ASYMPTOTIC PROPERTIES OF DERIVATIVES OF
STATIONARY MEASURES

SHU-TEH CHEN MOY
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1. Introduction. Let X be a non-empty set and $\mathcal{S}$ be a $\sigma$-algebra of subsets of X. Consider the infinite product space $\Omega = \prod_{n=-\infty}^{\infty} X_n$ where $X_n = X$ for $n = 0, \pm 1, \pm 2, \cdots$ and the infinite product $\sigma$-algebra $\mathcal{T} = \prod_{n=-\infty}^{\infty} \mathcal{S}_n$ where $\mathcal{S}_n = \mathcal{S}$ for $n = 0, \pm 1, \pm 2, \cdots$. Elements of $\Omega$ are bilateral infinite sequences $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$ with $x_n \in X$. Let us denote the elements of $\Omega$ by $w$. If $w = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$, $x_n$ is called the nth coordinate of $w$ and shall be considered as a function on $\Omega$ to X. Let $T$ be the shift transformation on $\Omega$ to $\Omega$: the nth coordinate of $Tw$ is equal to the $n + 1$th coordinate of $w$. For any function $g$ on $\Omega$, $Tg$ is the function defined by $Tg(w) = g(Tw)$ so that $Tx_n = x_{n+1}$ for any integer $n$. We shall consider two probability measures $\mu, \nu$ defined on $\mathcal{T}$. For $n = 1, 2, \cdots$ let $\Omega_n = \prod_{i=1}^{n} X_i$ where $X_i = X$, $i = 1, 2, \cdots, n$, and $\mathcal{T}_n = \prod_{i=1}^{n} \mathcal{S}_i$ where $\mathcal{S}_i = \mathcal{S}$, $i = 1, 2, \cdots, n$. Then $\Omega_1 = X$ and $\mathcal{T}_1 = \mathcal{S}$. Let $\mathcal{T}_m, m \leq n, n = 0, \pm 1, \pm 2, \cdots$, be the $\sigma$-algebra of subsets of $\Omega$ consisting of sets of the form

$$[w = \{\cdots, x_{-1}, x_0, x_1, \cdots\}; (x_m, x_{m+1}, \cdots, x_n) \in E]$$

Where $E \in \mathcal{T}_{n-m+1}$. Then $\mathcal{T}_m \subset \mathcal{T}_{m+1} \subset \mathcal{T}$. Let $\mu_{m,n}$, $\nu_{m,n}$ be the contractions of $\mu, \nu$, respectively, to $\mathcal{T}_m$. If $\nu_{m,n}$ is absolutely continuous with respect to $\mu_{m,n}$, the derivative of $\nu_{m,n}$ with respect to $\mu_{m,n}$ is a function of $x_m, \cdots, x_n$ and shall be designated by $f_{m,n}(x_m, \cdots, x_n)$. Since $f_{m,n}(x_m, \cdots, x_n)$ is positive with $\nu$-probability one, $1/f_{m,n}(x_m, \cdots, x_n)$ is well defined with $\nu$-probability one. We shall let the function $1/f_{m,n}(x_m, \cdots, x_n)$ take on the value 0 when $f_{m,n}(x_m, \cdots, x_n) \leq 0$. Thus $1/f_{m,n}(x_m, \cdots, x_n)$ is well defined everywhere. In fact $1/f_{m,n}(x_m, \cdots, x_n)$ is the derivative of $\nu_{m,n}$-continuous part of $\mu_{m,n}$ with respect to $\nu_{m,n}$. According to the celebrated theorem of E. S. Anderson and B. Jessen [1] and J. L. Doob ([2]), pp. 343) $1/f_{m,n}(x_m, \cdots, x_n)$ converges with $\nu$-probability one as $n \to \infty$. If we assume that $\mu, \nu$ are stationary, i.e., $\mu, \nu$ are $T$ invariant, more precise results may be expected. A fundamental theorem of Information Theory, first proved by C. Shannon for stationary Markovian measures [5] and later generalized to any stationary measure by B. McMillan [4], may be considered as a theorem of this sort. In their theorem $X$ is assumed to be a finite set. In this paper we shall first treat Markovian stationary measures $\mu, \nu$ with $X$ being

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any set, finite or infinite, and \( \mathcal{S} \), any \( \sigma \)-algebra of subsets of \( X \). It will be proved that \( n^{-1} \log f_{m,n}(x_m, \ldots, x_n) \) converges as \( n \to \infty \) with \( \nu \)-probability one and also in \( L_1(\nu) \) under some integrability conditions. The case that \( \nu \) is only stationary is also treated. Similar convergence theorem is proved under the assumption that \( X \) is countable.

2. Asymptotic properties of derivatives of a Markovian measure with stationary transition probabilities with respect to another such measure.

Let \( X, \mathcal{X}, \Omega, \mathcal{F}, \Omega_n, \mathcal{F}_n, \mathcal{F}_m, \nu_{m,n}, \nu_{m,n} \) be as in \( \S \) 1. \( x_n, n = 0, \pm 1, \pm 2, \ldots \), are considered as functions or random variables on \( \Omega \) to \( X \). Notations for conditional probabilities and conditional expectations relative to one or several random variables will be as in [2], chapter 1, \( \S \) 7. Since we have two probability measures we shall use subscripts \( \mu, \nu \) to indicate conditional probabilities and conditional expectations taken under measures \( \mu, \nu \) respectively. In this section \( \mu, \nu \) are assumed to be Markovian i.e., for any \( A \in \mathcal{S}, m < n, n = 0 \pm 1, \pm 2, \ldots \),

1. \( P_\mu[x_n \in A | x_m, \ldots, x_{n-1}] = P_\mu[x_n \in A | x_{n-1}] \) with \( \mu \)-probability one and

2. \( P_\nu[x_n \in A | x_m, \ldots, x_{n-1}] = P_\nu[x_n \in A | x_{n-1}] \) with \( \nu \)-probability one. For any set \( E \subseteq \Omega \) let \( I_E \) be the real valued function on \( \Omega \) defined by

\[
I_E(w) = 1 \text{ if } w \in E \\
= 0 \text{ if } w \notin E.
\]

**Lemma 1.** If \( \nu_{n-1,n} \) is absolutely continuous with respect to \( \mu_{n-1,n} \) then for any \( A \in \mathcal{S} \)

3. \( P_\nu[x_n \in A | x_{n-1}] f_{n-1,n-1}(x_{n-1}) \\
\quad = E_\mu[I_{x_n \in A} f_{n-1,n-1}(x_{n-1}, x_n) | x_{n-1}] \) with \( \mu \)-probability one.

**Proof.** For any \( A, B \in \mathcal{S} \)

\[
\nu[x_n \in A, x_{n-1} \in B] \\
\quad = \int_{[x_{n-1} \in B]} P_\nu[x_n \in A | x_{n-1}] d\nu \\
\quad = \int_{[x_{n-1} \in B]} P_\nu[x_n \in A | x_{n-1}] f_{n-1,n-1}(x_{n-1}) d\mu.
\]

On the other hand

\[
\nu[x_n \in A, x_{n-1} \in B] \\
\quad = \int_{[x_{n-1} \in B]} I_{x_n \in A} f_{n-1,n-1}(x_{n-1}, x_n) | x_{n-1}) d\mu.
\]
Hence for any $B \in \mathcal{F}$

$$
= \int_{\{x_{n-1} \in B\}} E_\mu[I_{x_n \in A} f_{n-1}(x_{n-1}, x_n) \mid x_{n-1}] d\mu.
$$

therefore (3) is true with $\mu$-probability one. Dividing both sides of (3) by $f_{n-1}(x_{n-1})$ we then have

$$(4) \quad P_\mu[x_n \in A \mid x_{n-1}] = \frac{E_\mu[I_{x_n \in A} f_{n-1}(x_{n-1}, x_n) \mid x_{n-1}]}{f_{n-1}(x_{n-1})}.
$$

With $\mu$-probability one on the set $[f_{n-1}(x_{n-1}) > 0]$. Since $\nu[f_{n-1}(x_{n-1}) > 0] = 1$, (4) is true with $\nu$-probability one.

**Theorem 1.** If $\nu_{n-1}$ is absolutely continuous with respect to $\mu_{n-1}$ for $n = 0, \pm 1, \pm 2, \cdots$ then $\nu_{mn}$ is absolutely continuous with respect to $\mu_{mn}$ for $n = 0, \pm 1, \pm 2, \cdots$ and $m \leq n$ with

$$(5) \quad f_{mn}(x_m, \ldots, x_n) = f_{m+1}(x_m, x_{m+1}) f_{m+2}(x_{m+1}, x_{m+2}) \cdots f_{n-1}(x_{n-1}, x_n) f_{n-1}(x_{n-1})
$$

with $\mu$-probability one.

**Proof.** We shall prove the theorem for the case that $m = 1, n = 2, 3, \cdots$. The proof for the general case that $m$ is any integer is similar. Since $\nu_{12}$ is absolutely continuous with respect to $\mu_{12}$ by hypothesis, (5) is trivially true for $m = 1, n = 2$. Suppose $\nu_{1k}$ ($k \geq 2$) is absolutely continuous with respect to $\mu_{1k}$ and $f_{1k}(x_1, \cdots, x_k)$ is given by (5) with $\mu$-probability one. For any $A \in \mathcal{F}$, $B \in \mathcal{F}_k$

$$
\nu[x_{k+1} \in A, (x_1, \cdots, x_k) \in B] = \int_{\{(x_1, \cdots, x_k) \in B\}} P_\nu[x_{k+1} \in A \mid x_1, \cdots, x_k] d\nu.
$$

Since $\nu$ is Markovian and by (4)

$$
\nu[x_{k+1} \in A, (x_1, \cdots, x_k) \in B] = \int_{\{(x_1, \cdots, x_k) \in B\}} P_\nu[x_{k+1} \in A \mid x_k] d\nu
$$
\[ = \int_{\{(x_1, \ldots, x_k) \in B\}} \frac{E_{\mu}[I_{x_k+1} \in A f_k(x_k, x_{k+1}) | x_k]}{f_k(x_k)} d\nu \]
\[ = \int_{\{(x_1, \ldots, x_k) \in B\}} \frac{E_{\mu}[I_{x_k+1} \in A f_k(x_k, x_{k+1}) | x_k]}{f_k(x_k)} f_1(x_1, \ldots, x_k) d\mu \]

Since \( \mu \) is Markovian
\[ E_{\mu}[I_{x_k+1} \in A f_k(x_k, x_{k+1}) | x_k] = E_{\mu}[I_{x_k+1} \in A f_k(x_k, x_{k+1}) | x_1, \ldots, x_k] \]

with \( \mu \)-probability one. Hence
\[ \nu[x_{k+1} \in A, (x_1, \ldots, x_k) \in B] \]
\[ = \int_{\{(x_1, \ldots, x_k) \in B\}} \frac{E_{\mu}[I_{x_k+1} \in A f_k(x_k, x_{k+1}) f_1(x_1, \ldots, x_k) | x_1, \ldots, x_k]}{f_k(x_k)} d\mu \]
\[ = \int_{\{(x_1, \ldots, x_k) \in B\}} I_{x_k+1} \in A f_1(x_1, \ldots, x_k) f_{k+1}(x_k, x_{k+1}) f_k(x_k) d\mu \]

Hence
\[ \nu[x_{k+1} \in A, (x_1, \ldots, x_k) \in B] \]
\[ = \int_{\{(x_1, \ldots, x_k) \in B\}} f_1(x_1, \ldots, x_k) f_{k+1}(x_k, x_{k+1}) f_k(x_k) d\mu \]

for any \( A \in \mathcal{F}, B \in \mathcal{F}_k \). Hence for any \( E \in \mathcal{F}_{k+1} \)
\[ \nu(E) = \int_E f_1(x_1, \ldots, x_k) f_{k+1}(x_k, x_{k+1}) f_k(x_k) d\mu \]

Therefore \( \nu_{k+1} \) is absolutely continuous with respect to \( \mu_{k+1} \) and

(6) \[ f_{k+1}(x_1, \ldots, x_{k+1}) = f_1(x_1, \ldots, x_k) f_{k+1}(x_k, x_{k+1}) f_k(x_k) \]

with \( \mu \)-probability one. (6) together with the supposition that (5) holds true for \( m = 1, n = k \) implies that (5) holds true for \( m = 1, n = k + 1 \). Thus the theorem for the case that \( m = 1 \) is proved.

Any Markovian probability measure on \( \mathcal{F} \) is said to have stationary transition probabilities if \( E \) being a set of probability one implies that \( TE, T^{-1}E \) are also of probability one and for any \( A \in \mathcal{F} \) and any \( n \)
\[ P[x_{n+1} \in A | x_n] = TP[x_n \in A | x_{n-1}] \]

with probability one. Thus for a Markovian probability measure with stationary transition probabilities we have for any pair of integers \( m, n \) and any \( A \in \mathcal{F} \).
(7) \[ P[x_n \in A \mid x_{n-1}] = T^{n-m}P[x_m \in A \mid x_{m-1}] \] with probability one and

(8) \[ E[g(x_{n-1}, x_n) \mid x_{n-1}] = T^{n-m}E[g(x_{m-1}, x_m) \mid x_{m-1}] \] with probability one for any real valued \( \mathcal{F}_m \)-measurable function \( g \) on \( \Omega_2 \).

**Theorem 2.** Let both \( \mu, \nu \) have stationary transition probabilities. If \( \nu_{n,n} \) is absolutely continuous with respect to \( \mu_{n,n} \) for \( n = 0, \pm 1, \pm 2, \cdots \) and \( \nu_{1,2} \) is absolutely continuous with respect to \( \mu_{1,2} \) then \( \nu_{m,n} \) is absolutely continuous with respect to \( \mu_{m,n} \) for \( m \leq n, n = 0, \pm 1, \pm 2, \cdots \) and

(9) \[ f_{m,n}(x_m, \cdots, x_n) = f_{m,m}(x_m) \frac{f_{1,2}(x_m, x_{m+1})}{f_{1,1}(x_m)} \cdots \frac{f_{1,2}(x_{n-1}, x_n)}{f_{1,1}(x_{n-1})} \]

with \( \mu \)-probability one.

**Proof.** By Lemma 1, for any \( A \in \mathcal{S} \)

(10) \[ P_x[x_2 \in A \mid x_1] = \frac{E_{\mu}[I_{x_2 \in A,f_{1,2}(x_1, x_2) \mid x_1}]}{f_{1,1}(x_1)} \]

with \( \nu \)-probability one. For any \( A, B \in \mathcal{S} \)

\[ \nu[x_n \in A, x_{n-1} \in B] = \int_{[x_{n-1} \in B]} P_x[x_n \in A \mid x_{n-1}]d\nu \]

\[ = \int_{[x_{n-1} \in B]} T^{n-2}P_x[x_2 \in A \mid x_1]d\nu \]

\[ = \int_{[x_{n-1} \in B]} \{T^{n-2}P_x[x_2 \in A \mid x_1]\}f_{n-1,n-1}(x_{n-1})d\mu \]

Hence by (10) and (8)

\[ \nu[x_n \in A, x_{n-1} \in B] \]

\[ = \int_{[x_{n-1} \in B]} T^{n-1} \left\{ \frac{E_{\mu}[I_{x_2 \in A,f_{1,2}(x_1, x_2) \mid x_1}]}{f_{1,1}(x_1)} \right\} f_{n-1,n-1}(x_{n-1})d\mu \]

\[ = \int_{[x_{n-1} \in B]} \frac{E_{\mu}[I_{x_2 \in A,f_{1,2}(x_n, x_{n-1}) \mid x_{n-1}}]}{f_{1,1}(x_{n-1})} f_{n-1,n-1}(x_{n-1})d\mu \]

\[ = \int_{[x_{n-1} \in B]} I_{x_n \in A}f_{n-1,n-1}(x_{n-1}) \frac{f_{1,2}(x_{n-1}, x_n)}{f_{1,1}(x_{n-1})} d\mu \]

\[ = \int_{[x_{n-1} \in B]} f_{n-1,n-1}(x_{n-1}) \frac{f_{1,2}(x_{n-1}, x_n)}{f_{1,1}(x_{n-1})} d\mu \].

Thus for any \( E \in \mathcal{F}_{n-1,n} \)
Hence for any integer $n$, $\nu_{n-1} \propto \nu$ is absolutely continuous with respect to $\mu_{n-1}$ and Theorem 1 is applicable. (11) also implies that

$$f_{n-1}(x_{n-1}, x_n) = f_{n-1}(x_{n-1}) \frac{f_1(x_{n-1}, x_n)}{f_1(x_{n-1})}$$

with $\mu$-probability one. Hence

$$\frac{f_{n-1}(x_{n-1}, x_n)}{f_{n-1}(x_{n-1})} = \frac{f_1(x_{n-1}, x_n)}{f_1(x_{n-1})}$$

with $\mu$-probability one on the set $[f_{n-1}(x_{n-1}) > 0]$. However, except that $w$ belongs to a set of $\mu$-probability 0, $n > 1$, $f_{n-1}(x_{n-1}(w)) = 0$ imply that $f_1(x_1(w), \cdots, x_{n-1}(w)) = 0$, hence

$$f_1(x_1, \cdots, x_{n-1}) \frac{f_{n-1}(x_{n-1}, x_n)}{f_{n-1}(x_{n-1})} = f_1(x_1, \cdots, x_{n-1}) \frac{f_1(x_{n-1}, x_n)}{f_1(x_{n-1})}$$

with $\mu$-probability one. Thus by (6)

$$f_1(x_1, \cdots, x_n) = f_1(x_1, \cdots, x_{n-1}) \frac{f_1(x_{n-1}, x_n)}{f_1(x_{n-1})}$$

with $\mu$-probability one. Combining (12) (13) and by induction, if $n > 1$

$$f_1(x_1, \cdots, x_n) = f_1(x_1) \frac{f_1(x_{n-1}, x_n)}{f_1(x_{n-1})} \cdots \frac{f_1(x_2, x_n)}{f_1(x_2)}$$

with $\mu$-probability one. Thus we have proved the theorem for the case that $m = 1$. For the general case the proof is similar.

**Theorem 3.** If $\mu$ has stationary transition probabilities and $\nu$ is stationary and if

$$\int |\log f_{m, n+1}(x_m, x_{m+1})| \, d\nu < \infty \text{ then }$$

$$\int |\log f_{m, n}(x_m, \cdots, x_n)\, d\nu < \infty \text{ for } n = m, m + 1, m + 2, \cdots \text{ and } n^{-1} \log f_{m, n}(x_m, \cdots, x_n) \text{ converges as } n \to \infty \text{ with } \nu\text{-probability one}$$

and also in $L_1(\nu)$ to a function $g$ with $\int g \, d\nu = a$ where

$$a = \int [\log f_1(x_1, x_2) - \log f_1(x_1)] \, d\nu \geq 0$$

In particular, if $\nu$ is ergodic, $g = a$ with $\nu$-probability one.
Proof. We shall first prove the theorem for the case that $m = 1$. Since for any $A \in \mathcal{F}$
\[ \nu[x_1 \in A] = \int_{[x_1 \in A]} f_{1,1}(x_1) \, d\mu = \int_{[x_1 \in A]} f_{1,2}(x_1, x_2) \, d\mu, \]
hence
\[ E_\mu[f_{1,2}(x_1, x_2) \mid x_1] = f_{1,1}(x_1). \]
Since $\int |\log f_{1,2}(x_1, x_2)| \, d\nu < \infty$ hence
\[ \int |f_{1,2}(x_1, x_2) \log f_{1,2}(x_1, x_2)| \, d\mu = \int |\log f_{1,2}(x_1, x_2)| \, d\nu < \infty. \]
The real valued function $L(\xi) = \xi \log \xi$ defined for all real $\xi \geq 0$ [$L(0)$ is
taken to be 0] is convex. By Jensen’s inequality for conditional ex-
pectations ([2], pp. 33)
\[ (15) \quad E_\mu[L(f_{1,2}(x_1, x_2)) \mid x_1] \geq L(f_{1,1}(x_1)). \]
By (15) and the fact that $L(\xi)$ is a function bounded below by a con-
stant, we have
\[ \int |L(f_{1,1}(x_1))| \, d\mu = \int |\log f_{1,1}(x_1)| \, d\nu < \infty \]
and
\[ \int \log f_{1,2}(x_1, x_2) d\nu - \int \log f_{1,1}(x_1) d\nu = a \geq 0. \]
Now by Theorem 2
\[ \log f_{1,n}(x_1, \ldots, x_n) = \log f_{1,1}(x_1) + \sum_{i=2}^{n} \{\log f_{1,2}(x_{i-1}, x_i) - \log f_{1,1}(x_{i-1})\}. \]
Since $\nu$ is stationary, $\log f_{1,n}(x_1, \ldots, x_n)$ is $\nu$-integrable. Applying the
ergodic theorem $n^{-1} \log f_{1,n}(x_1, \ldots, x_n)$ converges with $\nu$-probability one
and also in $L_1(\nu)$ to a function $g$ with
\[ \int g d\nu = \int [\log f_{1,2}(x_1, x_2) - \log f_{1,1}(x_1)] d\nu = a \geq 0. \]
For $m$ being any integer, we only need to mentioned that by (13),
\[ \log f_{m, m+1}(x_m, x_{m+1}) - \log f_{m, m}(x_m) = \log f_{1,2}(x_1, x_2) - \log f_{1,1}(x_1) \]
with $\nu$-probability one and therefore the same conclusion follows with
a similar proof.
COROLLARY 1. Suppose \( \mu, \nu \) satisfy the hypothesis of Theorem 3 for \( m = 1 \). If \( \nu \) is ergodic and if there is an \( A \in \mathcal{F} \) such that

\[
\nu\{P_\mu[x_2 \in A \mid x_1] \neq P_\mu[x_2 \in A \mid x_1]\} > 0
\]

then \( \nu \) is singular with respect to \( \mu \).

Proof. First we shall show that follows from (16)

\[
\mu[f_{i_1}(x_i) \neq f_{i_2}(x_i, x_2)] > 0.
\]

For, if \( f_{i_1}(x_i) = f_{i_2}(x_i, x_2) \) with \( \mu \)-probability one then by Lemma 1

\[
P_\mu[x_2 \in A \mid x_1]f_{i_1}(x_i) = P_\mu[x_2 \in A \mid x_1]f_{i_2}(x_i)
\]

with \( \mu \)-probability one. Thus \( P_\mu[x_2 \in A \mid x_1] = P_\mu[x_2 \in A \mid x_1]f_{i_1}(x_i) \) with \( \nu \)-probability one for every \( A \in \mathcal{F} \). Now the function \( L(\xi) = \xi \log \xi \) is strictly convex, hence it follows from (17) that

\[
a = \int \left[ L(f_{i_2}(x_1, x_2)) - L(f_{i_1}(x_1)) \right] d\mu > 0.
\]

Applying Theorem 3 \( f_{i_1} (x_1, \ldots, x_n) \to \infty \) with \( \nu \)-probability one as \( n \to \infty \). Hence \( 1/f_{i_n} (x_1, \ldots, x_n) \to 0 \) with \( \nu \)-probability one as \( n \to \infty \). Let \( \mathcal{F}' \) be the \( \sigma \)-algebra generated by \( \bigcup_{n=1}^\infty \mathcal{F}_{1,n} \) and \( \mu', \nu' \) be the contractions of \( \mu, \nu \) to \( \mathcal{F}' \) respectively. Since \( 1/f_{i_1} (x_1, \ldots, x_n) \) is the derivative of \( \nu_{1,n} \)-continuous part of \( \mu_{1,n} \) with respect to \( \nu_{1,n} \), \( 1/f_{i_n} (x_1, \ldots, x_n) \) converges with \( \nu \)-probability one as \( n \to \infty \) to the derivative of \( \nu' \)-continuous part of \( \mu' \) with respect to \( \nu' \) ([2], pp. 343). Now \( 1/f_{i_1} (x_1, \ldots, x_n) \) converges to 0 with \( \nu \)-probability one, hence the \( \nu' \)-continuous part of \( \mu' \) is 0 and \( \mu' \), \( \nu' \) are mutually singular. Hence \( \mu, \nu \) are mutually singular.

3. Extension to \( k \)-Markovian measures. The results of the preceding section can be extended to \( k \)-Markovian measures immediately. We shall state the theorems only since the proofs in the preceding section with obvious modifications apply as well.

THEOREM 4. Let \( \mu, \nu \) be any two \( k \)-Markovian measures on \( \mathcal{F} \). If \( \nu_{n-k} \) is absolutely continuous with respect to \( \mu_{n-k} \) for \( n = 0, \pm 1, \pm 2, \cdots \), then \( \nu_m \) is absolutely continuous with respect to \( \mu_m \) for \( n = 0, \pm 1, \pm 2, \cdots \) and \( m \leq n \) with

\[
f_{m,n}(x_m, \ldots, x_n) = f_{m-k}(x_{m-k}, \ldots, x_{m+k}) f_{m+1,m+1+k}(x_{m+1}, \ldots, x_{m+1+k}) f_{m+1,m+k}(x_{m+1}, \ldots, x_{m+k}) \ldots \frac{f_{n-k,m+k}(x_{n-k}, \ldots, x_n)}{f_{n-k,n}(x_{n-k}, \ldots, x_n)}
\]

with \( \mu \)-probability one.
THEOREM 5. Let $\mu, \nu$ be two $k$-Markovian measures on $\mathcal{F}$ with stationary transition probabilities. If $\nu_{n-k+1,n}$ is absolutely continuous with respect to $\mu_{n-k+1,n}$ for $n = 0, \pm 1, \pm 2, \cdots$ and $\nu_{1,k+1}$ is absolutely continuous with respect to $\mu_{1,k+1}$ then $\nu_{m,n}$ is absolutely continuous with respect to $\mu_{m,n}$ for $n = 0, \pm 1, \pm 2, \cdots, m \leq n$ and

\begin{align}
(19) \quad f_{m,n}(x_m, \cdots, x_n) &= f_{m,m+k-1}(x_m, \cdots, x_{m+k-1}) \frac{f_{1,k+1}(x_{m+1}, \cdots, x_{m+k+1})}{f_{1,k}(x_{m+1}, \cdots, x_{m+k})} \\
& \quad \times \frac{f_{1,k+1}(x_{n-k}, \cdots, x_n)}{f_{1,k}(x_{n-k}, \cdots, x_{n-1})}
\end{align}

with $\mu$-probability one.

THEOREM 6. Let $\mu, \nu$ be two $k$-Markovian measures such that $\nu$ is stationary and $\mu$ has stationary transition probabilities. If

\[ \int |\log f_{m,m+k}(x_m, \cdots, x_{m+k})| \, d\nu < \infty \]

then $\int |\log f_{m,n}(x_m, \cdots, x_n)| \, d\nu < \infty$ for $n = m, m+1, m+2, \cdots$ and $n^{-1}\log f_{m,n}(x_m, \cdots, x_n)$ converges as $n \to \infty$ with $\nu$-probability one to a function $g$ with $\int g \, d\nu = a \geq 0$ where

\[ a = \int |\log f_{1,k+1}(x_1, \cdots, x_{k+1}) - \log f_{1,k}(x_1, \cdots, x_k)| \, d\nu \geq 0. \]

In particular, if $\nu$ is ergodic, $g = a$ with $\nu$-probability one.

COROLLARY 2. Suppose $\mu, \nu$ satisfy the hypothesis of Theorem 6 for $m = 1$. If $\nu$ is ergodic and if there is a set $A \in \mathcal{F}$ such that

\[ \nu\{[P_m(x_{k+1} \in A \mid x_1, \cdots, x_k) \neq P_m(x_{k+1} \in A) \mid x_1, \cdots, x_k]\} > 0 \]

Then $\nu$ is singular with respect to $\mu$.

4. A generalization of McMillan’s theorem. In the setting of this paper, McMillan’s Theorem may be stated as the following. Let $X$ be a finite set of $K$ points and $\mathcal{F}$ be the $\sigma$-algebra of all subsets of $X$. Let $\nu$ be any stationary probability measure on $\mathcal{F}$ and $\mu$ the measure on $\mathcal{F}$ such that $\mu[X_m = a_0, X_{m+1} = a_1, \cdots, X_n = a_{n-m}] = K^{-(n-m+1)}$ for any intergers $m, n$ and $a_0, a_1 \cdots a_{n-m}$ in $X$. $\mu$ may be described as the equally distributed independent measure on $\mathcal{F}$. Then $n^{-1} f_{1,n}(x_1, \cdots, x_n)$ converges as $n \to \infty$ in $L_1(\nu)$. In particular, if $\nu$ is ergodic, the limit function is equal to $\log K - H$ with $\nu$-probability one where $H$ is the entropy of $\nu$ measure [4]. We shall generalize this theorem to the case that $X$ is countable and $\mu$ is Markovian with stationary transition probabilities.
THEOREM 7. Let the totality of elements of $X$ be $a_1, a_2, \cdots$ and $\nu$ be a stationary probability measure on $\mathcal{F}$ such that $\int -\log \nu_1(x) d\nu < \infty$ where $\nu_1$ is the function defined on $X$ by $\nu_1(a_i) = \nu[x_1 = a_i]$. Let $\mu$ be a Markovian measure on $\mathcal{F}$ with stationary transition probabilities. Let $p(a_i, a_j)$ be the value of $P_\mu[x_1 = a_j | x_0 = a_i]$. Let $\nu_{1n}$ be absolutely continuous with respect to $\mu_{1n}$ for $n = 1, 2, \cdots$. If

$$
\int -\log p(x_1, x_0) d\nu < \infty
$$

and $\int |\log f_1(x_i)| d\nu < \infty$ then $\int |\log f_{1n}(x_i, \cdots, x_n)| d\nu < \infty$ for $n = 1, 2, \cdots$ and $n^{-1} \log f_{1n}(x_1, \cdots, x_n)$ converges as $n \to \infty$ in $L_1(\nu)$. In particular, if $\nu$ is ergodic, the limit is equal to a constant with $\nu$-probability one.

Proof. Let

$$
\nu_n(a_{i_1}, a_{i_2}, \cdots, a_{i_n}) = \nu[x_1 = a_{i_1}, x_2 = a_{i_2}, \cdots, x_n = a_{i_n}]
$$

and

$$
\mu_n(a_{i_1}, a_{i_2}, \cdots, a_{i_n}) = \mu[x_1 = a_{i_1}, x_2 = a_{i_2}, \cdots, x_n = a_{i_n}].
$$

Then

$$
f_{1n}(x_1, \cdots, x_n) = \frac{\nu_n(x_1, \cdots, x_n)}{\mu_n(x_1, \cdots, x_n)}
$$

with $\mu$-probability one and

$$
\nu_n(a_{i_1} \mid x_{n-1}, \cdots, x_1) = \frac{\nu_n(x_1, \cdots, x_{n-1}, a_i)}{\nu_{n-1}(x_1, \cdots, x_{n-1})}
$$

with $\nu$-probability one and

$$
P_\mu[x_n = a_i \mid x_{n-1}] = \frac{\mu_n(x_1, \cdots, x_{n-1}, a_i)}{\mu_n(x_1, \cdots, x_{n-1})}
$$

with $\mu$-probability one. Hence

$$
\frac{f_{1n}(x_1, \cdots, x_n)}{f_{1n-1}(x_1, \cdots, x_{n-1})} = \sum_{i=1}^{\infty} P_\mu[x_n = a_i \mid x_{n-1}, \cdots, x_1] I_{x_n = a_i}
$$

with $\nu$-probability one and

$$
\log \frac{f_{1n-1}(x_1, \cdots, x_{n-1})}{f_{1n-1}(x_1, \cdots, x_{n-1})} = \sum_{i=1}^{\infty} \log P_\mu[x_n = a_i \mid x_{n-1}, \cdots, x_1] I_{x_n = a_i}
$$

$$
-\log p(x_{n-1}, x_n)
$$

$$
= T^n g_n
$$
with \( \nu \)-probability one where

\[
g_n = \sum_{i=1}^{\infty} \log P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}]I_{x_0=a_i} - \log p(x_{-1}, x_0) .
\]

We know that \( P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}] \) converges with \( \nu \)-probability one as \( n \to \infty \) to \( P[x_0 = a_i \mid x_{-1}, x_{-2}, \ldots] \) by Doob's Martingale Convergence Theorem. Hence \( L\{P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}]\} \) converges with \( \nu \)-probability one to \( L\{P[x_0 = a_i \mid x_{-1}, x_{-2}, \ldots]\} \). But \( L(\xi) \) is a bounded function for \( 0 \leq \xi \leq 1 \), hence \( L\{P[x_0 = a_i \mid x_{-1}, x_{-(n-1)}]\} \) are uniformly bounded with \( \nu \)-probability one. Hence \( L\{P[x_0 = x_i \mid x_{-1}, \ldots, x_{-(n-1)}]\} \) also converges in \( L_1(\nu) \) to \( L\{P[x_0 = x_i \mid x_{-1}, x_{-2}, \ldots]\} \) as \( n \to \infty \). Now by Jensen's inequality \( \int -L(P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}])d\nu \leq -L(P[x_0 = a_i]) \). Since

\[
\sum_{i=1}^{\infty} - L\{P[x_0 = a_i]\} = \int - \log \nu(x_0)d\nu < \infty
\]

converges in \( L_1(\nu) \), as \( m \to \infty \), to

\[
\sum_{i=1}^{\infty} - L\{P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}]\}
\]

uniformly in \( n \). Hence

\[
\sum_{i=1}^{\infty} - L\{P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}]\}
\]

converges in \( L_1(\nu) \) to

\[
\sum_{i=1}^{\infty} - L\{P[x_0 = a_i \mid x_{-1}, x_{-2}, \ldots]\} \text{ as } n \to \infty .
\]

Now

\[
\int - \sum_{i=1}^{\infty} \log P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}]I_{x_0=a_i} d\nu
\]

\[
= \int - \sum_{i=1}^{\infty} L\{P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}]\}d\nu \text{ and}
\]

\[
\int - \sum_{i=1}^{\infty} \log P[x_0 = a_i \mid x_{-1}, x_{-2}, \ldots]I_{x_0=a_i} d\nu
\]

\[
= \int - \sum_{i=1}^{\infty} L\{P[x_0 = a_i \mid x_{-1}, x_{-2}, \ldots]\}d\nu , \text{ hence}
\]

(23) \[
\lim_{n \to \infty} \int - \sum_{i=1}^{\infty} \log P[x_0 = a_i \mid x_{-1}, \ldots, x_{-(n-1)}]I_{x_0=a_i} d\nu
\]

\[
= \int - \sum_{i=1}^{\infty} \log P[x_0 = a_i \mid x_{-1}, x_{-2}, \ldots]I_{x_0=a_i} d\nu .
\]
(23) together with the facts that the sequence
\[
- \sum_{t=1}^{\infty} \log P_t[x_0 = x_t | x_{-1}, \cdots, x_{-(n-1)}] I_{x_0 = a_t}
\]
is also convergent with \(\nu\)-probability one and that the functions
\[
- \sum_{t=1}^{\infty} \log P_t[x_0 = x_t | x_{-1}, \cdots, x_{-(n-1)}] I_{x_0 = a_t}
\]
are non-negative with \(\nu\)-probability one imply that
\[
\sum_{t=1}^{\infty} P_t[x_0 = a_t | x_{-1}, \cdots, x_{-(n-1)}] I_{x_0 = a_t}
\]
converges as \(n \to \infty\) in \(L_\lambda(\nu)\) to
\[
\sum_{t=1}^{\infty} P_t[x_0 = a_t | x_{-1}, x_{-2}, \cdots] I_{x_0 = a_t}.
\]
Thus we have \(\{g_n\}\) to be an \(L_\lambda(\nu)\) convergent sequence. Let the limit of the sequence be \(h\). Let \(\bar{h}\) be the \(L_\lambda(\nu)\) limit of \(1/n(h + Th + \cdots + T^n h)\) as \(n \to \infty\). Now by (21)
\[
\log f_{1,2}(x_1, \cdots, x_n) = \log f_{1,1}(x_1) + \sum_{t=2}^{n} T^t g_t.
\]
Thus
\[
\int \left| \frac{1}{n} \log f_{1,n}(x_1, \cdots, x_n) - \bar{h} \right| d\nu \\
\leq \frac{1}{n} \int \left| \log f_{1,1}(x_1) \right| d\nu + \left| \frac{1}{n} \left( \sum_{t=2}^{n} T^t g_t - \bar{h} \right) \right| d\nu \\
+ \left| \frac{1}{n} \sum_{t=2}^{n} T^t h - \bar{h} \right| d\nu
\]
\[
= \frac{1}{n} \int \left| \log f_{1,1}(x_1) \right| d\nu + \frac{1}{n} \sum_{t=2}^{n} \left| g_t - \bar{h} \right| d\nu \\
+ \left| \frac{1}{n} \sum_{t=2}^{n} T^t h - \bar{h} \right| d\nu \to 0 \text{ as } n \to \infty.
\]

**Corollary 3.** Under the hypothesis of Theorem 7, if \(\nu\) is ergodic and not Markovian then \(\nu\) is singular to \(\mu\).

**Proof.** If \(\nu\) is ergodic then the \(L_\lambda(\nu)\) limit, \(\bar{h}\), of \(\{1/n \log f_{1,n}(x_1, \cdots, x_n)\}\) is equal with \(\nu\) probability one to
\[
\int \sum_{t=1}^{\infty} L[P_t[x_0 = a_t | x_{-1}, x_{-2}, \cdots]] d\nu - \int \log p(x_{-1}, x_0) d\nu
\]
which is greater or equal to
\[
\int \sum_{i=1}^{\infty} L[P[x_0 = a_i | x_{-1}, x_{-2}]]d\nu - \int \log p(x_{-1}, x_0)d\nu.
\]

Hence by (21)
\[
\bar{h} \geq \int \sum_{i=1}^{\infty} \log P[x_0 = a_i | x_{-1}, x_{-2}]I_{x_0 = a_i}d\nu - \int \log p(x_{-1}, x_0)d\nu
\]
\[
= \int \log f_1(x_1, x_2, x_3)d\nu - \int \log f_1(x_1, x_2)d\nu.
\]

However \[
\int \log f_1(x_1, x_2, x_3)d\nu - \int \log f_1(x_1, x_2)d\nu = 0 \text{ if and only if}
\]

(24) \[
\mu[f_1(x_1, x_2) \neq f_1(x_1, x_2, x_3)] = 0.
\]

(24) implies that
\[
P_s[x_3 \in A | x_1, x_2] = P_\mu[x_3 \in A | x_1, x_2]
\]
with $\nu$-probability one for any $A \in \mathcal{S}$. This is impossible since $\mu$ is Markovian and $\nu$ is not. Hence $\bar{h} > 0$ with $\nu$-probability one. Hence $f_1(x_1, \cdots, x_n) \to \infty$ with $\nu$ probability one and $\nu$ is singular to $\mu$ by the same argument used in the proof in Corollary 1.

The extensions of Theorem 7 and Corollary 3 to $k$-Markovian $\mu$ is obvious.

**References**


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