can be rewritten as

$$(3.1) \quad \left| \frac{f(z)}{z} - \frac{4}{2 + \rho^\alpha + 2\sqrt{1 - \rho^\alpha}} \right| \leq \frac{2(1 - \sqrt{1 - \rho^\alpha})}{2 + \rho^\alpha + 2\sqrt{1 - \rho^\alpha}},$$

where $|z| = \rho$ and $f(z) \in K_\alpha$. Thus for $|z| = \rho$ the following inequalities, which provide a means of comparison between K_α and various classes of univalent functions, are obtained:

$$(3.2) \quad \frac{2}{3 - \sqrt{1 - \rho^\alpha}} \leq \Re \left\{ \frac{f(z)}{z} \right\} \leq \frac{2}{1 + \sqrt{1 - \rho^\alpha}},$$

$$(3.3) \quad \left| \Im \frac{f(z)}{z} \right| \leq \frac{2(1 - \sqrt{1 - \rho^\alpha})}{2 + \rho^\alpha + 2\sqrt{1 - \rho^\alpha}},$$

$$(3.4) \quad \frac{2\rho}{3 - \sqrt{1 - \rho^\alpha}} \leq |f(z)| \leq \frac{2(1 - \sqrt{1 - \rho^\alpha})}{\rho^{\alpha-1}},$$

$$(3.5) \quad \left| \arg \frac{f(z)}{z} \right| \leq \arcsin \frac{1 - \sqrt{1 - \rho^\alpha}}{2}.$$

Each of the inequalities (3.2)-(3.5) is sharp. This fact follows at once from Theorem 1.1 since equality in any one of (3.2)-(3.5) depends on the attainment by $f(z)/z$ of a suitable boundary value for the disc (3.1) or (1.2).

The following theorem is an immediate consequence of (3.4) and Theorem 2.1:

THEOREM 3.1. *If $f(z) \in K_\alpha$, then the image of $|z| < U(\alpha)$ by $w = f(z)$ contains the disc*

$$(3.6) \quad |w| < \frac{2U(\alpha)}{3 - \sqrt{1 - [U(\alpha)]^\alpha}},$$

and is contained in the disc

$$(3.7) \quad |w| < 2 \frac{1 - \sqrt{1 - [U(\alpha)]^\alpha}}{[U(\alpha)]^{\alpha-1}}.$$

These results are sharp.

4. A lower bound for $S(\alpha)$. An upper bound for $S(\alpha)$, the starlike radius for the class K_α , is evidently the value $U(\alpha)$ determined in § 2. In this section a lower bound for $S(\alpha)$ is found by determining a number

$\rho_1(\alpha)$ such that every function of K_α is starlike in the disc $|z| < \rho_1(\alpha)$.

LEMMA 4.1. *If $f(z) \in K_\alpha$ and $|a| \leq 1/4$, then*

$$(4.1) \quad w(z) = - \frac{az^{\alpha-1}f(z)}{1 + az^{\alpha-1}f(z)}$$

satisfies

$$(4.2) \quad \left| w - \frac{r^2}{1-r^2} \right| \leq \frac{r}{1-r^2}$$

whenever $|z|^\alpha \leq 4r(1-r)$, $0 \leq r \leq 1/2$.

Proof. The lemma is obvious when $a = 0$. For $0 < |a| \leq 1/4$, (4.1) yields

$$\frac{f(z)}{z} = \frac{1}{az^\alpha} \cdot \frac{-w(z)}{1+w(z)},$$

and the desired result is easily obtained by applying the inequality $|f(z)/z| \leq 1/(1-r)$, which is a consequence of Theorem 1.1.

LEMMA 4.2. *If α is a positive integer and if for fixed r , $0 < r < 1/2$, c and d are numbers such that*

$$(4.3) \quad 0 \leq c \leq \frac{1 + (\alpha - 2)r^2}{1 - 2r^2}, \quad 0 < d = \frac{1 + (\alpha - 2)r}{1 - 2r} - c,$$

then $\sigma = 1$ satisfies

$$(4.4) \quad |\sigma - c| \leq d.$$

Moreover, if w is a parameter satisfying (4.2) and if σ_0 satisfies (4.4), then σ_1 satisfies (4.4) where

$$(4.5) \quad \sigma_1 = 1 + w(\sigma_0 + \alpha - 1).$$

Proof. It is obvious that $1 - c \leq d$ holds for all r , $0 < r < 1/2$, and that $-d \leq 1 - c$ holds provided

$$c \leq \frac{2 + (\alpha - 4)r}{2(1 - 2r)}.$$

The fact that $\sigma = 1$ satisfies (4.4) may be verified by noting that the upper bound of c in this last inequality exceeds the upper bound on c in (4.3) for all r , $0 < r < 1/2$.

The proof of the second statement is obtained by using (4.2), (4.3),

(4.4), (4.5), and the triangle inequality to show that

$$\begin{aligned}
 |\sigma_1 - c| &\leq \left| 1 - c + \frac{(c + \alpha - 1)r^2}{1 - r^2} \right| \\
 &\quad + (c + \alpha - 1) \left| w - \frac{r^2}{1 - r^2} \right| + |w| |\sigma_0 - c| \\
 &\leq \frac{1 + (\alpha - 2)r^2 - (1 - 2r^2)c}{1 - r^2} + \frac{(c + \alpha - 1)r}{1 - r^2} + \frac{rd}{1 - r^2} = d.
 \end{aligned}$$

LEMMA 4.3. *If (4.3) holds for $0 < r < 1/2$, there is a value of c satisfying $c \geq d$ if and only if $0 < r \leq r_1(\alpha)$, where $r_1(\alpha)$ is the smallest positive root of*

$$(4.6) \quad 1 - (\alpha + 2)r + 2(\alpha - 1)r^2 - 2(\alpha - 2)r^3 = 0.$$

Proof. By (4.3) the inequality $c \geq d$ holds if and only if

$$\frac{1 + (\alpha - 2)r^2}{1 - 2r^2} \geq \frac{1 + (\alpha - 2)r}{2(1 - 2r)},$$

which is equivalent to the statement that the left member of (4.6) is nonnegative. Clearly $r_1(\alpha) < 1/2$.

THEOREM 4.1. *If $f(z) \in K_\alpha$ and c, d satisfy (4.3), where $|z|^\alpha = \rho^\alpha \leq 4r(1 - r)$, then*

$$(4.7) \quad \left| z \frac{f'(z)}{f(z)} - c \right| \leq d.$$

Proof. For the functions $f_{p,n}(z)$ of (2.1) put

$$\sigma_{p,n} = z \frac{f'_{p,n}}{f_{p,n}}, \quad w_{p,n} = - \frac{a_{n-p} z^{\alpha-1} f_{p+1,n}}{1 + a_{n-p} z^{\alpha-1} f_{p+1,n}},$$

and note by differentiation that $\sigma_{p+1,n} = 1 + w_{p,n}(\sigma_{p,n} + \alpha - 1)$. For $|z| = \rho$ inductive application of Lemmas 4.1 and 4.2 shows that (4.7) holds for $f_{n,n}$, and the validity of (4.7) in this case for $|z| \leq \rho$ follows from the maximum property for harmonic functions. Inasmuch as $f_{n,n}$ is the $(n + 1)$ th approximant of (1.1) the theorem holds for functions of K_α having terminating C -fraction expansions. The validity of the theorem in the case of non-terminating C -fractions (1.1) is an immediate consequence of the uniform convergence of $f_{n,n}$ to f on any closed subset of $|z| < 1$.

THEOREM 4.2. *The starlike radius of K_α satisfies $S(\alpha) \geq \rho_1(\alpha)$ where*

$[\rho_1(\alpha)]^\alpha = 4r_1(\alpha)[1 - r_1(\alpha)]$ and where $r_1(\alpha)$ is the smallest positive root of (4.6).

Proof. For $r \leq r_1(\alpha)$ Lemma 4.3 shows that Theorem 4.1 can be applied to any function $f(z) \in K_\alpha$ with $c \geq d$, and hence that

$$\Re z \frac{f'(z)}{f(z)} \geq 0, \quad |z| \leq \rho_1(\alpha).$$

Since this inequality insures that $f(z)$ is starlike for $|z| < \rho_1(\alpha)$ the proof is complete.

In particular, $r_1(1) = (\sqrt{3} - 1)/2$ and $S(1) \geq 4\sqrt{3} - 6$ which improves the lower bound of 8/9 obtained for $S(1)$ in [3].

5. A lower bound for $C(\alpha)$. It is clear that $S(\alpha)$ and $U(\alpha)$ are upper bounds for $C(\alpha)$, the radius of convexity of K_α . In this section a lower bound for $C(\alpha)$ is found by determining a number $\rho_2(\alpha)$ such that every function of K_α is convex for $|z| < \rho_2(\alpha)$.

LEMMA 5.1. *Let α denote a positive integer and let $r_2(\alpha)$ be the smallest positive root of the equation:*

$$(5.1) \quad 1 - (\alpha^2 + 2\alpha + 6)r + 6(\alpha^2 + \alpha + 2)r^2 - 4(3\alpha^2 + 2)r^3 + 12(\alpha - 1)\alpha r^4 - 4\alpha(\alpha - 2)r^5 = 0.$$

If for fixed r , $0 < r \leq r_2(\alpha)$, σ_0 and σ_1 are numbers which satisfy

$$(5.2) \quad |\sigma_0 - c| \leq d, \quad |\sigma_1 - c| \leq d,$$

where

$$(5.3) \quad \frac{1 + (\alpha - 2)r}{2(1 - 2r)} \leq c \leq \frac{1 + (\alpha - 2)r^2}{1 - 2r^2}, \quad d = \frac{1 + (\alpha - 2)r}{1 - 2r} - c,$$

and if

$$(5.4) \quad \gamma_1 = 2(\sigma_1 - 1) + \frac{\sigma_1 - 1}{\sigma_1} \left[\gamma_0 \frac{\sigma_0}{\sigma_0 + \alpha - 1} + (\alpha - 1) \frac{2\sigma_0 + \alpha - 2}{\sigma_0 + \alpha - 1} \right],$$

then $|\gamma_0| \leq 1$ implies $|\gamma_1| \leq 1$.

Proof. For $0 < r < r_1(\alpha)$, where $r_1(\alpha)$ is as determined in Theorem 4.2, $0 < d < c$ and

$$c^2 - d^2 - c \leq -\frac{\alpha r^2 [(\alpha - 1) - 2(\alpha - 2)r + 2(\alpha - 2)r^2]}{(1 - 2r)^2(1 - 2r^2)} \leq 0.$$

Thus by (5.2)

$$\left| \frac{\sigma_1 - 1}{\sigma_1} - \frac{c^2 - d^2 - c}{c^2 - d^2} \right| \leq \frac{d}{c^2 - d^2}$$

and it follows that

$$\left| \frac{\sigma_1 - 1}{\sigma_1} \right| \leq \frac{1}{c - d} - 1.$$

Similarly, (5.2) can be used to show that

$$\begin{aligned} \left| \frac{\sigma_0}{\sigma_0 + \alpha - 1} \right| &\leq \frac{c + d}{c + d + \alpha - 1}, \\ \left| (\alpha - 1) \frac{2\sigma_0 + \alpha - 2}{\sigma_0 + \alpha - 1} \right| &\leq (\alpha - 1) \frac{2(c + d) + \alpha - 2}{c + d + \alpha - 1}. \end{aligned}$$

For $|\gamma_0| \leq 1$ application to (5.4) of the triangle inequality, (5.2) and the bounds determined above lead to the inequality

$$(5.5) \quad |\gamma_1| \leq 2(c + d - 1) + \left[\frac{1}{c - d} - 1 \right] \frac{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2)}{c + d + \alpha - 1}.$$

The desired inequality, $|\gamma_1| \leq 1$, will hold for those values of $r < r_1(\alpha)$ for which the right member of (5.5) does not exceed 1, or equivalently, for which

$$(5.6) \quad c - d \geq \frac{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2)}{(2\alpha - 1)(c + d) + (\alpha - 1)(\alpha - 2) + [3 - 2(c + d)][c + d + \alpha - 1]} = D.$$

Since $2c = (c + d) + (c - d)$, (5.3) shows that the existence of a value of c satisfying (5.6) is insured for all $r < r_1(\alpha)$ for which

$$(5.7) \quad 2 \frac{1 + (\alpha - 2)r^2}{1 - 2r^2} \geq (c + d) + D.$$

This last inequality is equivalent to the requirement that the polynomial in the left member of (5.1) be non-negative.

The proof of the lemma will be completed by establishing the existence of a smallest positive zero, $r_2(\alpha)$ of (5.1) for which $r_2(\alpha) < r_1(\alpha)$. Since the equation (4.7) determining $r_1(\alpha)$ is equivalent to

$$2 \frac{1 + (\alpha - 2)r^2}{1 - 2r^2} = c + d,$$

and since $D > 0$ for $r = r_1(\alpha)$, it follows that (5.7) fails to hold for $r = r_1(\alpha)$. The desired conclusion about $r_2(\alpha)$ is then easily obtained by noting that (5.7) holds with strict inequality for $r = 0$.

THEOREM 5.1. *The radius of convexity of K_α satisfies*

$$(5.8) \quad [C(\alpha)]^\alpha \geq 4r_2(\alpha)[1 - r_2(\alpha)] = [\rho_2(\alpha)]^\alpha$$

where $r_2(\alpha)$ is the smallest positive root of (5.1)

Proof. For the functions $f_{p,n}(z)$ of (2.1) put

$$\sigma_{p,n} = z \frac{f'_{p,n}}{f_{p,n}}, \quad \gamma_{p,n} = z \frac{f''_{p,n}}{f'_{p,n}}.$$

It is easily verified from (2.1) that

$$\gamma_{p+1} = 2(\sigma_{p+1} - 1) + \frac{\sigma_{p+1} - 1}{\sigma_{p+1}} \left[\frac{\gamma_p \sigma_p}{\sigma_p + \alpha - 1} + (\alpha - 1) \frac{2\sigma_p + \alpha - 2}{\sigma_p + \alpha - 1} \right]$$

where the subscript n has been omitted. Theorem 4.1 and the fact that $\gamma_{0,n} = 0$ show that the hypotheses of Lemma 5.1 are satisfied, and inductive application of the lemma yields $|\gamma_{n,n}| \leq 1$. It follows that

$$\Re[1 + \gamma_{n,n}] \geq 0, \quad |z| \leq \rho_2(\alpha),$$

which insures the convexity of the $(n + 1)$ th approximant of any C -fraction (1.1) for $|z| < \rho_2(\alpha)$, and the proof of the theorem may be completed, as in Theorem 4.1, by reference to uniform convergence.

It is found that $\rho_2(1) > .641$. An upper bound for $C(\alpha)$ can be obtained by finding for the function $f(z, \pi)$ of (1.3) the zeros of $zf''(z, \pi) + f'(z, \pi)$ with smallest modulus. For $\alpha = 1$ this smallest modulus is approximately .707.

REFERENCES

1. J. F. Paydon and H. S. Wall, *The continued fraction as a sequence of linear transformations*, Duke Math. J., **9** (1942), 360-372.
2. O. Perron, *Über ein Schlichtheitschranke von James S. Thale*, S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1956.
3. J. S. Thale, *Univalence of continued fractions and Stieltjes transforms*, Proc. Amer. Math. Soc., **7** (1956), 232-244.
4. H. S. Wall, *Analytic theory of continued fractions*, Van Nostrand, New York, 1948.

DE PAUL UNIVERSITY
NORTHWESTERN UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DAVID GILBARG
Stanford University
Stanford, California

F. H. BROWNELL
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

L. J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY
D. DERRY

E. HEWITT
A. HORN
L. NACHBIN

M. OHTSUKA
H. L. ROYDEN
M. M. SCHIFFER

E. SPANIER
E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 10, No. 4

December, 1960

M. Altman, <i>An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space</i>	1107
Nesmith Cornett Ankeny, <i>Criterion for rth power residuacity</i>	1115
Julius Rubin Blum and David Lee Hanson, <i>On invariant probability measures I</i>	1125
Frank Featherstone Bonsall, <i>Positive operators compact in an auxiliary topology</i>	1131
Billy Joe Boyer, <i>Summability of derived conjugate series</i>	1139
Delmar L. Boyer, <i>A note on a problem of Fuchs</i>	1147
Hans-Joachim Bremermann, <i>The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces</i>	1149
Andrew Michael Bruckner, <i>Minimal superadditive extensions of superadditive functions</i>	1155
Billy Finney Bryant, <i>On expansive homeomorphisms</i>	1163
Jean W. Butler, <i>On complete and independent sets of operations in finite algebras</i>	1169
Lucien Le Cam, <i>An approximation theorem for the Poisson binomial distribution</i>	1181
Paul Civin, <i>Involutions on locally compact rings</i>	1199
Earl A. Coddington, <i>Normal extensions of formally normal operators</i>	1203
Jacob Feldman, <i>Some classes of equivalent Gaussian processes on an interval</i>	1211
Shaul Foguel, <i>Weak and strong convergence for Markov processes</i>	1221
Martin Fox, <i>Some zero sum two-person games with moves in the unit interval</i>	1235
Robert Pertsch Gilbert, <i>Singularities of three-dimensional harmonic functions</i>	1243
Branko Grünbaum, <i>Partitions of mass-distributions and of convex bodies by hyperplanes</i>	1257
Sidney Morris Harmon, <i>Regular covering surfaces of Riemann surfaces</i>	1263
Edwin Hewitt and Herbert S. Zuckerman, <i>The multiplicative semigroup of integers modulo m</i>	1291
Paul Daniel Hill, <i>Relation of a direct limit group to associated vector groups</i>	1309
Calvin Virgil Holmes, <i>Commutator groups of monomial groups</i>	1313
James Fredrik Jakobsen and W. R. Utz, <i>The non-existence of expansive homeomorphisms on a closed 2-cell</i>	1319
John William Jewett, <i>Multiplication on classes of pseudo-analytic functions</i>	1323
Helmut Klingen, <i>Analytic automorphisms of bounded symmetric complex domains</i>	1327
Robert Jacob Koch, <i>Ordered semigroups in partially ordered semigroups</i>	1333
Marvin David Marcus and N. A. Khan, <i>On a commutator result of Tausky and Zassenhaus</i>	1337
John Glen Marica and Steve Jerome Bryant, <i>Unary algebras</i>	1347
Edward Peter Merkes and W. T. Scott, <i>On univalence of a continued fraction</i>	1361
Shu-Teh Chen Moy, <i>Asymptotic properties of derivatives of stationary measures</i>	1371
John William Neuberger, <i>Concerning boundary value problems</i>	1385
Edward C. Posner, <i>Integral closure of differential rings</i>	1393
Marian Reichaw-Reichbach, <i>Some theorems on mappings onto</i>	1397
Marvin Rosenblum and Harold Widom, <i>Two extremal problems</i>	1409
Morton Lincoln Slater and Herbert S. Wilf, <i>A class of linear differential-difference equations</i>	1419
Charles Robson Storey, Jr., <i>The structure of threads</i>	1429
J. François Treves, <i>An estimate for differential polynomials in $\partial/\partial z_1, \dots, \partial/\partial z_n$</i>	1447
J. D. Weston, <i>On the representation of operators by convolutions integrals</i>	1453
James Victor Whittaker, <i>Normal subgroups of some homeomorphism groups</i>	1469